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Sheffer stroke branching of BCK-algebras

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Abstract

The main objective of the study is to introduce branches of Sheffer stroke BCK-algebras due their specific elements. At the onset of the study, an atom of a Sheffer stroke BCK-algebra is defined and it is shown that the set of all atoms of the algebraic structure is its subalgebra. Then it is proved that specified subsets defined by atoms of a Sheffer stroke BCK-algebra are ideals but the inverses are not true in general. Moreover, a branch and a chain on a Sheffer stroke BCK-algebra are introduced and some properties are presented. Finally, relationships between aforementioned concepts are built and supported by illustrative examples.

Keywords: *Sheffer stroke, Sheffer stroke BCK-algebra, ideal, atom, branch.*

BCK-cebirlerinin Sheffer stroke dallanması

Öz

Bu çalışmanın temel amacı, belirli elemanları yardımıyla Sheffer stroke BCK-cebirlerinin dallarını tanıtmaktır. Çalışmanın başlangıcında, bir Sheffer stroke BCK-cebirinin bir atomu tanımlanarak bu cebirsel yapının tüm atomlarının kümesinin bu yapının bir altcebiri olduğu gösterilmiştir. Ardından bir Sheffer stroke BCK-cebirinin bir atomu yardımıyla tanımlanan özel altkümelerinin bu cebirsel yapının idealleri olduğu fakat bu ifadenin tersinin genelde doğru olmadığı ispatlanmıştır. Dahası, bir Sheffer stroke BCK-cebiri üzerinde bir dal ve zincir tanımlanarak bazı özellikleri sunulmuştur. Son olarak, bahsi geçen yapılar arasındaki bağlantılar inşa edilmiştir ve bu bağlantılar açıklayıcı örneklerle desteklenmiştir.

Anahtar kelimeler: *Sheffer stroke, Sheffer stroke BCK-cebiri, ideal, atom, dal.*

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1. Introduction

In 1966, Imai and Iséki introduced the class of BCI-algebras in two ways. The first way is about the set theory; especially, a generalization of set-theoretic difference. The second way is about classical and non-classical propositional calculi. Then they defined the concept of BCK-algebras as a proper subclass of these algebras in the same year [1]. Recently, the concept of BCK-algebra draws attentions of scientists studying on several areas in mathematics such as group theory, functional analysis, probability theory, fuzzy set theory, topology, etc. Although the literature consists of several studies on BCK-algebras, importance seems to have been particularly attach to the ideal theory of BCK-algebras. For more details, recent studies [2-10] can be suggested in literature.

On the other side, H. M. Sheffer introduced Sheffer stroke known as NAND gate in logic, and is one of the two operators that can be used by itself to construct a logical formal system without any other logical operators [11]. The other one is Pierce arrow, called NOR gate in this field. Especially, Sheffer operator is utilised in computer science and algebra. In the first, it is used to have the single chip forming processor in a computer and to store and start up in flash memory disks. Therefore, there exist many patents about this operator in this field. Since it is simpler and cheaper than to produce different diods for each distinct Boolean operations, the usage of Sheffer stroke is more and more preferable. In the second, this operator can be applied several logical algebras such as Boolean algebras [12], ortholattices [13], orthoimplication algebras [14], branches and obstinate SBE-filters of Sheffer stroke BE-algebras [15], Sheffer stroke BL-algebras and their neutrosophic structures [16-17], fuzzy filters of Sheffer stroke Hilbert algebras [18] and their fuzzy ideals with t-conorms [19], Sheffer stroke BCK-algebras [20] and their neutrosophic N-ideals [21]. Thus, Sheffer stroke provides to reduce the number of axioms in a system, and new and easily controllable axiom systems for many algebraic structures. However, this operator has some disadvantages so that the lengths of axioms or formulas with Sheffer stroke can be long, or the readability can be difficult.

The organization of the study is as follows: the next section presents essentials on BCK-algebras equipped with Sheffer stroke. In the third section, the main results are provided, and outcomes of the study are supported with illustrative examples. The results of the manuscript are new and novel, therefore, contribute the ongoing theory of pure mathematics regarding BCK-algebras and Sheffer stroke.

2. Preliminaries

In this section, fundamental concepts of BCK-algebras equipped by Sheffer operation are presented. In an attempt to facilitate readability, a statement $\check{x}|\check{y}$ defined by Sheffer operation | is represented in the form $\check{x}|\check{y} \coloneqq \mathfrak{So}_{\check{x}\check{y}}$, for any elements $\check{x}$ and $\check{y}$.

Definition 1.1. [13] Let $\mathcal{A} = (\mathfrak{Z}, \mathfrak{So})$ be a groupoid. The operation $\mathfrak{So}$ on $\mathfrak{Z}$ is said to be a Sheffer stroke *(Sheffer operation)* if it satisfies the following:

(S1) $\mathfrak{So}_{\check{x}\check{y}} = \mathfrak{So}_{\check{y}\check{x}}$,
(S2) $\mathfrak{So}_{(\check{x}\check{x})(\check{x}\check{y})} = \check{x}$,
(S3) $\mathfrak{So}_{\check{x}((\check{y}\check{z})(\check{y}\check{z}))} = \mathfrak{So}_{((\check{x}\check{y})(\check{x}\check{y}))\check{z}}$,
(S4) $\mathfrak{So}_{(\check{x}((\check{x}\check{x})|(\check{y}\check{y})))(\check{x}((\check{x}\check{x})(\check{y}\check{y})))} = \check{x}$

for all $\breve{x}, \breve{y}, \breve{z} \in \mathfrak{Z}$.

Lemma 1.2. [13] Let $\mathcal{A} = (\mathfrak{Z}, \mathfrak{So})$ be a groupoid. Then a binary relation $\leq$ defined on $\mathfrak{Z}$ by

$$\hat{x} \leq \hat{y} \Leftrightarrow \mathfrak{So}_{\breve{x}\breve{y}} = \mathfrak{So}_{\breve{x}\breve{x}} \quad (1)$$

is an order on $\mathfrak{Z}$.

