Padé approximations for solving differential equations of Lane-Emden type

Veyis Turut

Batman University, Faculty of Sciences and Arts, Department of Mathematics, Batman, Turkey

veyis.turut@batman.edu.tr

Abstract

In this paper unvariate Padé approximation is applied to second order differential equations of Lane-Emden type. Numerical results show that Padé approximation is reliable, efficient and easy to use from a numerical computational viewpoint.

Keywords: Padé approximation, Adomian decomposition method, Lane-Emden type equations.

Lane-Emden diferansiyel denklem türlerinin Padé yaklaşımları ile çözümü

Özet

Bu çalışmada ikinci mertebeden Lane-Emden türündeki diferansiyel denklemlere Padé yaklaşımı uygulanmıştır. Nümerik sonuçlar, Padé yaklaşımının güvenilir, etkili ve nümerik hesaplamalar için kullanışlı olduğunu göstermektedir.

Anahtar kelimeler: Padé yaklaşımı, Adomian ayrıştırma yöntemi, Lane-Emden türü denklemler.

1. Introduction

Padé approximation [1,3] has been applied for rational series solutions in many areas. It is also known that Padé approximants, show superior performance over series approximations. It can be seen in many papers that Padé approximants give better numerical results than approximation by polynomial.

A reliable algorithm based mainly on the Adomian decomposition method has been constructed and applied on differential equations of Lane-Emden type by Wazwaz in [4,5]. In these papers [4,5] series solutions of second order differential equations of Lane-Emden type has been obtained successfully. In this paper Padé approximation is applied to these series solutions of differential equations of Lane-Emden type that solved by Wazwaz in [5]. The algorithm, based mainly on the Adomian decomposition method, has been presented in [5]. So a short information will be given about that algorithm in section 2 as ‘’ Analysis of the method’’.

Lane–Emden-type equations formulated as [5]

$$\begin{aligned} & y'' + \frac{2}{x} y' + f(y) = 0, \qquad 0 < x \le 1 \\ & y(0) = A, \qquad y'(0) = \mathrm{B}. \end{aligned} \tag{1}$$

On the other hand, studies have been carried out on another class of singular initial value problems of the form [5]

$$\begin{aligned} & y'' + \frac{2}{x} y' + f(\mathrm{x}, y) = g(x), \qquad 0 < x \le 1, \\ & y(0) = A, \qquad y'(0) = \mathrm{B}. \end{aligned} \tag{2}$$

where A and B are constants, $f(x, y)$ is a continuous real valued function, and $g(x) \in [0,1]$ Eq. (2) differs from the classical Lane–Emden type equations (1) for the function $f(x, y)$ and for the inhomogeneous term $g(x)$ [5].

The information above about Lane–Emden type equations has been taken from [5]and a very large information about Lane–Emden-type equations can be found in [4,5].

2. Analysis of the method

The Adomian decomposition method usually defines the equation in an operator form by considering the highest-ordered derivative in the problem [5]. To overcome the singularity behavior, Wazwaz [5] defined the differential operator L in terms of the two derivatives contained in the problem. Following [6,7], Wazwaz [5] rewrote (2) in the form

$$Ly = -f(x,y) + g(x) \tag{3}$$

where the differential operator L is defined by

$$L = x^{-2}\frac{d}{dx}\left(x^2\frac{d}{dx}\right). \tag{4}$$

Wazwaz [5] therefore considered the inverse operator L^{-1} as a two-fold integral operatör defined by

$$L^{-1}(.) = \int_0^x x^{-2}\int_0^x x^2 (.)\,dxdx. \tag{5}$$

Operating with L^{-1} on (3), Wazwaz [5] obtained

$$y(x) = A + Bx + L^{-1}g(x) - L^{-1}f(x,y). \tag{6}$$

The Adomian decomposition method introduces the solution $y(x)$ by an infinite series of components [5]

