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THE MULTI-WAVE METHOD FOR EXACT SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

Yusuf Pandir^{a*}, Halime Ulusoy^b

^{a*}Department of Mathematics, Faculty of Science and Arts, Bozok University, Yozgat Turkey
yusuf.pandir@bozok.edu.tr (corresponding author)

^bGraduate School of Natural and Applied Sciences, Bozok University, Yozgat, Turkey
ulusoyhalime@gmail.com

Abstract

In this research, we use the multi-wave method to obtain new exact solutions for generalized forms of the 5th order KdV equation and the fifth order KdV (fKdV) equation with power law nonlinearity. Computations are performed with the help of the mathematics software Mathematica. Then periodic wave solutions, bright soliton solutions and rational function solutions with free parameters are obtained by this approach. It is shown that this method is very useful and effective.

Keywords: Multi-wave method, KdV equation, Soliton solutions.

1. Introduction

The research of exact solutions to nonlinear evolution equations is very important, because these problems appear in fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, and so on. For this reason, a lot of mathematical methods have been developed in this area. In order to find the wave solutions, many methods were attempted in literature such as solitary wave ansatz method [1], Hirota's bilinear transformation method [2,3], exp-function method [4,5], (G'/G) -expansion method [6], trial equation method [7], and so on. Also, Dai et al. [8] proposed extended three-soliton method. Then, Shi et al. [9-11] constructed a novel approach, namely, multi-wave method to find new exact solutions for nonlinear problems and extend Dai et al.'s work [8].

In this work, we apply the multi-wave method in order to find new results to nonlinear problems. We demonstrate an application of multi-wave method for finding exact solutions of two generalized nonlinear evolution equations. One of these equations is 5th order KdV equation [12]

$$q_t + aq^m q_x + bq^{2m} q_x + cq_{xxx} + kq_{xxxxx} = 0, \quad (1)$$

where a, b, c and k are all constants. The first term is the evolution term, while the second and third terms together form the nonlinear terms. Finally, the coefficients of c and k are the third and fifth order dispersion terms, respectively. The other one is the dimensionless form of the fifth order KdV (fKdV) equation with power law nonlinearity [13]

$$q_t + aq_{xxx} + bq_{xxxxx} - c(q^{n+1})_x = 0, \quad (2)$$

where a, b and c are constants parameters and $n > 0$.

2. The bilinear form of 5th order KdV equation

Let's suppose $m = 1$ in Eq. (1) and

$$w = (\ln f)_x, \quad q = w_x = (\ln f)_{xx} = 0. \quad (3)$$

Substituting $q = w_x$ into Eq. (1), we can get

$$w_{xt} + aw_x w_{xx} + bw_x^2 w_{xx} + cw_{xxx} + kw_{xxxxx} = 0. \quad (4)$$

Integrating Eq. (4) with respect to x and equating the integral constant to zero, we have

$$w_t + \frac{a}{2}w_x^2 + bw_x^3 + cw_{xxx} + kw_{xxxxx} = 0. \quad (5)$$

Substituting Eq. (3) into Eq. (5), we can reduce Eq. (1) into the bilinear forms as follows

$$\begin{aligned} \left(D_t D_x + cD_x^4 - \frac{b}{360}D_x^6\right)(f \cdot f) &= 0, \\ \left(\frac{a}{2}D_x^2 - 3cD_x^2 + \frac{b}{24}D_x^4\right)(f \cdot f) &= 0, \end{aligned} \quad (6)$$

where $f = f(x, t)$ is an unknown real function, the bilinear operator $D_x^m D_t^n$ is defined as

$$D_x^m D_t^n ab = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(x, t)b(x', t')|_{x=x', t=t'} = 0. \quad (7)$$

3. An application of the multi-wave method for 5th order KdV equation

We first consider the test function of extended four-soliton method as follows:

$$f(x, t) = a_1 \cos(\xi_1) + a_2 \sin(\xi_2) + a_3 \cosh(\xi_3) + \exp(-\xi_4) + a_4 \exp(\xi_4), \quad (8)$$

where $\xi_i = p_i x + r_i t, i = 1, 2, 3, 4$. Substituting (8) into (3), we can write the form of general solution of Eq. (1)

$$q = \frac{-a_1 p_1^2 \cos(p_1 x + r_1 t) - a_2 p_2^2 \sin(p_2 x + r_2 t) + a_3 p_3^2 \cosh(p_3 x + r_3 t) + p_4^2 \exp(-p_4 x - r_4 t) + a_4 p_4^2 \exp(p_4 x + r_4 t)}{a_1 \cos(p_1 x + r_1 t) + a_2 \sin(p_2 x + r_2 t) + a_3 \cosh(p_3 x + r_3 t) + \exp(-p_4 x - r_4 t) + a_4 \exp(p_4 x + r_4 t)} - \left(\frac{-a_1 p_1 \sin(p_1 x + r_1 t) + a_2 p_2 \cos(p_2 x + r_2 t) + a_3 p_3 \sinh(p_3 x + r_3 t) - p_4 \exp(-p_4 x - r_4 t) + a_4 p_4 \exp(p_4 x + r_4 t)}{a_1 \cos(p_1 x + r_1 t) + a_2 \sin(p_2 x + r_2 t) + a_3 \cosh(p_3 x + r_3 t) + \exp(-p_4 x - r_4 t) + a_4 \exp(p_4 x + r_4 t)}\right)^2. \quad (9)$$

Using Eqs. (9) and (6), we obtain a set of algebraic equations for $a_1, a_2, a_3, a_4, p_1, p_2, p_3, p_4, r_1, r_2, r_3, r_4$. Solving these systems with the aid of Mathematica, we obtain the following solutions, respectively:

Case 1.

