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GENERALIZED TOPOLOGICAL OPERATOR ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-OPERATOR) THEORY IN GENERALIZED TOPOLOGICAL SPACES ($\mathscr{T}_{\mathfrak{g}}$-SPACES) PART III. GENERALIZED DERIVED ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-DERIVED) AND GENERALIZED CODERIVED ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-CODERIVED) OPERATORS

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Abstract. In a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, the $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ can be characterized in the generalized sense by the novel $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, giving rise to novel generalized $\mathfrak{g}$-topologies on Ω. In this paper, which forms the third part on the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators in $\mathscr{T}_{\mathfrak{g}}$-spaces, we study the essential properties of $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in $\mathscr{T}_{\mathfrak{g}}$-spaces. We show that $(\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}) : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is a pair of both dual and monotone $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators that is $(\emptyset, \Omega)$, $(\cup, \cap)$-preserving, and $(\subseteq, \supseteq)$-preserving relative to $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-(open, closed) sets. We also show that $(\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}) : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is a pair of weaker and stronger $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators. Finally, we diagram various relationships amongst $\text{der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\text{cod}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and present a nice application to support the overall study.

1. Introduction

The ordinary and generalized derived operators as well as their duals, called ordinary and generalized coderived operators, respectively, in ordinary ($\mathfrak{a} = \mathfrak{o}$) or generalized ($\mathfrak{a} = \mathfrak{g}$) topological spaces ($\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*derived* and $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*coderived operators in* $\mathscr{T}_{\mathfrak{a}}$-*space*) can all play very important roles, yielding to nice characterizations in $\mathscr{T}_{\mathfrak{a}}$-spaces. For instance, ordinary and generalized characterizations of $\mathscr{T}_{\mathfrak{a}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ of a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ can be realized by specifying either the $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived operators $\text{der}_{\mathfrak{a}}$, $\mathfrak{g}\text{-Der}_{\mathfrak{a}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or the $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators $\text{cod}_{\mathfrak{a}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{a}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively. In actual fact, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-(derivedness, coderivedness) are the generalization of $\mathfrak{T}_{\mathfrak{a}}$-(derivedness,

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coderivedness) in $\mathscr{T}_{\mathfrak{a}}$-spaces while the latter are the generalization of $\mathbb{R}$-(derivedness, coderivedness) in $\mathbb{R}$, respectively.

In contrast to $(\mathrm{der}_{\mathfrak{a}}, \mathrm{cod}_{\mathfrak{a}})$, $(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a}}) : \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$ in $\mathscr{T}_{\mathfrak{a}}$-spaces, $(\mathrm{der}_{\mathbb{R}}, \mathrm{cod}_{\mathbb{R}}) : \mathscr{P}(\mathbb{R}) \times \mathscr{P}(\mathbb{R}) \longrightarrow \mathscr{P}(\mathbb{R}) \times \mathscr{P}(\mathbb{R})$ in $\mathbb{R}$ is the oldest concept. If one year can be specified as the time when $(\mathrm{der}_{\mathbb{R}}, \mathrm{cod}_{\mathbb{R}}) : \mathscr{P}(\mathbb{R}) \times \mathscr{P}(\mathbb{R}) \longrightarrow \mathscr{P}(\mathbb{R}) \times \mathscr{P}(\mathbb{R})$ in $\mathbb{R}$ was first introduced and considered, that year should probably be 1872, the year in which Georg Cantor [1, 2] investigated the convergence of Fourier series. Thereafter, various Mathematicians have introduced and considered some types of $\mathfrak{T}_{\mathfrak{o}}$, $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{o}}$-operators in $\mathscr{T}_{\mathfrak{o}}$-spaces and other abstract spaces, and other types left untouched [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28].

Cenzer and Mauldin [6] have discussed some properties of the $\mathfrak{T}_{\mathfrak{o}}$-derived operator $\mathrm{der}_{\mathfrak{o}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{o}}$-space. Caldas, Jafari and Kovár [5] have studied some properties of the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{o}}$-derived operator $\theta\text{-}\mathrm{D} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (θ-*derived operator*). Devamanoharan, Missier and Jafari [7] have investigated some properties of the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{o}}$-derived operator $\mathrm{D}_p : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (p-*derived operator*). Missier and Raj [17] and Modak [18] have discussed some properties of the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{o}}$-derived operator $\mathrm{D}_\lambda : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (λ-*derived operator*) in a $\mathscr{T}_{\mathfrak{o}}$-space. Rodigo, Theodore and Jansi [22] have investigated some properties of the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{o}}$-derived operator $\mathrm{D}_{\beta^*} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (β^*-*derived operator*). Rajendiran and Thamilselvan [21] have considered the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{o}}$-derived operator $\mathrm{g}^*\mathrm{s}^*\mathrm{D} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (g^*s^*-*derived operator*) for the study of g^*s^*-derived sets in a $\mathscr{T}_{\mathfrak{o}}$-space. Sekar and Rajakumari [25] have studied some properties of the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{o}}$-derived operator $\mathrm{D}_{\alpha\mathrm{g}^*\mathrm{p}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ ($\alpha\mathrm{g}^*\mathrm{p}$-*derived operator*) in a $\mathscr{T}_{\mathfrak{o}}$-space. Lei and Zhang [16] have presented alternative axiomatic definitions for the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{o}}$-derived and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{o}}$-coderived operators $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{o}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{o}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{o}}$-space.

In view of the above references, no Mathematician has studied $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces, though few Mathematicians have studied $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{o}}$-derived and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{o}}$-coderived operators in $\mathscr{T}_{\mathfrak{o}}$-spaces. In this paper, we study a new class of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

Hereafter, the paper is structured as thus: In § 2, the preliminary and main concepts are described in §§ 2.1 and §§ 2.2, respectively. The main results are reported in § 3. In § 4, the various relationships amongst the $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{a}}$-derived and $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{a}}$-coderived operators in $\mathscr{T}_{\mathfrak{a}}$-space are diagrammed in §§ 4.1, and a nice application supporting the overall study is presented in §§ 4.2. Finally, the work is concluded in § 5.