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{7}$$

and the nonlinear function $f(x, y)$ by an infinite series of polynomials

$$f(x,y)=\sum_{n=0}^{\infty}A_{n,} \tag{8}$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently, and A_n are Adomian polynomials that can be constructed for various classes of nonlinearity according to specific algorithms set by Adomian [8,9], and Adomian and Rach [10] and calculated by Wazwaz [11]. For a nonlinear function $F(u)$, the first few polynomials are given by

$$\begin{aligned}
A_0 &= F(u_0),\\
A_1 &= u_1F'(u_0),\\
A_2 &= u_2F'(u_0)+\frac{u_1^2}{2!}F''(u_0),\\
A_3 &= u_3F'(u_0)+u_1u_2F''(u_0)+\frac{u_1^3}{3!}F'''(u_0),\\
A_4 &= u_4F'(u_0)+\left(\frac{u_2^2}{2!}+u_1u_3\right)F''(u_0)+\frac{u_1^2}{2!}F'''(u_0)+\frac{1}{4!}u_1^4\,\mathrm{F}^{(iv)}(u_0),\\
&\vdots
\end{aligned} \tag{9}$$

Substituting (7) and (8) into (6) wazwaz [5] obtained

$$\sum_{n=0}^{\infty}y_n(x)=A+Bx+L^{-1}g(x)-L^{-1}\sum_{n=0}^{\infty}A_n. \tag{10}$$

To determine the components $y_n(x)$, Wawaz [5] used Adomian decomposition method that suggests the use of the recursive relation

$$\begin{aligned}
y_0(x) &= A+Bx+L^{-1}g(x),\\
y_{k+1}(x) &= L^{-1}(A_k), \qquad k\ge 0,
\end{aligned} \tag{11}$$

which gives

$$
\begin{aligned}
&y_0(x) = A + Bx + L^{-1}g(x),\\
&y_1(x) = L^{-1}(A_0),\\
&y_2(x) = -L^{-1}(A_1),\\
&y_3(x) = -L^{-1}(A_2),\\
&\vdots
\end{aligned}
\tag{12}
$$

Wazwaz [5] combined the scheme (12) with (9) that will enable us to determine the components $y_n(x)$ recursively, and hence the series solution of $y(x)$ defined by (7) follows immediately. For numerical purposes, the $n-term$ approximant

$$\phi_n = \sum_{k=0}^{n-1} y_k, \tag{13}$$

can be used to approximate the solution [5].

3. Padé approximation

Consider a formal power series

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots \tag{14}$$

with $(c_0 \neq 0)$ [12]. In this paper ∂p is written for the exact degre of a polynomial p and ωp for the order of a power series p [12]. (i.e the degree of the first nonzero term).

The Padé approximation problem of order (m,n) or $[m,n]$ for f consists in finding polynomials

$$p(x) = \sum_{i=0}^{m} a_i x^i, \quad q(x) = \sum_{i=0}^{n} b_i x^i \tag{15}$$

such that in the power series $(fq-p)(x)$ the coefficients of x^i for $i = 0, \ldots m+n$ disappar, i.e [12].

$$\begin{aligned} &\partial(p) \leq m \\ &\partial(q) \leq n \\ &\omega(fq-p) \geq m+n+1 \end{aligned} \tag{16}$$

Condition (16) is equivalent with the following two linear systems of equations

$$\begin{cases} c_0 b_0 = 0 \\ c_1 b_0 + c_0 b_1 = a_1 \\ \vdots \\ c_m b_0 + c_{m-1} b_1 + \cdots + c_{m-n} b_n = a_m \end{cases} \tag{17}$$

$$\begin{cases} c_{m+1} b_0 + c_m b_1 + \ldots + c_{m-n+1} b_n = a_m \\ \vdots \\ c_{m+n} b_0 + c_{m-n+1} b_1 + \ldots + c_m b_n = 0 \end{cases} \tag{18}$$

with $c_i = 0$ for $i < 0$ [12]. For $n = 0$ the systems of equations (18) is empty. In this case, $a_i = c_i \ (i = 0, \ldots, m)$ and $b_0 = 1$ satisfy (16), in the other words the partial sums of (14) solve the Padé approximation problem of order $(m,0)$.

In general a solution for the coefficients a_i is known after substitution of a solution for the b_i in the left hand side of (17). So the crucial point is to solve the homogeneous system of n equations (18) in the $n+1$ unknowns b_i. This system has at least one nontrivial solution because one of the unknowns can be chosen freely [12].

