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SEMI Q-DISCRETE SURFACES OF REVOLUTION

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Abstract

Discrete differential geometry considers all kinds of discrete objects. It has a lot of applications in geometry. One kind of applications is semi q– discrete surfaces. Semi q- discrete surfaces consist of bivariate function of one discrete and one continuous variable. Such mixed continuous- discrete objects can be seen as semi- discretization of smooth surfaces. Rather than the constant discretization methods, Quantum Calculus can be effective to discretize smooth surfaces. In this study, we briefly introduce such semi q- discretization of smooth surfaces. We also investigate semi q-discrete of revolution. Then, we give some definitions of semi q-discrete surface by using the q- trigonometric functions. Finally, we discuss basic theorems about the study.
Keywords: Discrete Surfaces, Surface of Revolution, Semi q- Discrete Surface.

YARI Q-DİSKRET DÖNEL YÜZEYLER

Öz

Diskret diferansiyel geometri, diskret objelere ilgilenir. Aynı zamanda, geometride çok fazla uygulaması vardır. Bu uygulamalardan bir tanesi yarı q-diskret yüzeylerdir. Yarı q- diskret yüzeyler, bir diskret ve bir sürekli değişkenden oluşan iki değişkenli fonksiyondan oluşur. Böyle sürekli- diskret objeler düzgün yüzeylerin yarı diskretleştirmesi olarak görülebilir. Sabit diskretleştirme metotlarından ziyade, Kuantum analizi düzgün yüzeyleri diskretleştirmede oldukça etkilidir. Bu çalışmada, kısaca düzgün yüzeylerin böyle bir yarı q-diskretleştirilmesini tanıttık. Aynı zamanda yarı q- diskret dönel yüzeylerden bahsetik. Sonrasında q- tigonometrik fonksiyonlar yardımıyla yarı q- diskret yüzeylerin bazı tanımlarını verdik. Çalışma hakkında bazı temel teoremleri tartıştık.
Anahtar Kelimeler: Diskret Yüzeyler, Dönel Yüzeyler, Yarı q- Diskret Yüzey.

1 Introduction

A new field of discrete differential geometry is emerging on the border between differential and discrete geometry. Whereas classical differential geometry studies geometric shapes (such as surfaces), discrete differential geometry studies geometric shapes with a finite number of elements. Current progress in this field is to a large extent stimulated for applications in computer graphics, architectural design, etc. Recent progress in discrete differential geometry has led not only to the discretization of a large body of classical results, but also to understanding of some fundamental structures. There are a lot of applications in this field. In [1], discrete surface of mean curvature have been studied from a variety of different points of view. The authors give a definition of discrete constant mean curvature in space forms as special isothermic nets. In [2], Kemmotsu studies surface of revolution with periodic mean curvature in order to extend the theory of constant mean curvature surfaces. Also, the mean curvature of a periodic surface of revolution has been shown. Finally, Bobenko and Pinkall show that two approaches yield the same definition of the discrete surfaces with constant negative curvature, which is called discrete K- surfaces. We study mappings of the form $x: Z \times IR \to IR^3$ which can be seen as the limit case of discrete surfaces. Surfaces of revolution are obtained by rotating about their axes the generating curves.

Definition 1. Given a value of $q > 0$ we define $[r]$, where $r \in IN$, as

$$[r] = \begin{cases} (1-q^r)/(1-q), & q \neq 1 \\ r, & q = 1 \end{cases} \tag{1}$$

and call $[r]$ a q-integer. Clearly, we can extend this definition, allowing r to be any real number in [1]. We then call $[r]$ a q-real [3]. For any given > 0, let us define

$$IN_q=\{[r], with\ r \in IN\} \tag{2}$$

and we can see from definition (1) that

$$IN_q = \{0, 1, 1+q, 1+q+q^2, \dots\}. \tag{3}$$

Definition 2. We define a q- binomial coefficient as

$$\begin{bmatrix} t \\ r \end{bmatrix} = \frac{[t][t-1]\dots[t-r+1]}{[r]!} \tag{4}$$

for all real t and integers r ≥ 0, and as zero otherwise [3].

Definition 3. For any integers n and r, we define

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n][n-1]\dots[n-r+1]}{[r]!} = \frac{[n]!}{[r]![n-r]!} \tag{5}$$

for $n \geq r \geq 0$, and as zero otherwise. These are called Gaussian polynomials.

The Gaussian polynomial satisfy the Pascal- type relations

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + q^r \begin{bmatrix} n-1 \\ r \end{bmatrix} \tag{6}$$

and

$$\begin{bmatrix} n \\ r \end{bmatrix} = q^{n-r} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ r \end{bmatrix}. \tag{7}$$

Theorem 1. Let $A_n = \{1,2,\dots,n\}$ and let $A_{n,j}$ be the collection of all subsets of A_n with j elements $0 \le j \le n$. Then,

$$\begin{bmatrix} n \\ j \end{bmatrix} = \sum_{S \in A_{n,j}} q^{w(S)-j(j+1)/2} \tag{8}$$

where $w(S) = \sum_{s \in S} s$.