Definition 1.3. [20] A Sheffer stroke BCK-algebra is a structure $(\mathfrak{Z}, \mathfrak{So}, 0)$ of type $(2, 0)$, where $\mathfrak{Z}$ is a nonempty set and $\mathfrak{So}$ is Sheffer stroke on $\mathfrak{Z}$ such that the following identities are satisfied for all $\breve{x}, \breve{y}, \breve{z} \in \mathfrak{Z}$:

(sBCK-1) $\mathfrak{So}_{((((\breve{x}(\breve{y}\breve{y}))(\breve{x}(\breve{y}\breve{y})))(\breve{x}(\breve{z}\breve{z})))(((\breve{x}(\breve{y}\breve{y}))(\breve{x}(\breve{y}\breve{y})))(\breve{x}(\breve{z}\breve{z}))))(\breve{z}(\breve{y}\breve{y}))} = \mathfrak{So}_{00}$,
(sBCK-2) $\mathfrak{So}_{(\breve{x}(\breve{y}\breve{y}))(\breve{x}(\breve{y}\breve{y}))} = 0$ and $\mathfrak{So}_{(\breve{y}(\breve{x}\breve{x}))(\breve{y}(\breve{x}\breve{x}))} = 0$ imply $\breve{x} = \breve{y}$.

For short, the notion of Sheffer stroke BCK-algebra is written as ssBCK-algebra.

Lemma 1.4. [20] Let $(\mathfrak{Z}, \mathfrak{So}, 0)$ be a ssBCK-algebra. Then

(1) $\mathfrak{So}_{(\breve{x}(\breve{x}\breve{x}))(\breve{x}\breve{x})} = \hat{x}$,
(2) $\mathfrak{So}_{(\breve{x}(\breve{x}\breve{x}))(\breve{x}(\breve{x}\breve{x}))} = 0$,
(3) $\mathfrak{So}_{\breve{x}(((\breve{x}(\breve{y}\breve{y}))(\breve{y}\breve{y}))((\breve{x}(\breve{y}\breve{y}))(\breve{y}\breve{y})))} = \mathfrak{So}_{00}$,
(4) $\mathfrak{So}_{(00)(\breve{x}\breve{x})} = \breve{x}$,
(5) $\mathfrak{So}_{\breve{x}0} = \mathfrak{So}_{00}$,
(6) $\mathfrak{So}_{(\breve{x}(00))(\breve{x}(00))} = \breve{x}$,
(7) $\mathfrak{So}_{(0(\breve{x}\breve{x}))(\breve{0}(\breve{x}\breve{x}))} = 0$,
(8) $\mathfrak{So}_{(\breve{x}((\breve{y}(\breve{z}\breve{z}))(\breve{y}(\breve{z}\breve{z}))))((\breve{y}((\breve{x}(\breve{z}\breve{z}))(\breve{x}(\breve{z}\breve{z}))))(\breve{y}((\breve{x}(\breve{z}\breve{z}))(\breve{x}(\breve{z}\breve{z})))))} = \mathfrak{So}_{00}$,
(9) $\mathfrak{So}_{((\breve{x}(\breve{x}(\breve{y}\breve{y})))(\breve{x}(\breve{x}(\breve{y}\breve{y}))))(\breve{y}\breve{y})} = \mathfrak{So}_{00}$,

for all $\breve{x}, \breve{y}, \breve{z} \in \mathfrak{Z}$.

Lemma 1.5. [20] Let $(\mathfrak{Z}, \mathfrak{So}, 0)$ be a ssBCK-algebra. Then a relation $\leq$ defined on $\mathfrak{Z}$ by

$$\breve{x} \leq \breve{y} \text{ if and only if } \mathfrak{So}_{(\breve{x}(\breve{y}\breve{y}))(\breve{x}(\breve{y}\breve{y}))} = 0 \quad (2)$$

is a partial order on $\mathfrak{Z}$. With respect to this order, 0 is the least element of $\mathfrak{Z}$. Also,

$$\breve{y} \leq \mathfrak{So}_{\breve{x}(\breve{y}\breve{y})} \quad (3)$$

and

$$\breve{x} \leq \breve{z} \text{ implies } \mathfrak{So}_{(\breve{x}(\breve{y}\breve{y}))(\breve{x}(\breve{y}\breve{y}))} \leq \mathfrak{So}_{(\breve{z}(\breve{y}\breve{y}))(\breve{z}(\breve{y}\breve{y}))}, \quad (4)$$

for all $\breve{x}, \breve{y}, \breve{z} \in \mathfrak{Z}$.

A ssBCK-algebra is called bounded if it has the greatest element 1.

Lemma 1.6. [20] Let $(\mathfrak{Z}, \mathfrak{So}, 0)$ be a ssBCK-algebra and $\leq$ be an order on $\mathfrak{Z}$ as in Lemma 1.5. Then

(1) $\check{x} \leq \check{z}$ implies $\mathfrak{So}_{(\check{y}(\check{z}\check{z}))(\check{y}(\check{z}\check{z}))} \leq \mathfrak{So}_{(\check{y}(\check{x}\check{x}))(\check{y}(\check{x}\check{x}))}$,
(2) $\mathfrak{So}_{((\check{x}(\check{y}\check{y}))(\check{x}(\check{y}\check{y})))(\check{z}\check{z})} = \mathfrak{So}_{((\check{x}(\check{z}\check{z}))(\check{x}(\check{z}\check{z})))(\check{y}\check{y})}$,
(3) $\mathfrak{So}_{(\check{x}(\check{y}\check{y}))(\check{x}(\check{y}\check{y}))} \leq \check{z} \Leftrightarrow \mathfrak{So}_{(\check{x}(\check{z}\check{z}))(\check{x}(\check{z}\check{z}))} \leq \check{y}$,
(4) $\mathfrak{So}_{(\check{x}(\check{y}\check{y}))(\check{x}(\check{y}\check{y}))} \leq \check{x}$,
(5) $\check{x} \leq \mathfrak{So}_{\check{y}(\check{x}\check{x})}$,
(6) $\check{x} \leq \mathfrak{So}_{(\check{x}(\check{y}\check{y}))(\check{y}\check{y})}$,
(7) $\check{x} \leq \check{y}$ implies $\mathfrak{So}_{\check{z}(\check{x}\check{x})} \leq \mathfrak{So}_{\check{z}(\check{y}\check{y})}$,

for all $\check{x}, \check{y}, \check{z} \in \mathfrak{Z}$.

Unless otherwise specified, $\mathfrak{Z}$ denotes a ssBCK-algebra.