In short, by solving the equations (17) and (18) the coefficients a_i and b_i are found. Then the Padé equations (15) are found. After finding these polynomials we get The Padé approximation of order (m,n) or $[m,n]$ for f.

4. Applications and results

In this section Padé series solutions of differential equations of Lane-Emden type shall be illustrated by two examples.

Example 4.1.

Consider the nonlinear singular initial value problem[5]:

$$\begin{aligned} &y'' + \frac{2}{x} y' + 4\left(2e^{y} + e^{y/2}\right) = 0, \\ &y(0) = 0, \qquad y'(0) = 0. \end{aligned} \tag{19}$$

Wazwaz solved equation (19) by using the method that we mentioned in section 2 and

$$y(x) = -2\left(x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \frac{1}{5}x^{10} - \frac{1}{6}x^{12} + \cdots\right) \tag{20}$$

series solution has been obtained in [5]. For more information about the solution of equation (19) by Wazwaz can be seen in [5]. Exact solution for equation (20) is given as $f(x) = -2\ln(1 + x^2)$ in [5].

If the Padé approximation is applied to solution (20) then Padé series of different orders can be obtained. By applying Padé approximation the following Padé approximation of order $[4,4]$, $[6,4]$ and $[8,2]$ for equation (19) are obtained:

$$[4,4]_{y(x)} = \frac{-x^4 - 2x^2}{\frac{1}{6}x^4 + x^2 + 1} \tag{21}$$

$$[6,4]_{y(x)} = \frac{-\frac{1}{15}x^6 - \frac{1}{15}x^2 - 2x^2}{\frac{3}{10}x^4 + \frac{6}{5}x^2 + 1} \tag{22}$$

$$[8,2]_{y(x)} = \frac{-\frac{1}{30}x^8 + \frac{2}{15}x^6 - \frac{3}{5}x^4 - 2x^2}{\frac{4}{5}x^2 + 1} \tag{23}$$

To obtain the Padé series (21), (22), and (23) the solution tecnique that mentioned in section 3 for the linear systems of equations (17) and (18) is can be applied. If the numerical results are compared for example 1, the following tables and figures are obtained (Table 1 and figure 1, figure 2 ,figue 3);

Example 4.2.

Consider the nonlinear singular initial value problem[5]:

$$\begin{aligned} &y'' + \frac{6}{x}y' + 14y = -4y\ln y, \\ &y(0) = 1, \qquad y'(0) = 0. \end{aligned} \tag{24}$$

Wazwaz solved equation (23) by using the method that we mentioned in section 2 and

$$y(x) = 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 - \frac{1}{5!}x^{10} + \cdots \tag{25}$$

series solution has been obtained in [5]. More information about the solution of equation (24) by Wazwaz can be seen in [5]. Exact solution for equation (24) is given as $f(x) = e^{-x^2}$ in [5].

If the Padé approximation is applied to solution (25) then Padé series of different orders can be obtained. By applying Padé approximation the following Padé approximation of order $[4,4]$,$[6,4]$ and $[8,2]$ for equation (25) are obtained:

$$[4,4]_{y(x)} = \frac{\frac{1}{12}x^4 - \frac{1}{2}x^2 + 1}{\frac{1}{12}x^4 + \frac{1}{2}x^2 + 1} \tag{26}$$

$$[6,4]_{y(x)} = \frac{-\frac{3}{5}x^2 + \frac{3}{20}x^4 - \frac{1}{60}x^6 + 1}{\frac{1}{20}x^4 + \frac{2}{5}x^2 + 1} \tag{27}$$

$$[8,2]_{y(x)} = \frac{1 - \frac{4}{5}x^2 + \frac{3}{10}x^4 - \frac{1}{15}x^6 + \frac{1}{120}x^8}{\frac{1}{5}x^2 + 1} \tag{28}$$

To obtain the Padé series (26), (27), and (28) the solution tecnique that mentioned in section 3 for the linear systems of equations (17) and (18) is can be applied. If the numerical results are compared for example 2, the following tables and figures are obtained (Table 2 and figure 3, figure 4 ,figue 5);

5. Conclusion

As it is seen from the numerical results in the tables (Table 1, Table 2) and from figures (figure 1, figüre 2 figure 3 and figüre 4, figüre 5, figüre 6) The Padé approximations, show superior performance over series approximations, give results with no greater error bounds anda re reliable for more better numerical values.