Definition 4. Given a value of $q > 0$ we define $[r]!$, where $r \in IN$, as

$$[r]! = \begin{cases} [r][r-1]\dots 1, & r \ge 1 \\ 1, & r = 0 \end{cases} \tag{9}$$

and call [r]! a q-factorial.

Definition 5. Consider an arbitrary function f(x). Its q-differential is

$$d_q f(x) = f(qx) - f(x), \tag{10}$$

and its h-differential is

$$d_h f(x) = f(x+h) - f(x) \tag{11}$$

Definition 6. The following two expressions,

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x}, \tag{12}$$

$$D_h f(x) = \frac{d_h f(x)}{d_h x} = \frac{f(x+h) - f(x)}{h} \tag{13}$$

are called q- derivative and h-derivative, respectively, of the function f(x). Note that

$$\lim_{q \to 1} D_q f(x) = \lim_{h \to 0} D_h f(x) = \frac{df(x)}{dx} \tag{14}$$

In the ordinary calculus, a function, f(x) that possesses derivatives of all orders is analytic at x= a if it can be expressed as a power series about x= a. Taylor's theorem tells us the power series is

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(a) \frac{(x-a)^n}{n!} \tag{15}$$

Let us first consider a more general situation.

Theorem 2. Let a be a number, D be a linear operator on the space of polynomials, and $\{P_0(x), P_1(x), P_2(x), \dots\}$ be a sequence of polynomials satisfying three conditions:

(a) $P_0(a) = 1$ and $P_n(a) = 0$ for any $n \ge 1$;
(b) $degP_n = n$;
(c) $DP_n(x) = P_{n-1}(x)$ for any $n \ge 1$, and D(1)= 0.

Then, for any polynomial f(x) of degree N, one has the following generalized Taylor Formula:

$$f(x) = \sum_{n=0}^{N} (D^n f)(a) P_n(x) \tag{16}$$

Taylor's expansion of the classical exponential function is

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} \tag{17}$$

Definition 7. A q-analogue of the classical exponential function e^x is

$$e_q{}^x = \sum_{j=0}^{\infty} \frac{x^j}{[j]!} \tag{18}$$

The q- analogues of the sine and cosine functions can be defined in analogy with their well-known Euler expressions in terms of the exponential function.

Definition 8. The q-trigonometric functions are

$$sin_q x = \frac{e_q{}^{ix} - e_q{}^{-ix}}{2i}, \quad cos_q x = \frac{e_q{}^{ix} + e_q{}^{-ix}}{2} \tag{19}$$

To find the derivatives of the q-trigonometric functions, we apply the chain rule. Then, we obtain

$$D_q sin_q x = cos_q x$$
$$D_q cos_q x = -sin_q x \tag{20}$$

Let us explore what happens to this denominator when $x_j = [j]$. We have

$$x_{j+k+1} - x_j = \frac{1-q^{j+k+1}}{1-q} - \frac{1-q^j}{1-q} = q^j [k+1] \tag{21}$$

which is not independent of j, although it does have the common factor [k+1]. Now when k = 0 we have

$$f[x_j, x_{j+1}] = \frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j} = \frac{f(x_{j+1}) - f(x_j)}{q^j} \tag{22}$$

It is convenient to define

$$f(x_{j+1}) - f(x_j) = \Delta_q f(x_j) \tag{23}$$

so that

$$f[x_j, x_{j+1}] = \frac{\Delta_q f(x_j)}{q^j} \tag{24}$$

The second-order divided difference may be written as

$$f[x_j, x_{j+1}, x_{j+2}] = \frac{\Delta^2{}_q f(x_j)}{q^{2j+1}[2]} \tag{25}$$

Theorem 3. For all $j, k \ge 0$, we have

$$f[x_j, x_{j+1}, \dots, x_{j+k}] = \frac{\Delta^k{}_q f(x_j)}{q^{k(2j+k-1)/2}[k]!} \tag{26}$$

where each x_j equals [j], and [k]! = [k][k-1]...[1].