2. Theory

2.1. **Preliminary Concepts.** The standard reference for notations and preliminary concepts presented herein is the Ph.D. Thesis of Khodabocus, M. I. [38] (Cf. [31, 32, 33, 34, 35, 36, 37]).

The topological structure $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ denotes a $\mathscr{T}_{\mathfrak{a}}$-space on which no separation axioms are assumed unless otherwise mentioned [36, 37, 38]. By convention, the relation $(\alpha_1, \alpha_2, \ldots)\,\mathrm{R}\,\mathscr{A}_1 \times \mathscr{A}_2 \times \cdots$ means $\alpha_1\,\mathrm{R}\,\mathscr{A}_1$, $\alpha_2\,\mathrm{R}\,\mathscr{A}_2$, ... where $\mathrm{R} = \in, \subset, \supset, \ldots$. The pairs $(I_n^0, I_n^*) \subset \mathbb{Z}_+^0 \times \mathbb{Z}_+^*$ and $(I_\infty^0, I_\infty^*) \sim \mathbb{Z}_+^0 \times \mathbb{Z}_+^*$ are pairs of *finite* and *infinite index sets*, respectively, [37, 38]. The relations $\Gamma \subset \Omega$, $\mathscr{O}_{\mathfrak{a}} \in \mathscr{T}_{\mathfrak{a}}$, $\mathscr{K}_{\mathfrak{a}} \in \neg\mathscr{T}_{\mathfrak{a}} \stackrel{\text{def}}{=} \{\mathscr{K}_{\mathfrak{a}} : \complement_\Omega(\mathscr{K}_{\mathfrak{a}}) \in \mathscr{T}_{\mathfrak{a}}\}$ and $\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}}$ state that Γ, $\mathscr{O}_{\mathfrak{a}}$, $\mathscr{K}_{\mathfrak{a}}$ and $\mathscr{S}_{\mathfrak{a}}$ are a Ω-*subset*, $\mathscr{T}_{\mathfrak{a}}$-*open set*, $\mathscr{T}_{\mathfrak{a}}$-*closed set* and $\mathfrak{T}_{\mathfrak{a}}$-*set*, respectively [37, 38]. The operators $\begin{array}{rrcl}\mathrm{int}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}}: & \mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega)\\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto & \mathrm{int}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}),\ \mathrm{cl}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}})\end{array}$ are the $\mathfrak{T}_{\mathfrak{a}}$-*interior* and $\mathfrak{T}_{\mathfrak{a}}$-*closure operators*, respectively [37, 38]. The class $\mathscr{L}_{\mathfrak{a}}[\Omega] \stackrel{\mathrm{def}}{=} \{\mathbf{op}_{\mathfrak{a},\nu} = (\mathrm{op}_{\mathfrak{a},\nu}, \neg\,\mathrm{op}_{\mathfrak{a},\nu}) :\ \nu \in I_3^0\}$, where

$$\begin{aligned}\langle \mathrm{op}_{\mathfrak{a},\nu}:\ \nu \in I_3^0\rangle &= \langle \mathrm{int}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}}\circ\mathrm{int}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}}\circ\mathrm{cl}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}}\circ\mathrm{int}_{\mathfrak{a}}\circ\mathrm{cl}_{\mathfrak{a}}\rangle,\\ \langle \neg\,\mathrm{op}_{\mathfrak{a},\nu}:\ \nu \in I_3^0\rangle &= \langle \mathrm{cl}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}}\circ\mathrm{cl}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}}\circ\mathrm{int}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}}\circ\mathrm{cl}_{\mathfrak{a}}\circ\mathrm{int}_{\mathfrak{a}}\rangle,\end{aligned}$$

is the class of all possible pairs of compositions of these $\mathfrak{T}_{\mathfrak{a}}$-operators in $\mathfrak{T}_{\mathfrak{a}}$. Then, $\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}}$ is called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-set if and only if there exist $(\mathscr{O}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}) \in \mathscr{T}_{\mathfrak{a}} \times \neg\mathscr{T}_{\mathfrak{a}}$ and $\mathbf{op}_{\mathfrak{a}} \in \mathscr{L}_{\mathfrak{a}}[\Omega]$ such that the following statement holds:

$$(2.1)\qquad (\exists\xi)\big[(\xi \in \mathscr{S}_{\mathfrak{a}}) \wedge \big((\mathscr{S}_{\mathfrak{a}} \subseteq \mathrm{op}_{\mathfrak{a}}(\mathscr{O}_{\mathfrak{a}})) \vee (\mathscr{S}_{\mathfrak{a}} \supseteq \neg\,\mathrm{op}_{\mathfrak{a}}(\mathscr{K}_{\mathfrak{a}}))\big)\big].$$

The derived class $\mathfrak{g}$-ν-$\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}$-$\nu$-$\mathrm{E}[\mathfrak{T}_{\mathfrak{a}}]$ is called the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-sets of category $\nu \in I_3^0$ (briefly, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-*sets*) in $\mathfrak{T}_{\mathfrak{a}}$ [37, 38]. Accordingly, the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-sets [38] are

$$\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{(\nu,\mathrm{E})\in I_3^0\times\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\nu\text{-}\mathrm{E}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\mathrm{E}[\mathfrak{T}_{\mathfrak{a}}].$$

(2.2)

Evidently, $\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{(\nu,\mathrm{E})\in\{0\}\times\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\nu\text{-}\mathrm{E}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}} \mathrm{E}[\mathfrak{T}_{\mathfrak{a}}]$ is the class of all $\mathfrak{T}_{\mathfrak{a}}$-sets in $\mathfrak{T}_{\mathfrak{a}}$ [37, 38].