Definition 1.7. [21] Let $\mathfrak{B}$ be a nonempty subset of a ssBCK-algebra $\mathfrak{Z}$. Then a structure $(\mathfrak{B}, \mathfrak{So}, 0)$ is called a subalgebra of $\mathfrak{Z}$ if $\mathfrak{So}_{(\check{x}(\check{y}\check{y}))(\check{x}(\check{y}\check{y}))} \in \mathfrak{B}$, for all $\check{x}, \check{y} \in \mathfrak{B}$.

Definition 1.8. [21] A nonempty subset $\check{I}$ of a ssBCK-algebra $\mathfrak{Z}$ is called an ideal of $\mathfrak{Z}$ if it satisfies the following properties:

(I1) $0 \in \check{I}$,
(I2) $\mathfrak{So}_{(\check{x}(\check{y}\check{y}))(\check{x}(\check{y}\check{y}))} \in \check{I}$ and $\check{y} \in \check{I}$ imply $\check{x} \in \check{I}$, for all $\check{x}, \check{y} \in \mathfrak{Z}$.

Theorem 1.9. [21] Let $\check{I}$ be a subset of a ssBCK-algebra $\mathfrak{Z}$. Then $\check{I}$ is an ideal of $\mathfrak{Z}$ if and only if

(I3) $\check{x}, \check{y} \in \check{I}$ implies $(\hat{x}|\hat{x})|(\hat{y}|\hat{y})\mathfrak{So}_{(\check{x}\check{x})(\check{y}\check{y})} \in \check{I}$,
(I4) $\check{x} \leq \check{y}$ and $\check{y} \in \check{I}$ imply $\check{x} \in \check{I}$,

for all $\check{x}, \check{y} \in \mathfrak{Z}$.

3. Branches

In this section, atoms, branches and chains of ssBCK-algebras are introduced, and relationships between these concepts are investigated. Also, the concepts about branches were defined in the reference [22]. Unless otherwise specified, $\check{x}^{\check{y}} := \mathfrak{So}_{(\check{x}(\check{y}\check{y}))(\check{x}(\check{y}\check{y}))}$, for all $\check{x}$ and $\check{y}$ in these algebraic structures.

Definition 3.1. An element $\check{\varepsilon}$ of a ssBCK-algebra $\mathfrak{Z}$ is called an atom of $\mathfrak{Z}$ if $\check{x} \leq \check{\varepsilon}$ implies $\check{x} = 0$ or $\check{x} = \check{\varepsilon}$. The set of all atoms of $\mathfrak{Z}$ is denoted by $\widetilde{\mathfrak{Z}}$.

Example 3.2. Consider a ssBCK-algebra $\mathfrak{Z}$ where the set $\mathfrak{Z} = \{0,1,2,3,4,5,6,7\}$ with Hasse diagram in Fig. 1 and the Sheffer stroke $\mathfrak{So}$ on $\mathfrak{Z}$ has the Cayley table in Table 1 [20].

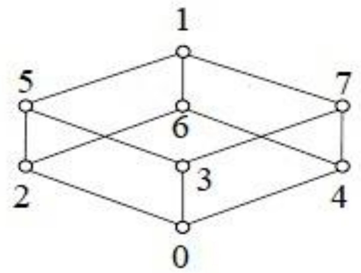

Figure 1. Hasse diagram of $\mathfrak{Z}$.

Table 1. Table of the Sheffer stroke $\mathfrak{S}\mathfrak{o}$ on $\mathfrak{Z}$.

$\mathfrak{S}\mathfrak{o}$	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	0	7	6	5	4	3	2
2	1	7	7	1	1	7	7	1
3	1	6	1	6	1	6	1	6
4	1	5	1	1	5	1	5	5
5	1	4	7	6	1	4	7	6
6	1	3	7	1	5	7	3	5
7	1	2	1	6	5	6	5	2

Then $0, 2, 3$ and 4 are atoms of $\mathfrak{Z}$. Also, $\overset{\frown}{\mathfrak{Z}} = \{0, 2, 3, 4\}$.

Lemma 3.3. A nonzero element $\breve{\varepsilon}$ of a ssBCK-algebra $\mathfrak{Z}$ is an atom of $\mathfrak{Z}$ if and only if a subset $\{0, \breve{\varepsilon}\}$ of $\mathfrak{Z}$ is an ideal of $\mathfrak{Z}$.

Proof. Let $\breve{\varepsilon}$ be a nonzero atom of a ssBCK-algebra $\mathfrak{Z}$. It is obvious that $0 \in \{0, \breve{\varepsilon}\}$. Suppose that $\breve{y} \in \{0, \breve{\varepsilon}\}$ and $\breve{x}^{\breve{y}} \in \{0, \breve{\varepsilon}\}$. If $\breve{y} = 0$, then $\breve{x} = \breve{x}^0 = \breve{x}^{\breve{y}} \in \{0, \breve{\varepsilon}\}$ from Lemma 1.4 (6). Assume that $\breve{x}^{\breve{y}} = 0$. Then $\breve{x} \leq \breve{y}$ from Lemma 1.5, and so, $\breve{x} \leq 0$ or $\breve{x} \leq \breve{\varepsilon}$. Since $\breve{\varepsilon}$ is an atom of $\mathfrak{Z}$, it follows that $\breve{x} = 0$ or $\breve{x} = \breve{\varepsilon}$, and so, $\breve{x} \in \{0, \breve{\varepsilon}\}$. Thus, $\{0, \breve{\varepsilon}\}$ is an ideal of $\mathfrak{Z}$.

Conversely, let $\{0, \breve{\varepsilon}\}$ be an ideal of $\mathfrak{Z}$, and $\breve{x} \leq \breve{\varepsilon}$, for any $\breve{x} \in \mathfrak{Z}$. Since $\breve{x}^{\breve{\varepsilon}} = 0 \in \{0, \breve{\varepsilon}\}$ from Lemma 1.5 and $\breve{\varepsilon} \in \{0, \breve{\varepsilon}\}$, it is obtained from (I2) that $\breve{x} \in \{0, \breve{\varepsilon}\}$. Hence, $\breve{x} = 0$ or $\breve{x} = \breve{\varepsilon}$. Therefore, $\breve{\varepsilon}$ is an atom of $\mathfrak{Z}$.

Lemma 3.4. Every ideal of a ssBCK-algebra $\mathfrak{Z}$ is a subalgebra of $\mathfrak{Z}$.