$$a_4 = p_1 = p_2 = p_3 = r_1 = r_2 = r_3 = 0, p_4 = 2\sqrt{\frac{18c-3a}{b}}, r_4 = -4\sqrt{3}(a+24c)\left(\frac{6c-a}{5b}\right)^{\frac{3}{2}}, \quad (10)$$

where a_1, a_3 are free parameters. Substituting (10) into Eq. (9) yields the following wave solution of Eq. (1)

$$q(x, t) = \frac{(72c-12a)(a_1+a_3)\exp(-\xi_4)}{b(a_1+a_3+\exp(\xi_4))}, \quad (11)$$

$$\text{where } \xi_4 = 2\sqrt{\frac{18c-3a}{b}}x - 4\sqrt{3}(a+24c)\left(\frac{6c-a}{5b}\right)^{\frac{3}{2}}t.$$

Case 2.

$$p_1 = p_2 = \sqrt{\frac{3a-18c}{b}}, p_3 = p_4 = \sqrt{\frac{18c-3a}{b}}, r_1 = r_2 = \sqrt{3}(a+54c)\left(\frac{a-6c}{5b}\right)^{\frac{3}{2}},$$

$$r_3 = r_4 = -\sqrt{3}(a+54c)\left(\frac{6c-a}{5b}\right)^{\frac{3}{2}}, \quad (12)$$

where a_1, p_2 and r_2 are free parameters. Substituting (12) into Eq. (9) yields the exact solution

$$q(x, t) = \frac{18c-3a}{b} + \left(\frac{a_1\sqrt{\frac{3a-18c}{b}}\sin(\xi_1) - a_2\sqrt{\frac{3a-18c}{b}}\cos(\xi_2) - a_3\sqrt{\frac{18c-3a}{b}}\sinh(\xi_3) + \sqrt{\frac{18c-3a}{b}}\exp(-\xi_4) - a_4\sqrt{\frac{18c-3a}{b}}\exp(\xi_4)}{a_1\cos(\xi_1) + a_2\sin(\xi_2) + a_3\cosh(\xi_3) + \exp(-\xi_4) + a_4\exp(\xi_4)} \right)^2, \quad (13)$$

$$\text{where } \xi_1 = \xi_2 = \sqrt{\frac{3a-18c}{b}}x + \sqrt{3}(a+54c)\left(\frac{a-6c}{5b}\right)^{\frac{3}{2}}t, \xi_3 = \xi_4 = \sqrt{\frac{18c-3a}{b}}x - \sqrt{3}(a+54c)\left(\frac{6c-a}{5b}\right)^{\frac{3}{2}}t.$$

Also, if we take $\frac{b}{3} = \pm(a-6c)$ and $a_3 = -1, a_4 = 1$, then Eq. (13) can be reduced to the following form:

$$q(x, t) = 1 - \left(\frac{-a_1\sin(\xi_1) + a_2\cos(\xi_2) + \sinh(\xi_4)}{a_1\cos(\xi_1) + a_2\sin(\xi_2) + \cosh(\xi_4)} \right)^2. \quad (14)$$

Case 3.

$$a_1 = a_4 = 0, p_2 = p_3 = 0, r_2 = r_3 = 0, p_4 = \sqrt{\frac{72c-12a}{b}}, r_4 = -2\sqrt{3}(a+54c)\left(\frac{6c-a}{5b}\right)^{\frac{3}{2}}, \quad (15)$$

where a_3 is a free parameter. Substituting (15) into Eq. (9), we can obtain the following exact solitary solution:

$$q(x, t) = \frac{(72c-12a)a_3\exp(-\xi_4)}{b(a_3+\exp(\xi_4))^2}, \quad (16)$$

where $\xi_4 = \sqrt{\frac{72c-12a}{b}}x - 2\sqrt{3}(a+54c)\left(\frac{6c-a}{5b}\right)^{\frac{3}{2}}t$.

Case 4.

$$\begin{aligned} a_1 = 0, p_2 = \sqrt{\frac{3a-18c}{b}}, \quad p_3 = p_4 = \sqrt{\frac{18c-3a}{b}}, \quad r_2 = \sqrt{3}(a+54c)\left(\frac{a-6c}{5b}\right)^{\frac{3}{2}}, \\ r_3 = r_4 = -\sqrt{3}(a+54c)\left(\frac{6c-a}{5b}\right)^{\frac{3}{2}}, \end{aligned} \quad (17)$$

where a_2, a_3 and a_4 are free parameters. Substituting (17) into Eq. (9) yields the exact solution

$$q(x, t) = \frac{18c-3a}{b} - \left(\frac{a_2 \sqrt{\frac{3a-18c}{b}} \cos(\xi_2) + a_3 \sqrt{\frac{18c-3a}{b}} \sinh(\xi_3) - \sqrt{\frac{18c-3a}{b}} \exp(-\xi_4) + a_4 \sqrt{\frac{18c-3a}{b}} \exp(\xi_4)}{a_2 \sin(\xi_2) + a_3 \cosh(\xi_3) + \exp(-\xi_4) + a_4 \exp(\xi_4)} \right)^2, \quad (18)$$

where $\xi_2 = \sqrt{\frac{3a-18c}{b}}x + \sqrt{3}(a+54c)\left(\frac{a-6c}{5b}\right)^{\frac{3}{2}}t$, $\xi_3 = \xi_4 = \sqrt{\frac{18c-3a}{b}}x - \sqrt{3}(a+54c)\left(\frac{6c-a}{5b}\right)^{\frac{3}{2}}t$.

Also, if we take $\frac{b}{3} = \pm(6c-a)$ and $a_3 = -1, a_4 = 1$, then Eq. (18) can be reduced to the following form:

$$q(x, t) = 1 - \left(\frac{a_2 \cos(\xi_2) + \sinh(\xi_4)}{a_2 \sin(\xi_2) + \cosh(\xi_4)} \right)^2. \quad (19)$$

Case 5.