Definition 2.1 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-*Interior,* $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-*Closure Operators* [31, 32])**.** In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$, the one-valued maps

$$(2.3)\qquad \begin{array}{rrcl}\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu}: & \mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega)\\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto & \displaystyle\bigcup_{\mathscr{O}_{\mathfrak{a}}\in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}]} \mathscr{O}_{\mathfrak{a}},\end{array}$$

$$(2.4)\qquad \begin{array}{rrcl}\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu}: & \mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega)\\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto & \displaystyle\bigcap_{\mathscr{K}_{\mathfrak{a}}\in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}]} \mathscr{K}_{\mathfrak{a}}\end{array}$$

where $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathscr{O}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}] :\ \mathscr{O}_{\mathfrak{a}} \subseteq \mathscr{S}_{\mathfrak{a}}\}$ and $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathscr{K}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}] :\ \mathscr{K}_{\mathfrak{a}} \supseteq \mathscr{S}_{\mathfrak{a}}\}$ are called $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-closure operators, respectively. Then, $\mathfrak{g}\text{-}\mathrm{I}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu} :\ \nu \in I_3^0\}$ and $\mathfrak{g}\text{-}\mathrm{C}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu} :\ \nu \in I_3^0\}$ are the classes of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closure operators, respectively.

Definition 2.2 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-*Vector Operator* [31, 32])**.** In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$, the two-valued map

$$(2.5)\qquad \begin{array}{rrcl}\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{a},\nu}: & \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega)\times\mathscr{P}(\Omega)\\ & (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) & \longmapsto & (\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu}(\mathscr{R}_{\mathfrak{a}}), \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu}(\mathscr{S}_{\mathfrak{a}}))\end{array}$$

is called a $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-vector operator. Then, $\mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\text{def}}{=} \{\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{a},\nu} = (\mathfrak{g}\text{-Int}_{\mathfrak{a},\nu}, \mathfrak{g}\text{-Cl}_{\mathfrak{a},\nu}) : \nu \in I_3^0\}$ is the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operators.

Remark 2.3. For every $\nu \in I_3^0$, $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{a},\nu} = \mathbf{ic}_{\mathfrak{a}} \stackrel{\text{def}}{=} (\text{int}_{\mathfrak{a}}, \text{cl}_{\mathfrak{a}})$ if based on $\text{O}[\mathfrak{T}_{\mathfrak{a}}]\times\text{K}[\mathfrak{T}_{\mathfrak{a}}]$. Then, $\begin{array}{rrcl}\mathbf{ic}_{\mathfrak{a}}: & \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \\ & (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) & \longmapsto & (\text{int}_{\mathfrak{a}}(\mathscr{R}_{\mathfrak{a}}), \text{cl}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}))\end{array}$ is a $\mathfrak{T}_{\mathfrak{a}}$-vector operator in a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$.

Definition 2.4 (*Complement* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*Operator* [31, 32])**.** Let $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ be a $\mathscr{T}_{\mathfrak{a}}$-space. Then, the one-valued map

$$\begin{array}{rrcl}\mathfrak{g}\text{-Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}}: & \mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto & \complement_{\mathscr{R}_{\mathfrak{a}}}(\mathscr{S}_{\mathfrak{a}}),\end{array} \tag{2.6}$$

where $\complement_{\mathscr{R}_{\mathfrak{a}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ denotes the relative complement of its operand with respect to $\mathscr{R}_{\mathfrak{a}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{a}}]$, is called a natural complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operator on $\mathscr{P}(\Omega)$.

For clarity, the notation $\mathfrak{g}\text{-Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}} = \mathfrak{g}\text{-Op}_{\mathfrak{a}}$ is employed whenever $\mathscr{R}_{\mathfrak{a}} = \Omega$ or $\mathscr{R}_{\mathfrak{a}}$ is understood from the context. When $\mathfrak{g}\text{-Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is with respect to $\mathscr{R}_{\mathfrak{a}} \in \text{S}[\mathfrak{T}_{\mathfrak{a}}]$, the term natural complement $\mathfrak{T}_{\mathfrak{g}}$-operator is employed and it stand for $\text{Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

2.2. **Main Concepts.** The main concepts underlying the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators in $\mathscr{T}_{\mathfrak{a}}$-spaces, $\mathfrak{a} \in \{\mathfrak{o}, \mathfrak{g}\}$, are now presented.

Definition 2.5 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-*Derived,* $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-*Coderived Operators*)**.** Suppose $\mathfrak{g}\text{-Int}_{\mathfrak{a},\nu}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, denote the $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-closure operators and, $\mathfrak{g}\text{-Op}_{\mathfrak{a}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ denote the absolute complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operator in a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$. Then, the one-valued maps of the types

$$\begin{array}{rrcl}\mathfrak{g}\text{-Der}_{\mathfrak{a},\nu}: & \mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto & \{\xi \in \mathfrak{T}_{\mathfrak{a}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{a},\nu}(\mathscr{S}_{\mathfrak{a}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{a}}(\{\xi\}))\},\end{array} \tag{2.7}$$

$$\begin{array}{rrcl}\mathfrak{g}\text{-Cod}_{\mathfrak{a},\nu}: & \mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto & \{\zeta \in \mathfrak{T}_{\mathfrak{a}} :\ \zeta \in \mathfrak{g}\text{-Int}_{\mathfrak{a},\nu}(\mathscr{S}_{\mathfrak{a}} \cup \{\zeta\})\}\end{array} \tag{2.8}$$

on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ are called, respectively, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived operator of category ν and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operator of category ν. The classes $\mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\text{def}}{=} \{\mathfrak{g}\text{-Der}_{\mathfrak{a},\nu} :\ \nu \in I_3^0\}$ and $\mathfrak{g}\text{-CD}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\text{def}}{=} \{\mathfrak{g}\text{-Cod}_{\mathfrak{a},\nu} :\ \nu \in I_3^0\}$ are called, respectively, the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived operators and the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators.