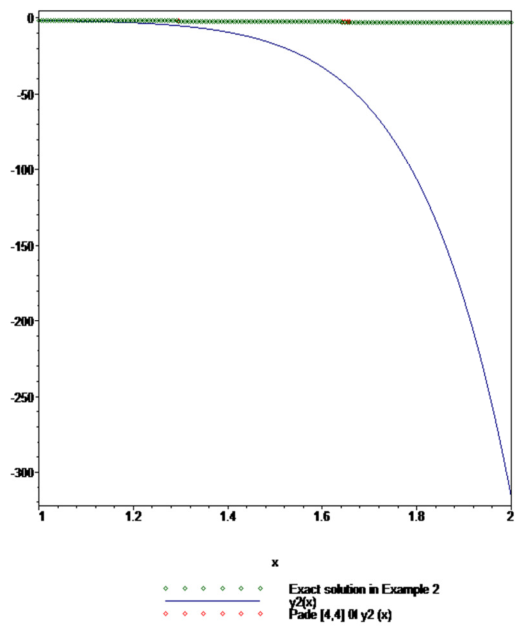

Figure 1. Exact solution of equation 19 in **Example 1**, $y(x)$ series solution in **Example 1**, $[4/4]$ Padé approximation of $y(x)$ series solution ($[4/4]_{y(x)}$).

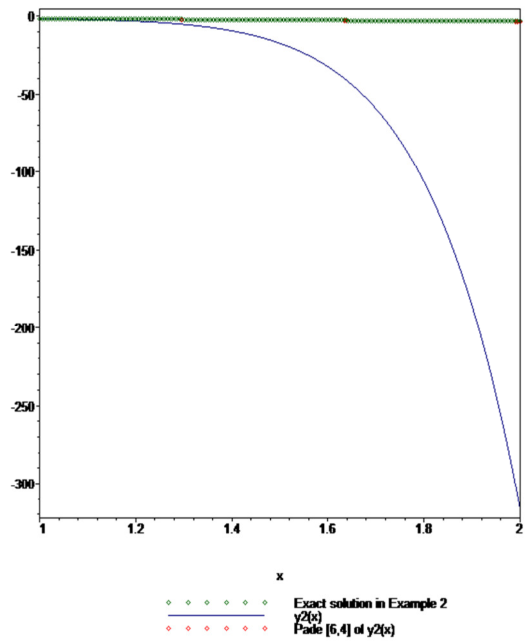

Figure 2. Exact solution of equation 19 in **Example 1**, $y(x)$ series solution in **Example 1**, $[6/4]$ Padé approximation of $y(x)$ series solution ($[6/4]_{y(x)}$).

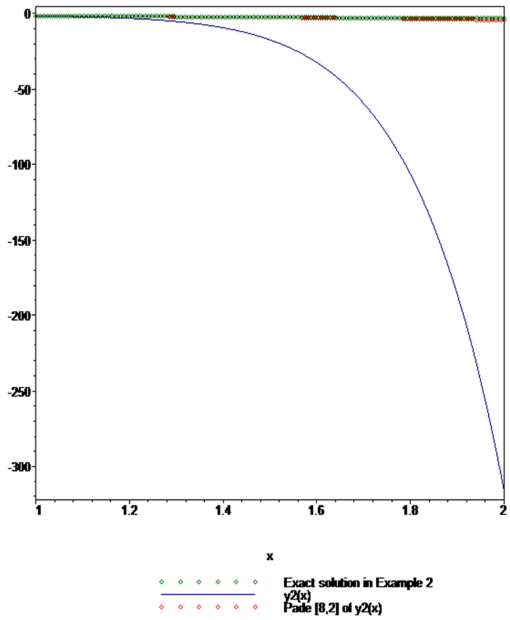

Figure 3. Exact solution of equation 19 in **Example 1**, $y(x)$ series solution in **Example 1**, $[8/2]$ Padé approximation of $y(x)$ series solution ($[8/2]_{y(x)}$).