$$\begin{aligned} a_2 = 0, p_1 = \sqrt{\frac{3a-18c}{b}}, \quad p_3 = p_4 = \sqrt{\frac{18c-3a}{b}}, \quad r_1 = \sqrt{3}(a+54c)\left(\frac{a-6c}{5b}\right)^{\frac{3}{2}}, \\ r_3 = r_4 = -\sqrt{3}(a+54c)\left(\frac{6c-a}{5b}\right)^{\frac{3}{2}}, \end{aligned} \quad (20)$$

where a_1, a_3 and a_4 are free parameters. Substituting (20) into Eq. (9), we can write the exact solution

$$q(x, t) = \frac{18c-3a}{b} - \left(\frac{-a_1 \sqrt{\frac{3a-18c}{b}} \sin(\xi_1) + a_3 \sqrt{\frac{18c-3a}{b}} \sinh(\xi_3) - \sqrt{\frac{18c-3a}{b}} \exp(-\xi_4) + a_4 \sqrt{\frac{18c-3a}{b}} \exp(\xi_4)}{a_1 \cos(\xi_1) + a_3 \cosh(\xi_3) + \exp(-\xi_4) + a_4 \exp(\xi_4)} \right)^2, \quad (21)$$

where $\xi_1 = \sqrt{\frac{3a-18c}{b}}x + \sqrt{3}(a+54c)\left(\frac{a-6c}{5b}\right)^{\frac{3}{2}}t$, $\xi_3 = \xi_4 = \sqrt{\frac{18c-3a}{b}}x - \sqrt{3}(a+54c)\left(\frac{6c-a}{5b}\right)^{\frac{3}{2}}t$.

Also, if we take $\frac{b}{3} = \pm(6c-a)$ and $a_3 = -1, a_4 = 1$, then Eq. (21) can be reduced to the following form:

$$q(x, t) = 1 - \left(\frac{-a_1 \sin(\xi_1) + \sinh(\xi_4)}{a_1 \cos(\xi_1) + \cosh(\xi_4)} \right)^2. \quad (22)$$

Case 6.

$$a_3 = 0, p_1 = p_2 = \sqrt{\frac{3a-18c}{b}}, \quad p_3 = p_4 = \sqrt{\frac{18c-3a}{b}}, \quad r_1 = r_1 = \sqrt{3}(a+54c) \left(\frac{a-6c}{5b}\right)^{\frac{3}{2}},$$

$$r_3 = r_4 = -\sqrt{3}(a+54c) \left(\frac{6c-a}{5b}\right)^{\frac{3}{2}}, \quad (23)$$

where a_1, a_2 and a_4 are free parameters. Substituting (23) into Eq. (9) yields the solitary solution

$$q(x, t) = \frac{18c-3a}{b} - \left(\frac{-a_1 \sqrt{\frac{3a-18c}{b}} \sin(\xi_1) + a_2 \sqrt{\frac{3a-18c}{b}} \cos(\xi_2) - \sqrt{\frac{18c-3a}{b}} \exp(-\xi_4) + a_4 \sqrt{\frac{18c-3a}{b}} \exp(\xi_4)}{a_1 \cos(\xi_1) + a_2 \sin(\xi_2) + \exp(-\xi_4) + a_4 \exp(\xi_4)} \right)^2, \quad (24)$$

where $\xi_1 = \xi_2 = \sqrt{\frac{3a-18c}{b}}x + \sqrt{3}(a+54c) \left(\frac{a-6c}{5b}\right)^{\frac{3}{2}}t$, $\xi_4 = \sqrt{\frac{18c-3a}{b}}x - \sqrt{3}(a+54c) \left(\frac{6c-a}{5b}\right)^{\frac{3}{2}}t$.

Also, if we take $\frac{b}{3} = \pm(6c-a)$ and $a_4 = 1$, then Eq. (24) can be reduced to the following form:

$$q(x, t) = 1 - \left(\frac{-a_1 \sin(\xi_1) + a_2 \cos(\xi_2) + 2 \sinh(\xi_4)}{a_1 \cos(\xi_1) + a_2 \sin(\xi_2) + 2 \cosh(\xi_4)} \right)^2. \quad (25)$$

Case 7.

$$a_1 = a_2 = 0, \quad p_3 = p_4 = \sqrt{\frac{18c-3a}{b}}, \quad r_3 = r_4 = -\sqrt{3}(a+54c) \left(\frac{6c-a}{5b}\right)^{\frac{3}{2}}, \quad (26)$$

where a_3, a_4 are free parameters. Substituting (26) into Eq. (9) yields the following wave solution to Eq.(1)

$$q(x, t) = \frac{18c-3a}{b} - \left(\frac{a_3 \sqrt{\frac{18c-3a}{b}} \sinh(\xi_3) - \sqrt{\frac{18c-3a}{b}} \exp(-\xi_4) + a_4 \sqrt{\frac{18c-3a}{b}} \exp(\xi_4)}{a_3 \cosh(\xi_3) + \exp(-\xi_4) + a_4 \exp(\xi_4)} \right)^2, \quad (27)$$

where $\xi_3 = \xi_4 = \sqrt{\frac{18c-3a}{b}}x - \sqrt{3}(a+54c) \left(\frac{6c-a}{5b}\right)^{\frac{3}{2}}t$. Also, if we take $\frac{b}{3} = \pm(6c-a)$ and $a_3 = -1, a_4 = 1$, then Eq. (27) can be reduced to the following form:

$$q(x, t) = 1 - \tanh^2(\xi_3) = \operatorname{sech}^2(\xi_3). \quad (28)$$

Case 8.

$$a_1 = a_3 = 0, p_2 = \sqrt{\frac{3a-18c}{b}}, \quad p_4 = \sqrt{\frac{18c-3a}{b}},$$

$$r_2 = \sqrt{3}(a+54c) \left(\frac{a-6c}{5b}\right)^{\frac{3}{2}}, \quad r_4 = -\sqrt{3}(a+54c) \left(\frac{6c-a}{5b}\right)^{\frac{3}{2}}, \quad (29)$$

where a_2 and a_4 are free parameters. Substituting (29) into Eq. (9), we have the following wave solution:

$$q(x, t) = \frac{18c-3a}{b} - \left(\frac{a_2 \sqrt{\frac{3a-18c}{b}} \cos(\xi_2) - \sqrt{\frac{18c-3a}{b}} \exp(-\xi_4) + a_4 \sqrt{\frac{18c-3a}{b}} \exp(\xi_4)}{a_2 \sin(\xi_2) + \exp(-\xi_4) + a_4 \exp(\xi_4)} \right)^2, \quad (30)$$

where $\xi_2 = \sqrt{\frac{3a-18c}{b}} x - \sqrt{3}(a+54c) \left(\frac{a-6c}{5b}\right)^{\frac{3}{2}} t$, $\xi_4 = \sqrt{\frac{18c-3a}{b}} x - \sqrt{3}(a+54c) \left(\frac{6c-a}{5b}\right)^{\frac{3}{2}} t$. Also, if we take $\frac{b}{3} = \pm(6c-a)$ and $a_4 = 1$, then Eq. (30) can be reduced to the following form:

$$q(x, t) = 1 - \left(\frac{a_2 \cos(\xi_2) + 2 \sinh(\xi_4)}{a_2 \sin(\xi_2) + 2 \cosh(\xi_4)} \right)^2. \quad (31)$$