Proof. Let $\check{I}$ be an ideal of a ssBCK-algebra $\mathfrak{Z}$, and $\breve{x}, \breve{y} \in \check{I}$. Since $\breve{x}^{\breve{y}} \leq \breve{x}$ from Lemma 1.6 (4), it follows from (I4) that $\breve{x}^{\breve{y}} \in \check{I}$. Then $\check{I}$ is a subalgebra of $\mathfrak{Z}$.

The following example illustrates that a subalgebra of a ssBCK-algebra is mostly not an ideal.

Example 3.5. Consider the ssBCK-algebra $\mathfrak{Z}$ in Example 3.2. Then $\{0,1,3,6\}$ is a subalgebra of $\mathfrak{Z}$ but it is not an ideal of $\mathfrak{Z}$ since $4 \notin \{0,1,3,6\}$ when $4^{6} = 0 \in \{0,1,3,6\}$ and $6 \in \{0,1,3,6\}$.

By the following lemma, we shows cases in which a subalgebra of a ssBCK-algebra is an ideal.

Lemma 3.6. Every element of a ssBCK-algebra $\mathfrak{Z}$ is an atom of $\mathfrak{Z}$ if and only if every subalgebra of $\mathfrak{Z}$ is an ideal of $\mathfrak{Z}$.

Proof. Let every element of a ssBCK-algebra $\mathfrak{Z}$ be an atom of $\mathfrak{Z}$, and $\mathfrak{B}$ be a subalgebra of $\mathfrak{Z}$. Then it follows from Lemma 1.4 (2) that $0 = \check{x}^{\check{x}} \in \mathfrak{B}$, for all $\check{x} \in \mathfrak{B}$. Assume that $\check{y}, \check{x}^{\check{y}} \in \mathfrak{B}$, for any $\check{x}, \check{y} \in \mathfrak{Z}$. Since $\check{x}^{\check{y}} \leq \check{x}$ from Lemma 1.6 (4), it is obtained that $\check{x}^{\check{y}} = 0$ or $\check{x}^{\check{y}} = \check{x}$. If $\check{x}^{\check{y}} = 0$, then $\check{x} \leq \check{y}$ from Lemma 1.5. Thus, $\check{x} = 0$ or $\check{x} = \check{y}$, and so, $\check{x} \in \mathfrak{B}$. If $\check{x}^{\check{y}} = \check{x}$, then $\check{x} \in \mathfrak{B}$. Hence, $\mathfrak{B}$ is an ideal of $\mathfrak{Z}$.

Conversely, let every subalgebra $\mathfrak{B}$ of $\mathfrak{Z}$ be an ideal of $\mathfrak{Z}$ and $\check{\varepsilon}$ be an element of $\mathfrak{Z}$ such that it is not an atom of $\mathfrak{Z}$. Then a subset $\{0, \check{\varepsilon}\}$ of $\mathfrak{Z}$ is not an ideal of $\mathfrak{Z}$ from Lemma 3.3, and so, it is not a subalgebra of $\mathfrak{Z}$. Since $\check{\varepsilon}^{0} = \check{\varepsilon} \in \{0, \check{\varepsilon}\}$ and $0^{\check{\varepsilon}} = 0 \in \{0, \check{\varepsilon}\}$ from Lemma 1.4 (6) and (7), we have that $\{0, \check{\varepsilon}\}$ is a subalgebra of $\mathfrak{Z}$, which is a contradiction. Thus, every element of $\mathfrak{Z}$ is an atom of $\mathfrak{Z}$.

Lemma 3.7. Let $\check{\varepsilon}_1$ and $\check{\varepsilon}_2$ be non-zero elements of a ssBCK-algebra $\mathfrak{Z}$ such that $\check{\varepsilon}_1 \neq \check{\varepsilon}_2$. If $\check{\varepsilon}_1$ and $\check{\varepsilon}_2$ are atoms of $\mathfrak{Z}$, then $\check{\varepsilon}_1^{\check{\varepsilon}_2} = \check{\varepsilon}_1$ and $\check{\varepsilon}_2^{\check{\varepsilon}_1} = \check{\varepsilon}_2$.

Proof. Let $\check{\varepsilon}_1$ and $\check{\varepsilon}_2$ be atoms of $\mathfrak{Z}$ such that $\check{\varepsilon}_1 \neq 0 \neq \check{\varepsilon}_2$. Then $\{0, \check{\varepsilon}_1\}$ and $\{0, \check{\varepsilon}_2\}$ are ideals of $\mathfrak{Z}$ from Lemma 3.3. Since $\check{\varepsilon}_1^{\check{\varepsilon}_2} \leq \check{\varepsilon}_1$ and $\check{\varepsilon}_2^{\check{\varepsilon}_1} \leq \check{\varepsilon}_2$ from Lemma 1.6 (4), it follows from (I4) that $\check{\varepsilon}_1^{\check{\varepsilon}_2} \in \{0, \check{\varepsilon}_1\}$ and $\check{\varepsilon}_2^{\check{\varepsilon}_1} \in \{0, \check{\varepsilon}_2\}$. If $\check{\varepsilon}_1^{\check{\varepsilon}_2} = 0$ or $\check{\varepsilon}_2^{\check{\varepsilon}_1} = 0$, i. e., $\check{\varepsilon}_1 \leq \check{\varepsilon}_2$ or $\check{\varepsilon}_2 \leq \check{\varepsilon}_1$, then it is obtained that $\check{\varepsilon}_1 = \check{\varepsilon}_2$, which is a contradiction. Thus, $\check{\varepsilon}_1^{\check{\varepsilon}_2} = \check{\varepsilon}_1$ and $\check{\varepsilon}_2^{\check{\varepsilon}_1} = \check{\varepsilon}_2$.

The inverse of Lemma 3.7 does not generally hold.

Example 3.8. In Example 3.2, $5^{4} = 5$ and $4^{5} = 4$ but $5 \notin \overset{\frown}{\mathfrak{Z}}$ when $4 \in \overset{\frown}{\mathfrak{Z}}$.

Lemma 3.9. The set $\overset{\frown}{\mathfrak{Z}}$ of all atoms of a ssBCK-algebra $\mathfrak{Z}$ is a subalgebra of $\mathfrak{Z}$.