Table 1. Numerical and absolute error values for **Example 1.**

x	Exact solution $f(x)=-2\ln(1+x^2)$	$y(x)$	$[4/4]_{y(x)}$	$[6/4]_{y(x)}$	$[8/2]_{y(x)}$	$\left\|-2\ln(1+x^2)-y(x)\right\|$	$\left\|-2\ln(1+x^2)-[4/4]_{y(x)}\right\|$	$\left\|-2\ln(1+x^2)-[6/4]_{y(x)}\right\|$	$\left\|-2\ln(1+x^2)-[8/2]_{y(x)}\right\|$
1.0	−1.386294361	−1.566666667	−1.384615385	−1.386666667	−1.388888889	0.180372306	0.001678976	0.000372306	0.002594528
1.1	−1.585985031	−2.102643246	−1.582752086	−1.586813940	−1.592329689	0.516658215	0.003232945	0.000828909	0.006344658
1.2	−1.783996079	−3.123842089	−1.778288340	−1.785660522	−1.798027896	1.339846010	0.005707739	0.001664443	0.014031817
1.3	−1.979082387	−5.177458734	−1.969699043	−1.982152956	−2.007651236	3.198376347	0.009383344	0.003070569	0.028568849
1.4	−2.170378536	−9.289331587	−2.155840308	−2.175658248	−2.224669607	7.118953051	0.014538228	0.005279712	0.054291071
1.5	−2.357309992	−17.28281250	−2.335877863	−2.365868263	−2.454659598	14.92550251	0.021432129	0.008558271	0.097349606
1.6	−2.539521090	−32.25683931	−2.509228476	−2.552718922	−2.705660640	29.71731822	0.030292614	0.013197832	0.166139550
	−2.716818316	−59.28058147	−2.675512194	−2.736324019	−2.988577883	56.56376315	0.041306122	0.019505703	0.271759567
1.8	−2.889126538	−106.3761022	−2.834513156	−2.916922342	−3.317627224	103.4869757	0.054613382	0.027795804	0.428500686
1.9	−3.056455714	−185.8766558	−2.986147188	−3.094836308	−3.710818814	182.8202001	0.070338526	0.038380594	0.654363100
2.0	−3.218875824	−316.2666667	−3.130434783	−3.270440252	−4.190476190	313.0477909	0.088441041	0.051564428	0.971600366

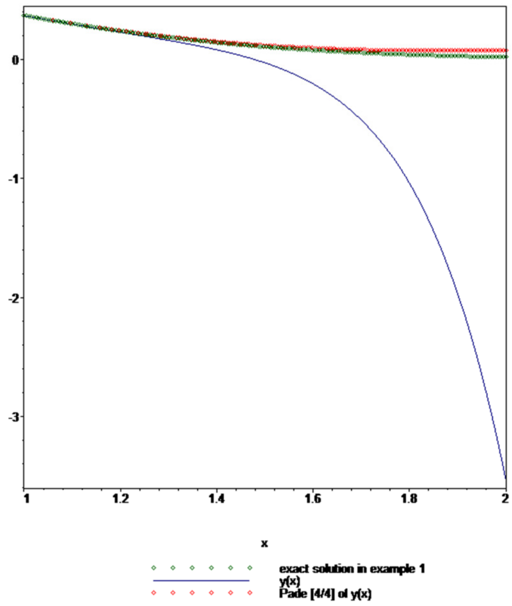

Figure 4. Exact solution of equation 24 in **Example 2**, $y(x)$ series solution in **Example 2**, $[4/4]$ Padé approximation of $y(x)$ series solution ($[4/4]_{y(x)}$).