Remark 2.6. If the notations $\mathfrak{g}\text{-Der}_{\mathfrak{a}}(\xi; \mathscr{S}_{\mathfrak{a}})$ and $\mathfrak{g}\text{-Cod}_{\mathfrak{a}}(\zeta; \mathscr{S}_{\mathfrak{a}})$, respectively, designate a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*derived point* $\xi \in \mathfrak{T}_{\mathfrak{a}}$ and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*coderived point* $\zeta \in \mathfrak{T}_{\mathfrak{a}}$ of some $\mathscr{S}_{\mathfrak{a}} \in \mathscr{P}(\Omega)$, then

$$\mathfrak{g}\text{-Der}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \stackrel{\text{def}}{=} \{\mathfrak{g}\text{-Der}_{\mathfrak{a}}(\xi; \mathscr{S}_{\mathfrak{a}}) :\ \xi \in \mathfrak{T}_{\mathfrak{a}}\}, \tag{2.9}$$

$$\mathfrak{g}\text{-Cod}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \stackrel{\text{def}}{=} \{\mathfrak{g}\text{-Cod}_{\mathfrak{a}}(\zeta; \mathscr{S}_{\mathfrak{a}}) :\ \xi \in \mathfrak{T}_{\mathfrak{a}}\}, \tag{2.10}$$

respectively, denote the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*derived set* and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*coderived set* of $\mathscr{S}_{\mathfrak{a}}$ in $\mathfrak{T}_{\mathfrak{a}}$.

It is interesting to view $\mathfrak{g}\text{-Der}_{\mathfrak{a}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{a}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ as the components of some so-called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operator, and the definition follows.

Definition 2.7 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-*Vector Operator*)**.** Let $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ be a $\mathscr{T}_{\mathfrak{a}}$-space. Then, an operator of the type

$$(2.11)\qquad \begin{array}{rcl} \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{a},\nu} : \times_{\alpha\in I_2^*} \mathscr{P}(\Omega) & \longrightarrow & \displaystyle\times_{\alpha\in I_2^*} \mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) & \longmapsto & \big(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\nu}(\mathscr{R}_{\mathfrak{a}}), \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\nu}(\mathscr{S}_{\mathfrak{a}})\big) \end{array}$$

on $\times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ ranging in $\times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ is called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operator of category ν and, $\mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\text{def}}{=} \big\{\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{a},\nu} = \big(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{a},\nu}, \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{a},\nu}\big) : \ \nu \in I_3^0\big\}$ is called the class of all such $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operators.

Remark 2.8. For every $\nu \in I_3^0$, $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g},\nu} = \mathbf{dc}_{\mathfrak{g}} \stackrel{\text{def}}{=} \big(\mathrm{der}_{\mathfrak{g}}, \mathrm{cod}_{\mathfrak{g}}\big)$ if based on $\big(\mathrm{cl}_{\mathfrak{g}}, \mathrm{int}_{\mathfrak{g}}\big)$. Then, $\begin{array}{rl} \mathbf{dc}_{\mathfrak{a}} : \ \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) & \longmapsto \big(\mathrm{der}_{\mathfrak{a}}(\mathscr{R}_{\mathfrak{a}}), \mathrm{cod}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}})\big) \end{array}$ is a $\mathfrak{T}_{\mathfrak{a}}$-vector operator in a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$.

Of the notations $\mathfrak{T}_{\mathfrak{o}} = (\Omega, \mathscr{T}_{\mathfrak{o}})$ and $\mathfrak{T} = (\Omega, \mathscr{T})$, either the first will be used instead of the second, or both will be used interchangeably.

3. Main Results

The main results of the study of the essential properties of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-(derived, coderived) operators $\big(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big) : \mathscr{P}(\Omega)\times\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$ in $\mathscr{T}_{\mathfrak{g}}$-spaces are presented below.

Theorem 3.1. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{dc}_{\mathfrak{g}} \in \mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$-*operators* $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

– I. $\big(\forall\mathscr{R} \in \mathscr{P}(\Omega)\big)\big[\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big]$

– II. $\big(\forall\mathscr{S} \in \mathscr{P}(\Omega)\big)\big[\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big]$

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Suppose $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathbf{dc}_{\mathfrak{g}} \in \mathrm{DC}[\mathfrak{T}_{\mathfrak{g}}]$ be given and $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ be arbitrary. Then,

$$\begin{array}{rcl} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} : \mathscr{R}_{\mathfrak{g}} & \longmapsto & \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} : \ \xi \in \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})\big)\big\} \\ & \subseteq & \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} : \ \xi \in \mathrm{cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})\big)\big\} \longleftrightarrow \mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} & \longmapsto & \big\{\zeta \in \mathfrak{T}_{\mathfrak{g}} : \ \zeta \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big)\big\} \\ & \supseteq & \big\{\zeta \in \mathfrak{T}_{\mathfrak{g}} : \ \zeta \in \mathrm{int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\zeta\})\big)\big\} \longleftrightarrow \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \end{array}$$

Thus, $\big(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}), \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \subseteq \big(\mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}), \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$ for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$. The proof of the theorem is, therefore, complete. ☐

Remark 3.2. If $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \precsim \mathrm{der}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \succsim \mathrm{cod}_{\mathfrak{g}}$ stand for $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, respectively, then the outstanding facts are: $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathrm{der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathrm{der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

Proposition 1. If $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathfrak{g}\text{-Op}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be the natural complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator of their components in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:

$$\begin{aligned}&\big(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\big)\big[\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\\ &\wedge\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\big]\end{aligned} \tag{3.1}$$

Proof. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and let $\mathfrak{g}\text{-Op}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be the natural complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator of their components in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, for a $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ taken arbitrarily, it follows that

$$\begin{gathered}\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\\ \updownarrow\\ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \{\xi\}\big)\big\}\big)\\ \updownarrow\\ \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\}\\ \updownarrow\\ \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\}\\ \updownarrow\\ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\end{gathered}$$

Thus, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ for every $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. For a $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ taken arbitrarily, it follows that

$$\begin{gathered}\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\ \updownarrow\\ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\}\big)\\ \updownarrow\\ \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big)\big\}\\ \updownarrow\\ \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big)\big\}\\ \updownarrow\\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{gathered}$$

Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ for every $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. The proof of the proposition is, therefore, complete. ☐

Lemma 3.3. *If* $\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-CD}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$*, respectively, in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *then:*

– I. $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$

– II. $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega) \longrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$

Proof. Let $(\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}) \in \mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-CD}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and let $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ be an arbitrary pair such that $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$, it results that

$$\begin{aligned}
\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{R}_{\mathfrak{g}} \quad &\longmapsto \quad \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\} \\
&\subseteq \quad \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\} \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\
\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{R}_{\mathfrak{g}} \quad &\longmapsto \quad \big\{\zeta \in \mathfrak{T}_{\mathfrak{g}} :\ \zeta \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \{\zeta\}\big)\big\} \\
&\subseteq \quad \big\{\zeta \in \mathfrak{T}_{\mathfrak{g}} :\ \zeta \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big)\big\} \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

Hence, for every $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ such that $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the lemma is, therefore, complete. □

Theorem 3.4. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *and* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be given pairs of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *respectively, in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

– I. $\big(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\big)\big[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big]$

– II. $\big(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\big)\big[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big]$

Proof. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ be given pairs of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and suppose $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

– I. The relation $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ holds. Consequently, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ implying, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But, $\mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ holds and consequently, $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Therefore, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and hence, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ for all $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

– II. The relation $\mathscr{S}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ holds. Therefore, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ implying, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But, $\mathscr{S}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ holds and consequently, $\mathscr{S}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Therefore, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds and thus, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ for all $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. The proof of the theorem is, therefore, complete. □

Proposition 2. If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a strong $\mathscr{T}_{\mathfrak{g}}$-space, then:

$$\big(\forall\, \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]\big)\big[\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} : (\emptyset, \Omega) \longmapsto (\emptyset, \Omega)\big] \tag{3.2}$$

Proof. Suppose $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, $(\emptyset, \Omega) \longleftrightarrow \big(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\emptyset), \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\Omega)\big) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$. Consequently,

$$\big(\emptyset \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset), \Omega \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\Omega)\big) \longleftrightarrow \big(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\emptyset), \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\Omega)\big) \longleftrightarrow (\emptyset, \Omega)$$

Hence, $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} : (\emptyset, \Omega) \longmapsto (\emptyset, \Omega)$ for any $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$. The proof of the proposition is, therefore, complete. □

Remark 3.5. Relative to the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, $\{\xi\} \in \mathscr{P}(\Omega)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*closure unit set* of $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ if and only if $\{\xi\}$ is a $\mathfrak{T}_{\mathfrak{g}}$-*unit set* or a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*derived unit set* of $\mathscr{S}_{\mathfrak{g}}$; $\{\xi\} \in \mathscr{P}(\Omega)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*interior unit set* of $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ if and only if, relative to the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, $\{\xi\}$ is a $\mathfrak{T}_{\mathfrak{g}}$-*unit set* and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*coderived unit set* of $\mathscr{S}_{\mathfrak{g}}$. Relative to the $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$, $\{\xi\} \in \mathscr{P}(\Omega)$ is a $\mathfrak{g}$-$\mathfrak{T}$-*closure unit set* of $\mathscr{S} \in \mathscr{P}(\Omega)$ if and only if $\{\xi\}$ is a $\mathfrak{T}$-*unit set* or a $\mathfrak{g}$-$\mathfrak{T}$-*derived unit set* of $\mathscr{S}$; $\{\xi\} \in \mathscr{P}(\Omega)$ is a $\mathfrak{g}$-$\mathfrak{T}$-*interior unit set* of $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ if and only if $\{\xi\}$ is a $\mathfrak{T}$-*unit set* and a $\mathfrak{g}$-$\mathfrak{T}$-*coderived unit set* of $\mathscr{S}$.

Corollary 3.6. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and,* $\{\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\xi;\mathscr{R}_{\mathfrak{g}}) : \ \xi \in \mathfrak{T}_{\mathfrak{g}}\}$ *and* $\{\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\zeta;\mathscr{R}_{\mathfrak{g}}) : \ \zeta \in \mathfrak{T}_{\mathfrak{g}}\}$, *respectively, be the corresponding collections of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*derived and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*coderived points of some* $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then:*

- I. $\big(\exists\, \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\xi;\mathscr{R}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\big[\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\xi;\mathscr{R}_{\mathfrak{g}}) \notin \mathscr{R}_{\mathfrak{g}}\big]$
- II. $\big(\forall\, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\zeta;\mathscr{R}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\big[\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\zeta;\mathscr{R}_{\mathfrak{g}}) \in \mathscr{R}_{\mathfrak{g}}\big]$

Proposition 3. If $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:

$$\big(\forall \mathscr{S} \in \mathscr{P}(\Omega)\big)\big[\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big] \tag{3.3}$$

Proof. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, for all $\mathscr{S} \in \mathscr{P}(\Omega)$, $\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But, for all $\mathscr{S} \in \mathscr{P}(\Omega)$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}}$. Hence, for all $\mathscr{S} \in \mathscr{P}(\Omega)$, $\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is, therefore, complete. □

Corollary 3.7. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *and* $\mathbf{dc}_{\mathfrak{g}} \in \text{DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be given pairs of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\text{der}_{\mathfrak{g}}$, $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, *respectively, in a* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, *then the following logical implication holds for any* $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$*:*

$$\begin{array}{c} \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ \downarrow \\ \mathscr{S}_{\mathfrak{g}} \cap \text{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \cup \text{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}). \end{array} \tag{3.4}$$

Proposition 4. If $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, respectively, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:

- I. $\big(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\big)\big[\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{V}_{\mathfrak{g}} \ \longleftarrow \ \mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{V}_{\mathfrak{g}}\big]$
- II. $\big(\forall \mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\big)\big[\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{U}_{\mathfrak{g}} \ \longleftarrow \ \mathscr{R}_{\mathfrak{g}} \supseteq \mathscr{U}_{\mathfrak{g}}\big]$