Case 9.

$$a_2 = a_3 = 0, p_1 = \sqrt{\frac{3a-18c}{b}}, p_4 = \sqrt{\frac{18c-3a}{b}},$$

$$r_1 = \sqrt{3}(a+54c) \left(\frac{a-6c}{5b}\right)^{\frac{3}{2}}, r_4 = -\sqrt{3}(a+54c) \left(\frac{6c-a}{5b}\right)^{\frac{3}{2}}, \quad (32)$$

where a_1 and a_4 are free parameters. Substituting (32) into Eq. (9) yields the solitary solution

$$q(x, t) = \frac{18c-3a}{b} - \left(\frac{-a_1 \sqrt{\frac{3a-18c}{b}} \sin(\xi_1) - \sqrt{\frac{18c-3a}{b}} \exp(-\xi_4) + a_4 \sqrt{\frac{18c-3a}{b}} \exp(\xi_4)}{a_1 \cos(\xi_1) + \exp(-\xi_4) + a_4 \exp(\xi_4)} \right)^2, \quad (33)$$

where $\xi_1 = \sqrt{\frac{3a-18c}{b}} x - \sqrt{3}(a+54c) \left(\frac{a-6c}{5b}\right)^{\frac{3}{2}} t$, $\xi_4 = \sqrt{\frac{18c-3a}{b}} x - \sqrt{3}(a+54c) \left(\frac{6c-a}{5b}\right)^{\frac{3}{2}} t$. Also, if we take $\frac{b}{3} = \pm(6c-a)$ and $a_4 = 1$, then Eq. (33) can be reduced to the following form:

$$q(x, t) = 1 - \left(\frac{-a_1 \sin(\xi_1) + 2 \sinh(\xi_4)}{a_1 \cos(\xi_1) + 2 \cosh(\xi_4)} \right)^2. \quad (34)$$

Case 10.

$$a_1 = a_2 = a_3 = 0, p_4 = \sqrt{\frac{18c-3a}{b}}, r_4 = -\sqrt{3}(a+54c) \left(\frac{6c-a}{5b}\right)^{\frac{3}{2}}, \quad (35)$$

where a_4 is a free parameter. Substituting (35) into Eq. (9), we can obtain soliton solution as follows:

$$q(x, t) = \frac{18c-3a}{b} - \left(\frac{-\sqrt{\frac{18c-3a}{b}} \exp(-\xi_4) + a_4 \sqrt{\frac{18c-3a}{b}} \exp(\xi_4)}{\exp(-\xi_4) + a_4 \exp(\xi_4)} \right)^2, \quad (36)$$

where $\xi_4 = \sqrt{\frac{18c-3a}{b}} x - \sqrt{3}(a+54c) \left(\frac{6c-a}{5b}\right)^{\frac{3}{2}} t$. Also, if we take $\frac{b}{3} = \pm(6c-a)$ and $a_4 = 1$, then Eq. (36) can be reduced to the following soliton solution

$$q(x, t) = \operatorname{sech}^2(\xi_4). \quad (37)$$

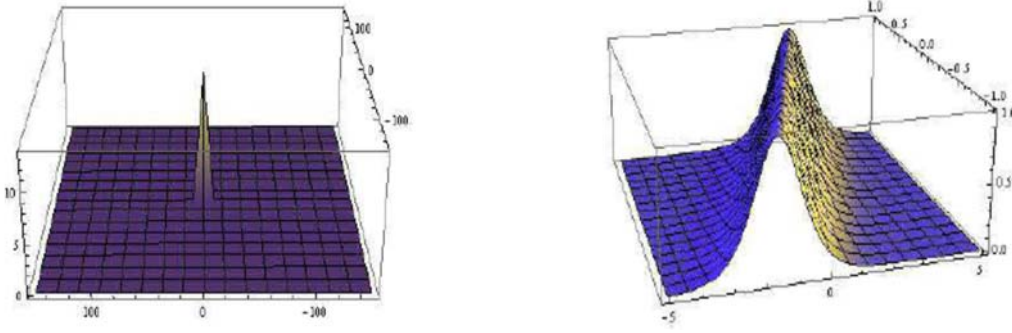


Figure 1. Solitary solution; Bright soliton

4. The bilinear form of fKdV equation

If $n = 2$, then Eq. (2) can be converted into

$$q_t + aq_{xxx} + bq_{xxxxx} - c(q^3)_x = 0. \quad (38)$$

Suppose that

$$w = (\ln f)_x, \quad q = w_x = (\ln f)_{xx}, \quad (39)$$

and substituting it into Eq. (38), we can get

$$w_{xt} + aw_{xxxx} + bw_{xxxxx} - c(w_x^3)_x = 0. \quad (40)$$

Integrate x one time, and let the integral constant is zero, we have

$$w_t + aw_{xxx} + bw_{xxxxx} - cw_x^3 = 0. \quad (41)$$

Substituting Eq. (39) into Eq. (41), we can reduce Eq. (38) into the bilinear forms as follows

$$\begin{aligned} (D_t D_x + aD_x^4 + \frac{c}{15}D_x^6)(f \cdot f) &= 0, \\ D_x^2(f \cdot f) \cdot (-3aD_x^2 - cD_x^4)(f \cdot f) &= 0, \end{aligned} \quad (42)$$

where $f = f(x, t)$ is an unknown real function, $15b = c$ and the bilinear operator $D_x^m D_t^n$ is defined as

$$D_x^m D_t^n ab = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n a(x, t) b(x', t') \Big|_{x=x', t=t'} = 0. \quad (43)$$