Definition 9. The following two expressions,

$$D_q f(x) = \frac{d_q f(x)}{d_q(x)} = \frac{f(qx) - f(x)}{(q-1)x} \tag{27}$$

$$D_h f(x) = \frac{d_h f(x)}{d_h(x)} = \frac{f(x+h) - f(x)}{h} \tag{28}$$

are called the q- derivative and h- derivative, respectively, of the function $f(x)$. Note that

$$\lim_{q\to 1} D_q f(x) = \lim_{h\to 0} D_h f(x) = \frac{df(x)}{dx} \quad (29)$$

If f(x) is differentiable. It is clear that as with the ordinary derivative, teh action of taking the q- or h- derivative of a function is a linear operator. In other words, D_q and D_h have the property that for any constants a and b.

$$D_q\big(af(x)+bg(x)\big) = aD_q f(x) + bD_q g(x),$$
$$D_h\big(af(x)+bg(x)\big) = aD_h f(x) + bD_h g(x).$$

Proposition 1. For any integer n,

$$D_q {(x-a)_q}^n = [n]{(x-a)_q}^{n-1} \quad (30)$$

Theorem 4. For any polynomial $f(x)$ of degree N and any number c, we have the following q- Taylor expansion:

$$f(x) = \sum_{j=0}^{N} ({D^j}_q f)(c)\frac{{(x-a)_q}^j}{[j]!} \quad (31)$$

Theorem 5. If yx= qyx, where q is a number commuting with both x and y, then

$$(x+y)^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} x^j y^{n-j} \quad (32)$$

2 Surfaces of Revolution

The use of surface of revolution is essential I physics and engineering. Surfaces of revolution are obtained by rotating about their axes the generating curves. Examples of surface of revolution include cylinder, sphere, torus, etc.

The curve is given by

$$x = \varphi(t),\ z = \Psi(t) \quad (33)$$

with t lies in [a, b] and is parametrized by arclength, so that

$$\dot{\varphi}^2 + \dot{\Psi}^2 = 1 \quad (34)$$

Then, the surface of revolution is the point set

$$M = \{(\varphi(t)cos\theta, \varphi(t)sin\theta, \Psi(t)) : t\epsilon(a,b), \theta\epsilon[0,2\pi)\} \quad (35)$$

In Quantum Calculus, the surface of revolution is written in the following:

$$x(\theta,t) = \big(\varphi(t)cos_q\theta, \varphi(t)sin_q\theta, \Psi(t)\big) : \theta \in [0,2\pi) \quad (36)$$

From Taylor's expansion, $\varphi(t)$ and $\Psi(t)$ are written

$$\varphi(t) = \frac{D^0\varphi(a)}{[0]!}[t-a]^0 + \frac{D^1\varphi(a)}{[1]!}[t-a]^1 + \frac{D^2\varphi(a)}{[2]!}[t-a]^2 + \dots \quad (37)$$

and

$$\Psi(t) = \frac{D^0\Psi(a)}{[0]!}[t-a]^0 + \frac{D^1\Psi(a)}{[1]!}[t-a]^1 + \frac{D^2\Psi(a)}{[2]!}[t-a]^2 + \dots \quad (38)$$

According to q-exponential function,

$$cos_q\theta = \frac{{e_q}^{i\theta} + {e_q}^{-i\theta}}{2} = \frac{\sum_{j=0}^{\infty}\frac{(i\theta)^j}{[j]!}}{2} + \frac{\sum_{j=0}^{\infty}\frac{(-i\theta)^j}{[j]!}}{2} \quad (39)$$

and

$$sin_q\theta = \frac{{e_q}^{i\theta} - {e_q}^{-i\theta}}{2i} = \frac{\sum_{j=0}^{\infty}\frac{(i\theta)^j}{[j]!}}{2i} - \frac{\sum_{j=0}^{\infty}\frac{(-i\theta)^j}{[j]!}}{2i} \quad (40)$$

Finally, the surface of revolution is written in the following:

$$x(\theta,t) = [(\varphi(a) + \frac{(\varphi(a+1)-\varphi(a))}{[1]!}[t-a] + \frac{(\varphi(a+2)-2\varphi(a+1)+\varphi(a))}{[2]!}[t-a]^2 + \cdots)\left(1 - \frac{\theta^2}{[2]!} + \frac{\theta^4}{[4]!} - \frac{\theta^6}{[6]!} + \cdots\right), (\varphi(a) + \frac{(\varphi(a+1)-\varphi(a))}{[1]!}[t-a] + \frac{(\varphi(a+2)-2\varphi(a+1)+\varphi(a))}{[2]!}[t-a]^2 + \cdots)\left(\theta - \frac{\theta^3}{[3]!} + \frac{\theta^5}{[5]!} - \frac{\theta^7}{[7]!} + \cdots\right), \Psi(a) + \frac{(\Psi(a+1)-\Psi(a))}{[1]!}[t-a] + \frac{(\Psi(a+2)-2\Psi(a+1)+\Psi(a))}{[2]!}[t-a]^2 + \cdots) \quad (41)$$

3 Conclusion

In differential geometry, there are a lot of calculations for the equation of surface of revolution. The best well- known method is Taylor expansion method. It is very useful method. We use different method for calculations for surface of revolution in this paper. In this study, we calculate the equation of surface of revolution by using q- trigonometric functions and q- integers. We use basic definitions and theorems for algebraic calculations.

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