Proof. Let $\overset{\frown}{\mathfrak{Z}}$ be the set of all atoms of a ssBCK-algebra $\mathfrak{Z}$. Since 0 is the least element of $\mathfrak{Z}$, we have that $0 \in \overset{\frown}{\mathfrak{Z}}$. Assume that $\check{\varepsilon}_1, \check{\varepsilon}_2 \in \overset{\frown}{\mathfrak{Z}}$. If $\check{\varepsilon}_1 = \check{\varepsilon}_2$, then $\check{\varepsilon}_1^{\check{\varepsilon}_2} = 0 \in \overset{\frown}{\mathfrak{Z}}$ or $\check{\varepsilon}_2^{\check{\varepsilon}_1} \in \overset{\frown}{\mathfrak{Z}}$ from Lemma 1.4 (2). If $\check{\varepsilon}_1 \neq \check{\varepsilon}_2$, then $\check{\varepsilon}_1^{\check{\varepsilon}_2} = \check{\varepsilon}_1 \in \overset{\frown}{\mathfrak{Z}}$ and $\check{\varepsilon}_2^{\check{\varepsilon}_1} = \check{\varepsilon}_2 \in \overset{\frown}{\mathfrak{Z}}$ from Lemma 3.7.

However, the set $\overset{\frown}{\mathfrak{Z}}$ of all atoms of a ssBCK-algebra $\mathfrak{Z}$ is generally not an ideal of $\mathfrak{Z}$.

Example 3.10. In Example 3.2, $\overset{\frown}{\mathfrak{Z}} = \{0,2,3,4\}$ is a subalgebra of $\mathfrak{Z}$ but it is not ideal of $\mathfrak{Z}$ since $7 \notin \overset{\frown}{\mathfrak{Z}}$ when $7^{3} = 4 \in \overset{\frown}{\mathfrak{Z}}$ and $3 \in \overset{\frown}{\mathfrak{Z}}$.

Lemma 3.11. For each atom $\breve{\alpha}$ of a ssBCK-algebra $\mathfrak{Z}$, a subset $\underline{\breve{\alpha}} = \{\check{x} \in A: \check{x} \leq \breve{\alpha}\}$ of $\mathfrak{Z}$ is an ideal of $\mathfrak{Z}$.

Proof. Let $\breve{\alpha}$ be an atom of a ssBCK-algebra $\mathfrak{Z}$ and $\underline{\breve{\alpha}} = \{\check{x} \in A: \check{x} \leq \breve{\alpha}\}$ be a subset of $\mathfrak{Z}$. Then $\underline{\breve{\alpha}} = \{0, \breve{\alpha}\}$. Thus, $\underline{\breve{\alpha}}$ is an ideal of $\mathfrak{Z}$ by Lemma 3.3.

However, it is not necessary that $\breve{\alpha}$ is not an atom of a ssBCK-algebra when $\underline{\breve{\alpha}}$ is its ideal.

Example 3.12. Consider the ssBCK-algebra $\mathfrak{Z}$ in Example 3.2. Then 5 is not an atom of $\mathfrak{Z}$ when $\underline{5} = \{0, 2, 3, 5\}$ is an ideal of $\mathfrak{Z}$.

Lemma 3.13. Let $\mathfrak{Z}$ be a ssBCK-algebra and $\leq$ be a partial order on $\mathfrak{Z}$ as in Lemma 1.5. Then $\check{x} \leq \check{y}$ implies $\mathfrak{So}_{\check{y}\check{z}} \leq \mathfrak{So}_{\check{x}\check{z}}$, for all $\check{x}, \check{y}, \check{z} \in \mathfrak{Z}$.

Proof. Let $\check{x} \leq \check{y}$. Then $\check{x}^{\check{y}} = 0$ from Lemma 1.5. Since

$$[(\mathfrak{So}_{\check{y}\check{z}})^{\mathfrak{So}_{\check{x}\check{z}}}]^{\check{x}^{\check{y}}} = \left([\check{z}^{\mathfrak{So}_{\check{x}\check{x}}}]^{\check{z}^{\mathfrak{So}_{\check{y}\check{y}}}}\right)^{(\mathfrak{So}_{\check{y}\check{y}})^{(\mathfrak{So}_{\check{x}\check{x}})}} = 0 \tag{5}$$

from (S1), (S2) and (sBCK-1), it follows Lemma 1.5 that $(\mathfrak{So}_{\check{y}\check{z}})^{\mathfrak{So}_{\check{x}\check{z}}} \leq \check{x}^{\check{y}} = 0$. Since 0 is the least element of $\mathfrak{Z}$, $(\mathfrak{So}_{\check{y}\check{z}})^{\mathfrak{So}_{\check{x}\check{z}}} = 0$. Thus, $\mathfrak{So}_{\check{y}\check{z}} \leq \mathfrak{So}_{\check{x}\check{z}}$, for all $\check{x}, \check{y}, \check{z} \in \mathfrak{Z}$.

Lemma 3.14. Let $\mathfrak{Z}$ be a ssBCK-algebra and $\leq$ be a partial order on $\mathfrak{Z}$ as in Lemma 1.5. Then

(1) $\check{x}^{\check{y}} = 0$ if and only if $\mathfrak{So}_{\check{x}\check{y}} = \mathfrak{So}_{\check{x}\check{x}}$,
(2) $\check{x} \leq \check{y}$ if and only if $\mathfrak{So}_{\check{y}\check{y}} \leq \mathfrak{So}_{\check{x}\check{x}}$,

for all $\check{x}, \check{y} \in \mathfrak{Z}$.

Proof.

(1) Let $\check{x}^{\check{y}} = 0$. Then $\check{x} \leq \check{y}$ from Lemma 1.5. Since $\mathfrak{So}_{\check{x}\check{y}} = \mathfrak{So}_{\check{y}\check{x}} \leq \mathfrak{So}_{\check{x}\check{x}}$ and $\mathfrak{So}_{\check{x}\check{x}} \leq \mathfrak{So}_{\check{y}^{(\mathfrak{So}_{\check{x}\check{x}})}\check{y}^{(\mathfrak{So}_{\check{x}\check{x}})}} = \mathfrak{So}_{\check{x}\check{y}}$ from (S1), (S2), Lemma 3.13 and Lemma 1.6 (5), it follows that $\mathfrak{So}_{\check{x}\check{y}} = \mathfrak{So}_{\check{x}\check{x}}$, for all $\check{x}, \check{y} \in \mathfrak{Z}$.