Proof. Suppose $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and

$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, respectively, and let $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})$. Suppose $(\mathscr{V}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}) \supseteq (\mathscr{S}_{\mathfrak{g}},\mathscr{U}_{\mathfrak{g}})$, then

$$\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}),\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \supseteq \left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right)$$

But $(\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\left(\mathscr{V}_{\mathfrak{g}},\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right) \supseteq \left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}),\mathscr{U}_{\mathfrak{g}}\right)$ and consequently,

$$\left(\mathscr{V}_{\mathfrak{g}},\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \supseteq \left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}),\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right) \supseteq \left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathscr{U}_{\mathfrak{g}}\right)$$

Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{V}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{U}_{\mathfrak{g}}$. The proof of the proposition is, therefore, complete. □

Theorem 3.8. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$*, respectively, in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})$*, then:*

– I. $\left(\forall\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\right)\left[\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \longleftrightarrow \mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\right]$

– II. $\left(\forall\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\right)\left[\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\right]$

Proof. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and let $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})$. Then:

– *Necessity.* Suppose $\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}),\mathscr{S}_{\mathfrak{g}}\right) \subseteq \left(\mathscr{R}_{\mathfrak{g}},\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)$ hold, then $\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \longleftrightarrow \mathscr{R}_{\mathfrak{g}}$ and $\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}}$. But, $\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$. Hence, $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$. The conditions of ITEMS I., II. are, therefore, necessary.

– *Sufficiency.* Conversely, suppose $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ holds. Then, $\mathscr{R}_{\mathfrak{g}} \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ and $\mathscr{S}_{\mathfrak{g}} \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \longleftrightarrow \mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Consequently, $\mathscr{R}_{\mathfrak{g}} \longleftrightarrow \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ and $\mathscr{S}_{\mathfrak{g}} \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Thus, $\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}),\mathscr{S}_{\mathfrak{g}}\right) \subseteq \left(\mathscr{R}_{\mathfrak{g}},\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)$. The conditions of ITEMS I., II. are, therefore, sufficient. The proof of the theorem is, therefore, complete. □

Proposition 5. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})$, and let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ holds for some $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$. Then:

– I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$

– II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$

Proof. Suppose $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega,\mathscr{T}_{\mathfrak{g}})$ and, let it be supposed that the condition $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ holds for some $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$. Then:

– I. Since $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, where $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operator in $\mathfrak{T}_{\mathfrak{g}}$. But,

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\\ &\subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\end{aligned}$$

Thus, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$.

– II. Since $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, where $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operator in $\mathfrak{T}_{\mathfrak{g}}$. But,

$$\begin{aligned} \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ &\supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \end{aligned}$$

Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is, therefore, complete. □

Theorem 3.9. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

$$\text{– I. } \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$$

$$\text{– II. } \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \bigcap_{\mathscr{V}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$$

for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$.

Proof. Suppose $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}, \mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and let $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}) \longleftrightarrow \bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} (\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))$ and $(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \cup \{\xi\} \longleftrightarrow \bigcap_{\mathscr{V}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} (\mathscr{V}_{\mathfrak{g}} \cup \{\xi\})$, it results that

$$\begin{aligned} \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \textstyle\bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} (\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})) &\longmapsto \bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})) \\ \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \textstyle\bigcap_{\mathscr{V}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} (\mathscr{V}_{\mathfrak{g}} \cup \{\xi\}) &\longmapsto \bigcap_{\mathscr{V}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}} \cup \{\xi\}) \end{aligned}$$

Consequently,

$$\begin{aligned} \mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}} &\longmapsto \left\{ \xi \in \mathfrak{T}_{\mathfrak{g}} : \ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\left(\bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathscr{U}_{\mathfrak{g}} \right) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}) \right) \right\} \\ &\longleftrightarrow \bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \{ \xi \in \mathfrak{T}_{\mathfrak{g}} : \ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})) \} \\ &\longleftrightarrow \bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \end{aligned}$$

and

$$\begin{aligned} \mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}} &\longmapsto \left\{ \xi \in \mathfrak{T}_{\mathfrak{g}} : \ \xi \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\left(\bigcap_{\mathscr{V}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathscr{U}_{\mathfrak{g}} \right) \cup \{\xi\} \right) \right\} \\ &\longleftrightarrow \bigcap_{\mathscr{V}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \{ \xi \in \mathfrak{T}_{\mathfrak{g}} : \ \xi \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}} \cup \{\xi\}) \} \\ &\longleftrightarrow \bigcap_{\mathscr{V}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}) \end{aligned}$$

Hence, it follows that $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})$ are equivalent to $\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ and $\bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$, respectively. The proof of the theorem is, therefore, complete. □

Corollary 3.10. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\in\mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$*, then:*

– I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})\subseteq\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$

– II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})\supseteq\bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$

for any $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$.

Theorem 3.11. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\in\mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$*, then:*

$$\begin{gathered}\big(\{\xi\},\{\zeta\}\big)\subset\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\\ \updownarrow\\ \big(\{\xi\},\{\zeta\}\big)\subset\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}),\mathscr{S}_{\mathfrak{g}}\cup\{\zeta\}\big)\end{gathered}\tag{3.5}$$

for any $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$.

Proof. Suppose $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\in\mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, and let $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then, since $\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\longleftrightarrow\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})$ and $\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\longleftrightarrow\big(\mathscr{S}_{\mathfrak{g}}\cup\{\xi\}\big)\cup\{\xi\}$, it results that

$$\begin{gathered}\big(\{\xi\},\{\zeta\}\big)\subset\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\\ \updownarrow\\ \big(\{\xi\},\{\zeta\}\big)\subset\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\times\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cup\{\zeta\}\big)\\ \updownarrow\\ \big(\{\xi\},\{\zeta\}\big)\subset\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}),\mathscr{S}_{\mathfrak{g}}\cup\{\zeta\}\big)\end{gathered}$$

holds for any $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$. Hence, $\big(\{\xi\},\{\zeta\}\big)\subset\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\longleftrightarrow\big(\{\xi\},\{\zeta\}\big)\subset\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}),\mathscr{S}_{\mathfrak{g}}\cup\{\zeta\}\big)$. The proof of the theorem is, therefore, complete. □

Corollary 3.12. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\in\mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$*, then:*

$$\begin{gathered}\big(\{\xi\},\{\zeta\}\big)\not\subset\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\\ \updownarrow\\ \big(\{\xi\},\{\zeta\}\big)\subset\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}),\mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\end{gathered}\tag{3.6}$$

for any $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$.