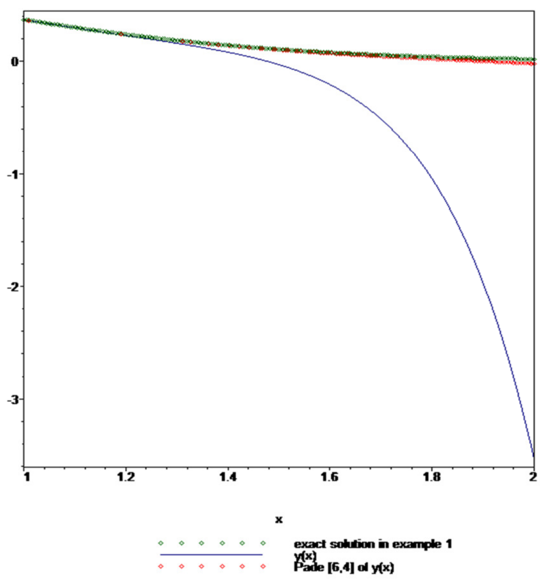

Figure 5. Exact solution of equation 24 **in Example 2**, $y(x)$ series solution in **Example 2**, $[6/4]$ Padé approximation of $y(x)$ series solution ($[6/4]_{y(x)}$).

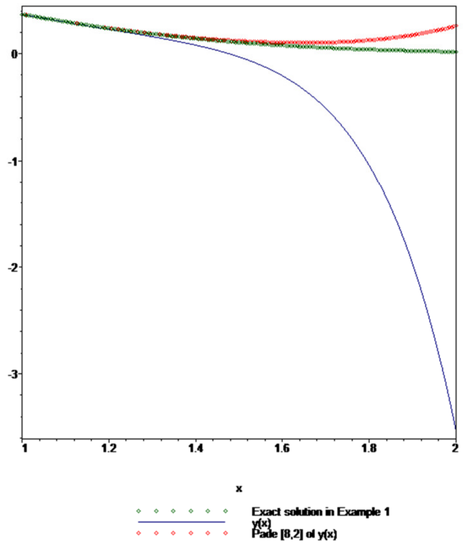

Figure 6. Exact solution of equation 24 in **Example 2**, $y(x)$ series solution in **Example 2**, $[8/2]$ Padé approximation of $y(x)$ series solution ($[8/2]_{y(x)}$).

Table 2. Numerical and absolute error values for **Example 2.**

x	*Exact solution* $f(x)=e^{-x^2}$	$y(x)$	$[4/4]_{y(x)}$	$[6/4]_{y(x)}$	$[8/2]_{y(x)}$	$\lvert e^{-x^2}-y(x)\rvert$	$\lvert e^{-x^2}-[4/4]_{y(x)}\rvert$	$\lvert e^{-x^2}-[6/4]_{y(x)}\rvert$	$\lvert e^{-x^2}-[8/2]_{y(x)}\rvert$
1.0	0.3678794412	0.3666666667	0.3684210526	0.3678160920	0.368055556	0.0012127745	0.0005416114	0.0000633492	0.0001761144
1.1	0.2981972794	0.2944915132	0.2993664381	0.2980269029	0.2987030382	0.0037057662	0.0011691587	0.0001703765	0.0005057588
1.2	0.2369277587	0.2266972365	0.2392223161	0.2365174319	0.2382346335	0.0102305222	0.0022945574	0.0004103268	0.0013068748
1.3	0.1845195240	0.1585875572	0.1886734331	0.1836195652	0.1876085900	0.0259319668	0.0041539091	0.0008999588	0.0030890660
1.4	0.1408584209	0.00797438939	0.1478754855	0.1390367461	0.1476245364	0.0611145270	0.0070170646	0.0018216748	0.0067661155
1.5	0.1053992246	−0.0298583982	0.1165644172	0.1019593614	0.1192753233	0.1352576228	0.0111651926	0.0034398632	0.0138760987
1.6	0.07730474044	−0.2060926500	0.09416871108	0.07119154532	0.1041751814	0.2833973904	0.01686397064	0.00611319512	0.02687044096
1.7	0.05557621261	−0.510307484	0.07991329748	0.04527586141	0.1050518714	0.5658836966	0.02433708487	0.01030035120	0.04947565879
1.8	0.3916389510	−1.043643114	0.07290832093	0.02260627889	0.1262915342	1.082807009	0.03374442583	0.01655761621	0.08712763910
1.9	0.02705184687	0.07221992578	0.07221992578	0.001523767826	0.1745266278	1.994719296	0.04516807891	0.02552807904	0.1474747809
2.0	0.01831563889	−3.533333333	0.07692307692	0.01960784314	0.2592592593	3.551648972	0.05860743803	0.03792348203	0.2409436204

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