Conversely, let $\mathfrak{So}_{\check{x}\check{y}} = \mathfrak{So}_{\check{x}\check{x}}$, for any $\check{x}, \check{y} \in \mathfrak{Z}$. Thus, $\check{x}^{\check{y}} = (\mathfrak{So}_{\check{y}\check{y}})^{(\mathfrak{So}_{\check{x}\check{x}})} = (\mathfrak{So}_{\check{y}\check{y}})^{(\mathfrak{So}_{\check{x}y})} = \check{x}^{(\mathfrak{So}_{\check{y}^{\check{y}}\check{y}^{\check{y}}})} = \check{x}^{(\mathfrak{So}_{00})} = 0$ from (S1)-(S3), Lemma 1.4 (2) and (5).

(2) $\check{x} \leq \check{y} \Leftrightarrow 0 = \check{x}^{\check{y}} = (\mathfrak{So}_{\check{y}\check{y}})^{(\mathfrak{So}_{\check{x}\check{x}})} \Leftrightarrow \mathfrak{So}_{\check{y}\check{y}} \leq \mathfrak{So}_{\check{x}\check{x}}$ from Lemma 1.5, (S1) and (S2).

Theorem 3.15. For each atom $\breve{\alpha}$ of a ssBCK-algebra $\mathfrak{Z}$, a subset $\mathfrak{Z}_{\breve{\alpha}} = \{\check{x} \in \mathfrak{Z}: \check{x}^{\breve{\alpha}} = \check{x}\}$ of $\mathfrak{Z}$ is an ideal of $\mathfrak{Z}$.

Proof. Let $\breve{\alpha}$ be an atom of a ssBCK-algebra $\mathfrak{Z}$, and $\mathfrak{Z}_{\breve{\alpha}} = \{\check{x} \in \mathfrak{Z}: \check{x}^{\breve{\alpha}} = \check{x}\}$ be a subset of $\mathfrak{Z}$. It is obvious from Lemma 1.4 (7) that $0 \in \mathfrak{Z}_{\breve{\alpha}}$. Assume that $\check{y} \in \mathfrak{Z}_{\breve{\alpha}}$ and $\check{x}^{\check{y}} \in \mathfrak{Z}_{\breve{\alpha}}$. Then $\mathfrak{So}_{\check{x}\check{x}} \leq \mathfrak{So}_{(\mathfrak{So}_{\breve{\alpha}\breve{\alpha}})^{(\mathfrak{So}_{\check{x}\check{x}})}(\mathfrak{So}_{\breve{\alpha}\breve{\alpha}})^{(\mathfrak{So}_{\check{x}\check{x}})}} = \mathfrak{So}_{\check{x}^{\breve{\alpha}}\check{x}^{\breve{\alpha}}}$ from Lemma 1.6 (5), (S1) and (S2). Since

$$(\breve{x}^{\breve{x}^{\breve{\alpha}}})^{\breve{\alpha}} = (\breve{x}^{\breve{\alpha}})^{\breve{x}^{\breve{\alpha}}} = 0 \quad (6)$$

from Lemma 1.6 (2), (S1) and Lemma 1.4 (2), it is obtained from Lemma 1.5 that $\breve{x}^{\breve{x}^{\breve{\alpha}}} \leq \breve{\alpha}$. Since $\breve{\alpha}$ is an atom of $\mathfrak{Z}$, it follows that $\breve{x}^{\breve{x}^{\breve{\alpha}}} = 0$ or $\breve{x}^{\breve{x}^{\breve{\alpha}}} = \breve{\alpha}$. Hence,

$$\mathfrak{So}_{\breve{x}^{\breve{\alpha}}\breve{x}^{\breve{\alpha}}} \leq \mathfrak{So}_{\breve{x}\breve{x}} \quad (7)$$

or

$$\breve{\alpha} = \breve{x}^{\breve{x}^{\breve{\alpha}}} = (\mathfrak{So}_{\breve{x}^{\breve{\alpha}}\breve{x}^{\breve{\alpha}}})^{(\mathfrak{So}_{\breve{x}\breve{x}})} \leq \mathfrak{So}_{\breve{x}\breve{x}} \quad (8)$$

from (S1), (S2), Lemma 1.4 (2), (7), Lemma 1.5 and Lemma 1.6 (2). So, $\breve{x}^{\breve{\alpha}} = \breve{x}$ from (S2) and Lemma 3.14. Therefore, $\breve{x} \in \mathfrak{Z}_{\breve{\alpha}}$, and so, $\mathfrak{Z}_{\breve{\alpha}}$ is an ideal of $\mathfrak{Z}$.

But, the inverse of Theorem 3.15 is usually not true.

Example 3. 16. Consider the ssBCK-algebra $\mathfrak{Z}$ in Example 3.2. Then $\mathfrak{Z}_6 = \{0, 3\}$ is an ideal of $\mathfrak{Z}$ but 6 is not an atom of $\mathfrak{Z}$.

Definition 3.17. Let $\mathfrak{Z}$ be a bounded ssBCK-algebra. Then a subset $B_{\breve{\varepsilon}} = \{\breve{x} \in \mathfrak{Z} : \breve{\varepsilon} \leq \breve{x}\}$ of $\mathfrak{Z}$ is called a branch of $\mathfrak{Z}$, where $\breve{\varepsilon}$ is an atom of $\mathfrak{Z}$. The element $\breve{\varepsilon}$ is called initial for $B_{\breve{\varepsilon}}$. If there exists $\breve{\varepsilon} \neq \breve{\alpha}$ such that $B_{\breve{\varepsilon}} \subset B_{\breve{\alpha}}$, then the branch $B_{\breve{\varepsilon}}$ is called improper. Also, the branch $B_{\breve{\varepsilon}}$ is called proper if there is not $\breve{\alpha} \in \mathfrak{Z}$ such that $\breve{\alpha} < \breve{\varepsilon}$. The set of all initial elements of proper branches of $\mathfrak{Z}$ is denoted by $\overset{\frown}{I}_{\mathfrak{Z}}$. Obviously, $\overset{\frown}{I}_{\mathfrak{Z}} \subseteq \overset{\frown}{\mathfrak{Z}}$.