Proposition 6. If $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:

$$\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \{\xi\}, \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\zeta\})\big) \tag{3.7}$$

for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$.

Proof. Suppose $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and let $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since

$$\begin{aligned}\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) &\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big((\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})) \cup (\{\xi\} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\}))\big)\\ &\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big((\mathscr{R}_{\mathfrak{g}} \cup \{\xi\}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\end{aligned}$$

it results that

$$\begin{aligned}\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) &\longleftrightarrow \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\}\\ &\longleftrightarrow \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big((\mathscr{R}_{\mathfrak{g}} \cup \{\xi\}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\}\\ &\longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \{\xi\}\big)\end{aligned}$$

Therefore, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \{\xi\}\big)$. Since

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big) &\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big((\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}) \cap (\{\zeta\} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\zeta\}))\big)\\ &\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big((\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\zeta\})) \cup \{\zeta\}\big)\end{aligned}$$

it follows that

$$\begin{aligned}\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \big\{\zeta \in \mathfrak{T}_{\mathfrak{g}} :\ \zeta \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big)\big\}\\ &\longleftrightarrow \big\{\zeta \in \mathfrak{T}_{\mathfrak{g}} :\ \zeta \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big((\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\zeta\})) \cup \{\zeta\}\big)\big\}\\ &\longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\zeta\})\big)\end{aligned}$$

Therefore, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\zeta\})\big)$. Hence, it follows that $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \{\xi\}, \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\zeta\})\big)$. The proof of the proposition is, therefore, complete. ☐

Proposition 7. If $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and $\big(\{\xi\}, \{\zeta\}\big) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ be a pair of $\mathfrak{T}_{\mathfrak{g}}$-unit sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:

$$\big(\{\xi\}, \{\zeta\}\big) \not\subset \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \ \longleftrightarrow\ \big(\{\xi\}, \{\zeta\}\big) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \tag{3.8}$$

for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$.

Proof. Suppose $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and let $\big(\{\xi\}, \{\zeta\}\big) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ be a pair of $\mathfrak{T}_{\mathfrak{g}}$-unit sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Suppose $\big(\{\xi\}, \{\zeta\}\big) \not\subset \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}})$ for an arbitrary $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$. Then, $\big(\{\xi\}, \{\zeta\}\big) \not\subset \mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}})$ implies $\big(\{\xi\}, \{\zeta\}\big) \not\subset \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \times \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big)$. Consequently, it follows that the relation $\big(\{\xi\}, \{\zeta\}\big) \subset \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \times \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big)$ holds. But, $\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \{\xi\}\big)$ and $\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)$ and on the other hand, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \{\xi\}\big) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. Therefore, $\big(\{\xi\}, \{\zeta\}\big) \subset (\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}})$ holds for some $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times$

$\mathfrak{g}$-K$[\mathfrak{T}_{\mathfrak{g}}]$ and consequently, $\big(\{\xi\},\{\zeta\}\big) \in \mathfrak{g}$-O$[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}$-K$[\mathfrak{T}_{\mathfrak{g}}]$. Hence, $\big(\{\xi\},\{\zeta\}\big) \not\subset \mathfrak{g}$-$\mathbf{Dc}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \big(\{\xi\},\{\zeta\}\big) \in \mathfrak{g}$-O$[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}$-K$[\mathfrak{T}_{\mathfrak{g}}]$. The proof of the proposition is, therefore, complete. ☐

Theorem 3.13. *If* $\mathfrak{g}$-$\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}$-DC$[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}$-Der$_{\mathfrak{g}}$, $\mathfrak{g}$-Cod$_{\mathfrak{g}}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

– I. $\big(\forall \mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\big)\big[\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big]$

– II. $\big(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\big)\big[\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big]$

Proof. Suppose $\mathfrak{g}$-$\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}$-DC$[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}$-Der$_{\mathfrak{g}}$, $\mathfrak{g}$-Cod$_{\mathfrak{g}}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and let $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

– I. Set $\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. Then,

$$\begin{aligned}\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \longleftrightarrow \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\end{aligned}$$

Hence, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$.

– II. Set $\mathscr{V}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Then,

$$\begin{aligned}\mathscr{V}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\end{aligned}$$

Thus, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the theorem is, therefore, complete. ☐

Proposition 8. If $\mathfrak{g}$-$\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}$-DC$[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}$-Der$_{\mathfrak{g}}$, $\mathfrak{g}$-Cod$_{\mathfrak{g}}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:

– I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$

– II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$

for any $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$.

Proof. Suppose $\mathfrak{g}$-$\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}$-DC$[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}$-Der$_{\mathfrak{g}}$, $\mathfrak{g}$-Cod$_{\mathfrak{g}}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, and let $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

– I. Since $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \longleftrightarrow \mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ holds for any $\mathscr{U}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, setting $\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ yields

$$\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \longleftrightarrow \big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)$$

But,

$$\begin{aligned}\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) &\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\\ &\longleftrightarrow \big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\\ &\quad\cup \big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\\ &\longleftrightarrow \big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\end{aligned}$$

Thus, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\longleftrightarrow\mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$.