If we take

$$D_x^4(f \cdot f) = \frac{-3a}{c} D_x^2(f \cdot f), \quad (44)$$

then Eq. (42) is converted into

$$\left(D_t D_x - \frac{3a}{c} D_x^2 + \frac{c}{15} D_x^6 \right) (f \cdot f) = 0. \quad (45)$$

5. The application of the multi-wave solutions method for solving the fKdV equation

For this problem, we can use test function of extended four-soliton method

$$f(x, t) = a_1 \cos(\xi_1) + a_2 \sin(\xi_2) + a_3 \cosh(\xi_3) + \exp(-\xi_4) + a_4 \exp(\xi_4), \quad (46)$$

where $\xi_i = p_i x + r_i t, i = 1, 2, 3, 4$. Substituting (46) into (45), we can write the form of general solution of the fKdV equation

$$q = \frac{-a_1 p_1^2 \cos(p_1 x + r_1 t) - a_2 p_2^2 \sin(p_2 x + r_2 t) + a_3 p_3^2 \cosh(p_3 x + r_3 t) + p_4^2 \exp(-p_4 x - r_4 t) + a_4 p_4^2 \exp(p_4 x + r_4 t)}{a_1 \cos(p_1 x + r_1 t) + a_2 \sin(p_2 x + r_2 t) + a_3 \cosh(p_3 x + r_3 t) + \exp(-p_4 x - r_4 t) + a_4 \exp(p_4 x + r_4 t)} - \left(\frac{-a_1 p_1 \sin(p_1 x + r_1 t) + a_2 p_2 \cos(p_2 x + r_2 t) + a_3 p_3 \sinh(p_3 x + r_3 t) - p_4 \exp(-p_4 x - r_4 t) + a_4 p_4 \exp(p_4 x + r_4 t)}{a_1 \cos(p_1 x + r_1 t) + a_2 \sin(p_2 x + r_2 t) + a_3 \cosh(p_3 x + r_3 t) + \exp(-p_4 x - r_4 t) + a_4 \exp(p_4 x + r_4 t)} \right)^2. \quad (47)$$

Substituting Eq. (47) into Eq. (45), and equating the coefficients of all powers of polynomials, we can find a set of algebraic equations for $a_1, a_2, a_3, a_4, p_1, p_2, p_3, p_4, r_1, r_2, r_3, r_4$. Solving these algebraic equations with the aid of Mathematica, we obtain the exact solutions as follows:

Case 2.

$$a_4 = 0, \quad p_1 = p_2 = p_3 = r_1 = r_2 = r_3 = 0, \quad r_4 = p_4 \left(\frac{3a^2 - bcp_4^4}{c} \right), \quad (48)$$

where a_1, a_3, p_4 are free parameters. Substituting (48) into Eq. (47) yields the following wave solution

$$q(x, t) = p_4^2 \left(\frac{(a_1 + a_3) \exp(-\xi_4)}{(a_1 + a_3 + \exp(\xi_4))^2} \right), \quad (49)$$

where $\xi_4 = p_4 \left(x - \frac{bcp_4^4 - 3a^2}{c} t \right)$.

Case 2.

$$a_4 = 0, \quad p_1 = ip_3, \quad p_2 = -ip_3, \quad p_4 = i\sqrt{3}p_3, \quad r_1 = ip_3 \left(\frac{3a^2 - 16bcp_3^4}{c} \right), \\ r_2 = ip_3 \left(\frac{16bcp_3^4 - 3a^2}{c} \right), \quad r_3 = p_3 \left(\frac{3a^2 - 16bcp_3^4}{c} \right), \quad r_4 = \frac{i\sqrt{3}p_3(3a^2 + 16bcp_3^4)}{c}, \quad (50)$$

where a_1, a_2, a_3 and p_3 are free parameters. Substituting (50) into Eq. (47), we can obtain

$$q(x, t) = \frac{a_1 p_3^2 \cosh(\xi_3) - ia_2 p_3^2 \sinh(\xi_3) + a_3 p_3^2 \cosh(\xi_3) - 3p_3^2 \exp(-\xi_4)}{a_1 \cosh(\xi_3) - ia_2 \sinh(\xi_3) + a_3 \cosh(\xi_3) + \exp(-\xi_4)} + \left(\frac{a_1 p_3 \sinh(\xi_3) - ia_2 p_3 \cosh(\xi_3) + a_3 p_3 \sinh(\xi_3) - i\sqrt{3} \exp(-\xi_4)}{a_1 \cosh(\xi_3) - ia_2 \sinh(\xi_3) + a_3 \cosh(\xi_3) + \exp(-\xi_4)} \right)^2, \quad (51)$$

where $\xi_3 = p_3 \left(x - \frac{16bcp_3^4 - 3a^2}{c} t \right)$ and $\xi_4 = i\sqrt{3}p_3 \left(x + \frac{3a^2 + 16bcp_3^4}{c} t \right)$.

Case 3.

$$p_1 = p_2 = ip_3, \quad p_3 = p_4, \quad r_1 = r_2 = ip_4 \left(\frac{3a^2 - 16bcp_4^4}{c} \right), \quad r_3 = r_4 = p_4 \left(\frac{3a^2 - 16bcp_4^4}{c} \right), \quad (52)$$

where a_1, a_2, a_3, a_4 are free parameters. Substituting (52) into Eq. (47), we have the following solutions:

$$q(x, t) = p_4^2 \left(1 - \frac{a_1 \sinh(\xi_4) + ia_2 \cosh(\xi_4) + a_3 \sinh(\xi_4) - \exp(-\xi_4) + a_4 \exp(\xi_4)}{a_1 \cosh(\xi_4) + ia_2 \sinh(\xi_4) + a_3 \cosh(\xi_4) + \exp(-\xi_4) + a_4 \exp(\xi_4)} \right)^2, \quad (53)$$

where $\xi_3 = p_3 \left(x - \frac{16bcp_4^4 - 3a^2}{c} t \right)$. Also, if $a_1 = a_3 = -1, a_4 = 1$, then Eq. (53) can be reduced to the following form:

$$q(x, t) = -p_4^2 \operatorname{cosech}^2(\xi_4). \quad (54)$$

Case 4.