Example 3.18. Consider the ssBCK-algebra $\mathfrak{Z}$ in Example 3.2. Then $\overset{\frown}{\mathfrak{Z}} = \{0, 2, 3, 4\}$, $B_0 = \mathfrak{Z}$, $B_2 = \{1, 2, 5, 6\}$, $B_3 = \{1, 3, 5, 7\}$ and $B_4 = \{1, 4, 6, 7\}$ shown in Fig. 2. Also, the branches B_2, B_3 and B_4 are improper but B_0 is proper. Clearly, $\overset{\frown}{I}_{\mathfrak{Z}} = \{0\}$ is an ideal of $\mathfrak{Z}$.

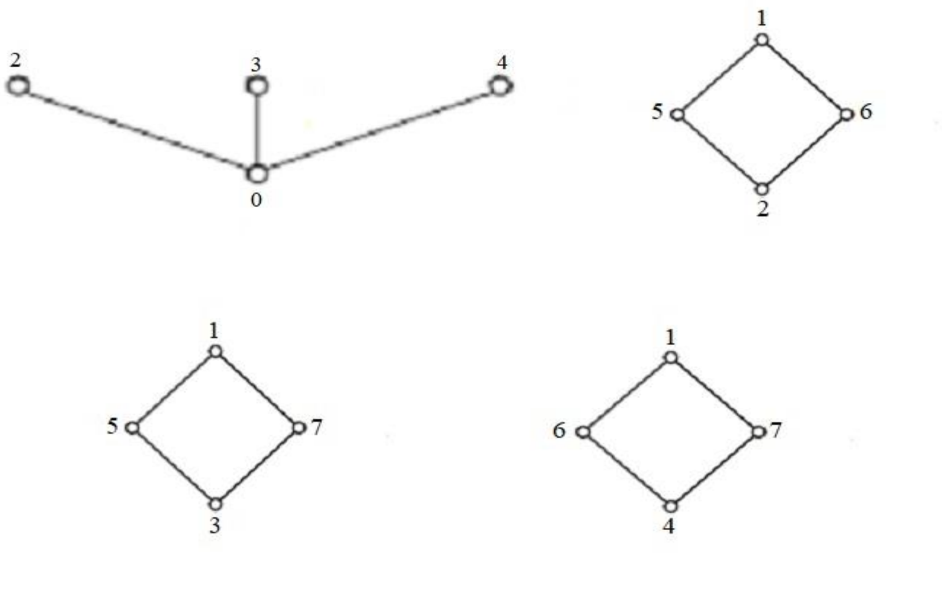

Figure 2. Hasse diagrams of $\overset{\frown}{\mathfrak{Z}}$, B_2, B_3 and B_4, respectively.

Lemma 3.19. Let $\mathfrak{Z}$ be a bounded ssBCK-algebra. Then $B_0 = \mathfrak{Z}$ and is a proper branch of $\mathfrak{Z}$.

Proof. Let $\mathfrak{Z}$ be a bounded ssBCK-algebra. Since 0 is the least element of $\mathfrak{Z}$ and $0 \in \widetilde{\mathfrak{Z}}$, it follows that $B_0 = \mathfrak{Z}$, and so, it is a proper branch of $\mathfrak{Z}$.

Theorem 3.20. Let $\mathfrak{Z}$ be a bounded ssBCK-algebra. Then $1 \in B_{\check{\varepsilon}}$, for all initials $\check{\varepsilon}$ of $\mathfrak{Z}$.

Proof. Let $\mathfrak{Z}$ be a bounded ssBCK-algebra. Since 1 is the greatest element of $\mathfrak{Z}$, it is obtained that $\check{\varepsilon} \leq 1$, for all initials $\check{\varepsilon}$ of $\mathfrak{Z}$. Then $1 \in B_{\check{\varepsilon}}$, for all initials $\check{\varepsilon}$ of $\mathfrak{Z}$.

Corollary 3.21. Let $\mathfrak{Z}$ be a bounded ssBCK-algebra. Then $\mathfrak{Z} = \bigcup B_{\check{\varepsilon}}$, for all initials $\check{\varepsilon}$ of $\mathfrak{Z}$.

Theorem 3.22. Let $\mathfrak{Z}$ be a bounded ssBCK-algebra. Then $\mathfrak{Z} - B_{\check{\varepsilon}}$ is an ideal of $\mathfrak{Z}$, for all nonzero initials $\check{\varepsilon}$ of $\mathfrak{Z}$.

Proof. Let $\mathfrak{Z}$ be a bounded ssBCK-algebra. Since 0 is the least element of $\mathfrak{Z}$, it follows that $0 \notin B_{\check{\varepsilon}}$, for all nonzero initials $\check{\varepsilon}$ of $\mathfrak{Z}$. Then $0 \in \mathfrak{Z} - B_{\check{\varepsilon}}$, for all nonzero initials $\check{\varepsilon}$ of $\mathfrak{Z}$. Assume that $\check{y}, \check{x}^{\check{y}} \in \mathfrak{Z} - B_{\check{\varepsilon}}$. Since $\check{y}, \check{x}^{\check{y}} \notin B_{\check{\varepsilon}}$, it is obtained that $\check{y} < \check{\varepsilon}$ and $\check{x}^{\check{y}} < \check{\varepsilon}$. Since $\check{\varepsilon}$ is an atom of $\mathfrak{Z}$, we have that $\check{y} = 0$ and $\check{x}^{\check{y}} = 0$. Thus, $\check{x} = \check{x}^0 = \check{x}^{\check{y}} = 0 \in \mathfrak{Z} - B_{\check{\varepsilon}}$ from Lemma 1.4 (6). Hence, $\mathfrak{Z} - B_{\check{\varepsilon}}$ is an ideal of $\mathfrak{Z}$.

Theorem 3.23. Let $\mathfrak{Z}$ be a bounded ssBCK-algebra. Then $\widetilde{I_{\mathfrak{Z}}}$ is a subalgebra of $\mathfrak{Z}$.