$$a_1 = a_4 = 0, \quad p_2 = p_3 = 0, \quad r_3 = r_2 = 0, \quad r_4 = p_4 \left(\frac{3a^2 - bcp_4^4}{c} \right), \quad (55)$$

where a_2, a_3 are free parameters. Substituting (55) into Eq. (47) yields the exact solution

$$q(x, t) = p_4^2 \left(\frac{a_3 \exp(-\xi_4)}{(a_3 + \exp(\xi_4))^2} \right), \quad (56)$$

where $\xi_4 = p_4 \left(x - \frac{bcp_4^4 - 3a^2}{c} t \right)$. Also, if we take $a_3 = 1$, then Eq. (56) can be reduced to the following form:

$$q(x, t) = \frac{p_4^2}{2(1 + \cosh(\xi_4))}. \quad (57)$$

Case 5.

$$\begin{aligned} a_1 = a_4 = 0, \quad p_2 = ip_3, \quad p_4 = i\sqrt{3}p_3, \quad r_1 = ip_3 \left(\frac{3a^2 - 16bcp_3^4}{c} \right), \\ r_3 = p_3 \left(\frac{3a^2 - 16bcp_3^4}{c} \right), \quad r_4 = \frac{i\sqrt{3}p_3(3a^2 + 16bcp_3^4)}{c}, \end{aligned} \quad (58)$$

where a_2, a_3 are free parameters. Substituting (58) into Eq. (47) yields the solution

$$q(x, t) = \frac{ia_2 p_3^2 \sinh(\xi_3) + a_3 p_3^2 \cosh(\xi_3) - 3p_3^2 \exp(-\xi_4)}{ia_2 \sinh(\xi_3) + a_3 \cosh(\xi_3) + \exp(-\xi_4)} - \left(\frac{ia_2 p_3 \cosh(\xi_3) + a_3 p_3 \sinh(\xi_3) - i\sqrt{3} \exp(-\xi_4)}{ia_2 \sinh(\xi_3) + a_3 \cosh(\xi_3) + \exp(-\xi_4)} \right)^2, \quad (59)$$

where $\xi_3 = p_3 \left(x + \frac{16bcp_3^4 - 3a^2}{c} t \right)$ and $\xi_4 = i\sqrt{3}p_3 \left(x + \frac{3a^2 + 16bcp_3^4}{c} t \right)$.

Case 6.

$$\begin{aligned} a_2 = a_4 = 0, \quad p_1 = ip_3, \quad p_4 = i\sqrt{3}p_3, \quad r_1 = ip_3 \left(\frac{3a^2 - 16bcp_3^4}{c} \right), \\ r_3 = p_3 \left(\frac{3a^2 - 16bcp_3^4}{c} \right), \quad r_4 = \frac{i\sqrt{3}p_3(3a^2 + 16bcp_3^4)}{c}, \end{aligned} \quad (60)$$

where a_1, a_3 are free parameters. Substituting (60) into Eq. (47), we obtain solitary solution

$$q(x, t) = \frac{a_1 p_3^2 \cosh(\xi_3) + a_3 p_3^2 \cosh(\xi_3) - 3p_3^2 \exp(-\xi_4)}{a_1 \cosh(\xi_3) + a_3 \cosh(\xi_3) + \exp(-\xi_4)} - \left(\frac{a_1 p_3 \sinh(\xi_3) + a_3 p_3 \sinh(\xi_3) - i\sqrt{3} p_3 \exp(-\xi_4)}{a_1 \cosh(\xi_3) + a_3 \cosh(\xi_3) + \exp(-\xi_4)} \right)^2, \quad (61)$$

where $\xi_3 = p_3 \left(x + \frac{16bcp_3^4 - 3a^2}{c} t \right)$ and $\xi_4 = i\sqrt{3}p_3 \left(x + \frac{3a^2 + 16bcp_3^4}{c} t \right)$. Also, if we take $a_1 = a_3 = 1$, then Eq. (61) can be reduced to the following form:

$$q(x, t) = \frac{2p_3^2 \cosh(\xi_3) - 3p_3^2 \exp(-\xi_4)}{2 \cosh(\xi_3) + \exp(-\xi_4)} - \left(\frac{2p_3 \sinh(\xi_3) - i\sqrt{3}p_3 \exp(-\xi_4)}{2 \cosh(\xi_3) + \exp(-\xi_4)} \right)^2. \quad (62)$$

Case 7.

$$a_3 = a_4 = 0, \quad p_1 = p_2, \quad p_4 = \sqrt{3}p_2, \quad r_1 = r_2 = p_2 \left(\frac{3a^2 - 16bcp_2^4}{c} \right), \quad r_4 = \frac{\sqrt{3}p_2(3a^2 + 16bcp_2^4)}{c}, \quad (63)$$

where a_1, a_2 are free parameters. Substituting (63) into Eq. (47) yields the periodic solitary solution

$$q(x, t) = \frac{-a_1 p_2^2 \cos(\xi_2) - a_2 p_2^2 \sin(\xi_2) + 3p_2^2 \exp(-\xi_4)}{a_1 \cos(\xi_2) + a_2 \sin(\xi_2) + \exp(-\xi_4)} - \left(\frac{a_1 p_2 \sin(\xi_2) + a_2 p_2 \cos(\xi_2) - \sqrt{3}p_2 \exp(-\xi_4)}{a_1 \cos(\xi_2) + a_2 \sin(\xi_2) + \exp(-\xi_4)} \right)^2, \quad (64)$$

where $\xi_2 = p_2 \left(x + \frac{16bcp_2^4 - 3a^2}{c} t \right)$ and $\xi_4 = \sqrt{3}p_3 \left(x + \frac{3a^2 + 16bcp_3^4}{c} t \right)$.

Case 8.

$$a_1 = a_2 = a_4 = 0, \quad p_4 = i\sqrt{3}p_3, \quad r_3 = p_3 \left(\frac{3a^2 - 16bcp_3^4}{c} \right), \quad r_4 = \frac{i\sqrt{3}p_3(3a^2 + 16bcp_3^4)}{c}, \quad (65)$$

where a_3 is a free parameter. Substituting (65) into Eq. (47), we find

$$q(x, t) = \frac{a_3 p_3^2 \cosh(\xi_3) - 3p_3^2 \exp(-\xi_4)}{a_3 \cosh(\xi_3) + \exp(-\xi_4)} - \left(\frac{a_3 p_3 \sinh(\xi_3) - i\sqrt{3}p_3 \exp(-\xi_4)}{a_3 \cosh(\xi_3) + \exp(-\xi_4)} \right)^2, \quad (66)$$

where $\xi_3 = p_3 \left(x + \frac{16bcp_3^4 - 3a^2}{c} t \right)$ and $\xi_4 = i\sqrt{3}p_3 \left(x + \frac{3a^2 + 16bcp_3^4}{c} t \right)$.

Case 9.

$$a_1 = a_3 = a_4 = 0, \quad p_4 = \sqrt{3}p_2, \quad r_3 = p_2 \left(\frac{3a^2 - 16bcp_2^4}{c} \right), \quad r_4 = \frac{\sqrt{3}p_2(3a^2 + 16bcp_2^4)}{c}, \quad (67)$$

where a_2 is a free parameter. Substituting (67) into Eq. (47) yields the periodic solitary solution

$$q(x, t) = \frac{-a_2 p_2^2 \sin(\xi_2) + 3p_2^2 \exp(-\xi_4)}{a_2 \sin(\xi_2) + \exp(-\xi_4)} - \left(\frac{a_2 p_2 \cos(\xi_2) - \sqrt{3}p_2 \exp(-\xi_4)}{a_2 \sin(\xi_2) + \exp(-\xi_4)} \right)^2, \quad (68)$$

where $\xi_2 = p_2 \left(x + \frac{16bcp_2^4 - 3a^2}{c} t \right)$ and $\xi_4 = \sqrt{3}p_2 \left(x + \frac{3a^2 + 16bcp_2^4}{c} t \right)$.

Case 10.

$$a_1 = 0, \quad p_2 = ip_4, \quad p_3 = p_4, \quad r_2 = ip_4 \left(\frac{3a^2 - 16bcp_4^4}{c} \right), \quad r_3 = r_4 = p_4 \left(\frac{3a^2 - 16bcp_4^4}{c} \right), \quad (69)$$

where a_2, a_3, a_4 are free parameters. Substituting (69) into Eq. (47) yields the exact solution

$$q(x, t) = p_4^2 - \left(\frac{ia_2 p_4 \cosh(\xi_4) + a_3 p_4 \sinh(\xi_4) - p_4 \exp(-\xi_4) + a_4 p_4 \exp(\xi_4)}{ia_2 \sinh(\xi_4) + a_3 \cosh(\xi_4) + \exp(-\xi_4) + a_4 \exp(\xi_4)} \right)^2, \quad (70)$$

where $\xi_4 = p_4 \left(x + \frac{16bcp_4^4 - 3a^2}{c} t \right)$. Also, if we take $a_3 = -1$ and $a_4 = 1$, then Eq. (70) can be reduced to the solution

$$q(x, t) = p_4^2 \left(1 - \left(\frac{\sinh(\xi_4) + ia_2 \cosh(\xi_4)}{\cosh(\xi_4) + ia_2 \sinh(\xi_4)} \right)^2 \right) \quad (71)$$

Case 11.

$$a_2 = 0, \quad p_1 = ip_4, \quad p_3 = p_4, \quad r_1 = ip_4 \left(\frac{3a^2 - 16bcp_4^4}{c} \right), \quad r_3 = r_4 = p_4 \left(\frac{3a^2 - 16bcp_4^4}{c} \right), \quad (72)$$

where a_1, a_3, a_4 are free parameters. Substituting (72) into Eq. (47) yields the solitary solution

$$q(x, t) = p_4^2 - \left(\frac{a_1 p_4 \sinh(\xi_4) + a_3 p_4 \sinh(\xi_4) - p_4 \exp(-\xi_4) + a_4 p_4 \exp(\xi_4)}{a_1 \sinh(\xi_4) + a_3 \cosh(\xi_4) + \exp(-\xi_4) + a_4 \exp(\xi_4)} \right)^2, \quad (73)$$

where $\xi_4 = p_4 \left(x + \frac{16bcp_4^4 - 3a^2}{c} t \right)$. Also, if we take $a_1 = -1$ and $a_3 = a_4 = 1$, then Eq. (73) can be reduced to the solution

$$q(x, t) = p_4^2 (1 - \tanh^2(\xi_4)) = p_4^2 \operatorname{sech}^2(\xi_4). \quad (74)$$

Case 12.

$$a_3 = 0, \quad p_1 = p_2 = -ip_4, \quad r_1 = r_2 = -ip_4 \left(\frac{3a^2 - 16bcp_4^4}{c} \right), \quad r_3 = p_4 \left(\frac{3a^2 - 16bcp_4^4}{c} \right), \quad (75)$$

where a_1, a_2, a_4 are free parameters. Substituting (75) into Eq. (47), we have

$$q(x, t) = p_4^2 - \left(\frac{a_1 p_4 \sinh(\xi_4) - ia_2 p_4 \cosh(\xi_4) - p_4 \exp(-\xi_4) + a_4 p_4 \exp(\xi_4)}{a_1 \cosh(\xi_4) - ia_2 \sinh(\xi_4) + \exp(-\xi_4) + a_4 \exp(\xi_4)} \right)^2, \quad (76)$$

where $\xi_4 = p_4 \left(x + \frac{16bcp_4^4 - 3a^2}{c} t \right)$. Also, if we take $a_1 = -1$ and $a_4 = 1$, then Eq. (76) can be reduced to the solution

$$q(x, t) = p_4^2 \left(1 - \left(\frac{\sinh(\xi_4) - ia_2 \cosh(\xi_4)}{\cosh(\xi_4) - ia_2 \sinh(\xi_4)} \right)^2 \right). \quad (77)$$