Proof. Let $\mathfrak{Z}$ be a bounded ssBCK-algebra and $\widetilde{I_{\mathfrak{Z}}}$ be defined as in Definition 3.17. Since $B_0 = \mathfrak{Z}$ is a proper branch of $\mathfrak{Z}$ from Lemma 3.22, we have $0 \in \widetilde{I_{\mathfrak{Z}}}$. Assume that $\check{x}, \check{y} \in \widetilde{I_{\mathfrak{Z}}}$. Since $\check{x}$ and $\check{y}$ are initials for proper branches $B_{\check{x}}$ and $B_{\check{y}}$, respectively, it follows that $\check{x}$ and $\check{y}$ are atoms of $\mathfrak{Z}$. Since $\check{x}^{\check{y}} = \check{x} \in \widetilde{I_{\mathfrak{Z}}}$ and $\check{y}^{\check{x}} = \check{y} \in \widetilde{I_{\mathfrak{Z}}}$ from Lemma 3.7, it is obtained that $\widetilde{I_{\mathfrak{Z}}}$ is a subalgebra of $\mathfrak{Z}$.

Definition 3.24. A nonempty subset $\mathfrak{C}$ of a bounded ssBCK-algebra $\mathfrak{Z}$ is called a chain if any two elements of $\mathfrak{C}$ are comparable. A chain initiated by $\check{\varepsilon}$ is denoted by $\mathfrak{C}_{\check{\varepsilon}}$. If $B_{\check{\varepsilon}} = \mathfrak{C}_{\check{\varepsilon}}$, then the branch $B_{\check{\varepsilon}}$ is called linear. If $B_{\check{\varepsilon}}$ has at least two incomparable elements of $\mathfrak{Z}$, then it is called expanded.

Example 3.25. Consider the ssBCK-algebra $\mathfrak{Z}$ in Example 3.2. Then $\mathfrak{C}_0^1 = \{0, 1, 2, 5\}, \mathfrak{C}_0^2 = \{0, 1, 3, 5\}, \mathfrak{C}_0^3 = \{0, 1, 3, 6\}, \mathfrak{C}_0^4 = \{0, 1, 2, 6\}, \mathfrak{C}_0^5 = \{0, 1, 4, 6\}$ and $\mathfrak{C}_0^6 = \{0, 1, 4, 7\}$ are chains of $\mathfrak{Z}$, and $\mathfrak{Z} = \mathfrak{C}_0^1 \cup \mathfrak{C}_0^2 \cup \mathfrak{C}_0^3 \cup \mathfrak{C}_0^4 \cup \mathfrak{C}_0^5 \cup \mathfrak{C}_0^6$. Also, A does not contain a linear branch. In particular, the branches B_2, B_3 and B_4 are expanded branches of $\mathfrak{Z}$.

Example 3.26. Consider a ssBCK-algebra $\mathfrak{T}$ where $\mathfrak{T} = \{0, \hat{a}, \hat{b}, 1\}$ with Hasse diagram in Fig. 3 and Sheffer stroke $\mathfrak{So}$ on $\mathfrak{T}$ with the Cayley table in Table 2 [20]:

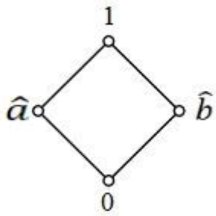

Figure 3. Hasse diagram of $\mathfrak{T}$.

Table 2. Table of the Sheffer stroke $\mathfrak{So}$ on $\mathfrak{T}$.

$\mathfrak{So}$	0	$\hat{a}$	$\hat{b}$	1
0	1	1	1	1
$\hat{a}$	1	$\hat{b}$	1	$\hat{b}$
$\hat{b}$	1	1	$\hat{a}$	$\hat{a}$
1	1	$\hat{b}$	$\hat{a}$	0

Then $B_{\hat{a}} = \mathfrak{C}_{\hat{a}}$ and $B_{\hat{b}} = \mathfrak{C}_{\hat{b}}$ are linear branches of $\mathfrak{T}$. Also, $\mathfrak{T} = B_0 = \mathfrak{C}_0^1 \cup \mathfrak{C}_0^2$ is an expanded branch.

Remark 3.27. Every bounded ssBCK-algebra is an expanded branch by Lemma 3.19.

Theorem 3.28. Let $\mathfrak{Z}$ be a bounded ssBCK-algebra. Then $B_{\breve{\varepsilon}} = \bigcup_{i=1}^{n} \mathfrak{C}_{\breve{\varepsilon}}^{i}$, for initials $\breve{\varepsilon}$ of $\mathfrak{Z}$.

Proof. Straightforward.

4. Conclusion

In this study, we introduce an atom, a branch and a chain of ssBCK-algebras and investigate several properties. Then it is shown that a nonzero element $\breve{\varepsilon}$ of a ssBCK-algebra is an atom if and only if a subset $\{0, \breve{\varepsilon}\}$ is an ideal, and that each element in a ssBCK-algebra is an atom if and only if each subalgebra of the structure is its ideal. We illustrate that ${\breve{\varepsilon}_1}^{\breve{\varepsilon}_2} = \breve{\varepsilon}_1$ and ${\breve{\varepsilon}_2}^{\breve{\varepsilon}_1} = \breve{\varepsilon}_2$ if $\breve{\varepsilon}_1$ and $\breve{\varepsilon}_2$ are nonzero distinct atoms of a ssBCK-algebra; however, the inverse does not usually hold, and that the set of all atoms of a ssBCK-algebra is its subalgebra. Also, we define specified sets $\underline{\breve{\alpha}}$ and $\mathfrak{Z}_{\breve{\alpha}}$, for an element $\breve{\alpha}$ in a ssBCK-algebra, and prove that these subsets are ideals if the element $\breve{\alpha}$ is an atom of this algebraic structure but the inverse is usually not true. After that improper and proper branches of a bounded ssBCK-algebra are built by atoms, we state that every branch contains the greatest element 1 of a bounded ssBCK-algebra. It is demonstrated that the branch originated by the atom 0 of a bounded ssBCK-algebra equals to the algebra itself, and that this algebra is a union of its branches. Moreover, we indicate that the complements of branches of a bounded ssBCK-algebra are ideals, and that the set of all initial elements of proper branches of a bounded ssBCK-algebra is the subalgebra but it is not an ideal. At the end of the study, a chain, linear and expanded branches of a bounded ssBCK-algebra are introduced and it is illustrated that every bounded ssBCK-algebra is an expanded branch, and any branch is a union of chains initiated by an atom. Therefore, we have achieved our aim to propound similarities and differences among other algebraic structures by studying on these specific subsets of ssBCK-algebras.

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