Case 13.

$$a_1 = a_2 = 0, \quad p_4 = p_3, \quad r_3 = r_4 = p_3 \left(\frac{3a^2 - 16bcp_3^4}{c} \right), \quad (78)$$

where a_3, a_4 are free parameters. Substituting (78) into Eq. (47), we get

$$q(x, t) = p_3^2 - \left(\frac{a_3 p_3 \sinh(\xi_3) - p_3 \exp(-\xi_3) + a_4 p_3 \exp(\xi_3)}{a_3 \cosh(\xi_3) + \exp(-\xi_3) + a_4 \exp(\xi_3)} \right)^2, \quad (79)$$

where $\xi_3 = p_3 \left(x + \frac{16bcp_3^4 - 3a^2}{c} t \right)$. Also, if we take $a_3 = -1$ and $a_4 = 1$, then Eq. (79) can be reduced to

$$q(x, t) = p_3^2(1 - \tanh^2(\xi_3)) = p_3^2 \operatorname{sech}^2(\xi_3). \quad (80)$$

Case 14.

$$a_1 = a_3 = 0, \quad p_4 = ip_2, \quad r_2 = p_2 \left(\frac{3a^2 - 16bcp_2^4}{c} \right), \quad r_4 = ip_2 \left(\frac{3a^2 - 16bcp_2^4}{c} \right), \quad (81)$$

where a_2, a_4 are free parameters. Substituting (81) into Eq. (47), we can obtain

$$q(x, t) = -p_2^2 - \left(\frac{a_2 p_2 \cos(\xi_2) - ip_2 \exp(-i\xi_2) + ia_4 p_2 \exp(i\xi_2)}{a_2 \sin(\xi_2) + \exp(-i\xi_2) + a_4 \exp(i\xi_2)} \right)^2, \quad (82)$$

where $\xi_2 = p_2 \left(x + \frac{16bcp_2^4 - 3a^2}{c} t \right)$. Also, if we take $a_4 = 1$, then Eq. (82) can be reduced to the wave solution

$$q(x, t) = -p_2^2 \left(1 + \frac{a_2 \cos(\xi_2) - 2\sin(\xi_2)}{a_2 \sin(\xi_2) + 2\cos(\xi_2)} \right)^2. \quad (83)$$

Case 15.

$$a_2 = a_3 = 0, \quad p_4 = ip_1, \quad r_1 = p_1 \left(\frac{3a^2 - 16bcp_1^4}{c} \right), \quad r_4 = -ip_1 \left(\frac{3a^2 - 16bcp_1^4}{c} \right), \quad (84)$$

where a_1, a_4 are free parameters. Substituting (84) into Eq. (47) yields the following exact solution

$$q(x, t) = -p_1^2 - \left(\frac{-a_1 p_1 \sin(\xi_1) + ip_1 \exp(i\xi_1) - ia_4 p_1 \exp(-i\xi_1)}{a_1 \cos(\xi_1) + \exp(i\xi_1) + a_4 \exp(-i\xi_1)} \right)^2, \quad (85)$$

where $\xi_1 = p_1 \left(x + \frac{16bcp_1^4 - 3a^2}{c} t \right)$. Also, if we take $a_1 = -1$ and $a_4 = 1$, then we can reduce Eq. (85) to the following form:

$$q(x, t) = -p_1^2 \operatorname{sec}^2(\xi_1). \quad (86)$$

Case 16.

$$a_1 = a_2 = a_3 = 0, \quad r_4 = p_4 \left(\frac{3a^2 - 16bcp_4^4}{c} \right), \quad (87)$$

where a_4 is a free parameter. Substituting (87) into Eq. (47) yields soliton solution

$$q(x, t) = p_4^2 - \left(\frac{-p_4 \exp(-\xi_4) + a_4 p_4 \exp(\xi_4)}{\exp(-\xi_4) + a_4 \exp(\xi_4)} \right)^2, \quad (88)$$

where $\xi_4 = p_4 \left(x + \frac{16bcp_4^4 - 3a^2}{c} t \right)$. Also, if we take $a_4 = 1$, then Eq. (88) can be reduced to the following form:

$$q(x, t) = p_4^2 \operatorname{sech}^2(\xi_4). \quad (89)$$

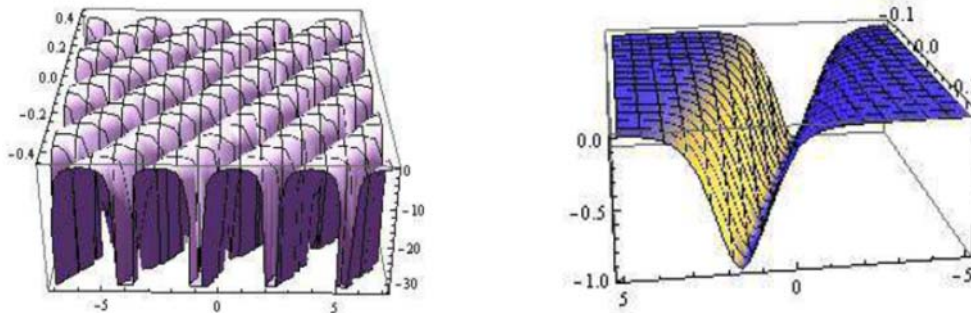


Figure 2. Periodic solution; Bright soliton

6. Conclusion

In this paper, we used the multi-wave method to obtain exact solutions to 5th order KdV equation and fifth order KdV (fKdV) equation. Using a useful test function, we obtain new exact solitary solutions, such as M-type wave solution, periodic solitary wave solution, triangular periodic wave solution, etc., for two nonlinear evolution equations. So, these applications illustrate the effectiveness of the suggested method with the help of the Mathematica. The obtained results show that this approach is possible for integrable equations or non-integrable equations to have periodic solitary waves.

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