– II. Since $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})\longleftrightarrow\mathscr{V}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$ holds for any $\mathscr{V}_{\mathfrak{g}}\in\mathscr{P}(\Omega)$, setting $\mathscr{V}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ yields

$$\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\longleftrightarrow\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$$

But,

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) &\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\ &\longleftrightarrow \big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\\ &\cap \big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\\ &\longleftrightarrow \big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\end{aligned}$$

Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\longleftrightarrow\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is, therefore, complete. □

Corollary 3.14. *If* $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\in\mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Der}_{\mathfrak{g}},\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$*, then:*

– I. $\big(\forall\mathscr{R}_{\mathfrak{g}}\in\mathscr{P}(\Omega)\big)\big[\mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\subseteq\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big]$

– II. $\big(\forall\mathscr{S}_{\mathfrak{g}}\in\mathscr{P}(\Omega)\big)\big[\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big]$

Proposition 9. If $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\in\mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}},\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, then the following logical implications holds:

– I. For any $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$,

$$\begin{gathered}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\subseteq\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\\ \wedge\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\\ \big\updownarrow\\ \mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\subseteq\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\end{gathered}\tag{3.9}$$

– II. For any $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$,

$$\begin{gathered}\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\supseteq\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\\ \wedge\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\\ \big\updownarrow\\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\end{gathered}\tag{3.10}$$

Proof. Suppose $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}}\in\mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}},\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, and let $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then:

– I. Substitute $\mathscr{S}_{\mathfrak{g}} = \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ in $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}\big) = \bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ and then take $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ into account. Consequently,

$$\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) = \bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}).$$

Thus, $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$.

– II. Substitute $\mathscr{R}_{\mathfrak{g}} = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ in $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\big) = \bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$ and then take $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ into account. Consequently,

$$\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) = \bigcap_{\mathscr{V}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$$

Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is, therefore, complete. □

Proposition 10. If $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathbf{dc}_{\mathfrak{g}} \in \text{DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given pair of $\mathfrak{T}_{\mathfrak{g}}$-operators $\text{der}_{\mathfrak{g}}$, $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:

– I. For any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$,

$$\begin{array}{ccc} \mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ & \searrow & \downarrow \\ & & \text{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \end{array} \tag{3.11}$$

– II. For any $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$,

$$\begin{array}{ccc} \mathscr{U}_{\mathfrak{g}} \supseteq \mathscr{V}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}) \\ & \searrow & \uparrow \\ & & \text{cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \text{cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}) \end{array} \tag{3.12}$$

Proof. Let $\mathfrak{g}\text{-}\mathbf{Dc}_{\mathfrak{g}} \in \mathfrak{g}\text{-DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathbf{dc}_{\mathfrak{g}} \in \text{DC}[\mathfrak{T}_{\mathfrak{g}}]$ be a given pair of $\mathfrak{T}_{\mathfrak{g}}$-operators $\text{der}_{\mathfrak{g}}$, $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then:

– I. Since $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$ implies $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\text{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, to prove the diagram it suffices to prove that, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ implies $\text{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Suppose $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, then

$$\begin{array}{rcl} \text{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) & \longleftrightarrow & \text{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \\ & \subseteq & \text{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ & \subseteq & \text{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \text{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \text{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \end{array}$$

Thus, $\mathrm{der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$.

– II. Since $\mathscr{U}_{\mathfrak{g}} \supseteq \mathscr{V}_{\mathfrak{g}}$ implies $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$ and $\mathrm{cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$, to prove the diagram it suffices to prove that, for any pair $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$, $\mathrm{cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$ implies $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$. Suppose $\mathrm{cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$, then

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \quad &\longleftrightarrow \quad \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}) \\
&\supseteq \quad \mathrm{cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}) \\
&\supseteq \quad \mathrm{cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})
\end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{V}_{\mathfrak{g}})$ for any $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$. The proof of the proposition is, therefore, complete. ☐

We conclude the present section with two corollaries and two axiomatic definitions derived from these two corollaries.

Corollary 3.15. *A necessary and sufficient condition for the set-valued map* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *to be a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-derived operator in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *is that, for every* $\left(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\right) \in \times_{\alpha\in I_3^*} \mathscr{P}(\Omega)$ *such that* $\{\xi\} \subset \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$*, it satisfies:*

– I. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\emptyset) = \emptyset$

– II. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})\big)$

– III. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$

– IV. $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$

Corollary 3.16. *A necessary and sufficient condition for the set-valued map* $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *to be a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-coderived operator in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *is that, for each* $\left(\{\zeta\}, \mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\right) \in \times_{\alpha\in I_3^*} \mathscr{P}(\Omega)$ *such that* $\{\zeta\} \subset \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$*, it satisfies:*

– I. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) = \Omega$

– II. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}\big)$

– III. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$

– IV. $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})$

Hence, in a strong $\mathscr{T}_{\mathfrak{g}}$-space, for a set-valued map $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ to be characterized as a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived operator it must necessarily and sufficiently satisfy a list of *derived set* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-derived operator conditions* (ITEMS I.–IV. of COR. 3.15), and similarly, for a set-valued map $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ to be characterized as a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operator it must necessarily and sufficiently satisfy a list of *derived set* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-coderived operator conditions* (ITEMS V.–VIII. of COR. 3.16).

Some nice Mathematical vocabulary follow. In COR. 3.15, ITEMS I., II., III. and IV. may well be taken as stating that the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-derived operator $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$:

$\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is $\emptyset$-*grounded* (alternatively, $\emptyset$-*preserving*), ξ-*invariant* (alternatively, ξ-*unaffected*), $\mathfrak{g}$-$\mathrm{Cl}_{\mathfrak{g}}$-*intensive* and $\cup$-*additive* (alternatively, $\cup$-*distributive*), respectively. On the other hand, Items i., ii., iii. and iv. of Cor. 3.16, may well be taken as stating that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator $\mathfrak{g}$-$\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is Ω-*grounded* (alternatively, Ω-*preserving*), ζ-*invariant* (alternatively, ζ-*unaffected*), $\mathfrak{g}$-$\mathrm{Int}_{\mathfrak{g}}$-*extensive* and $\cap$-*additive* (alternatively, $\cap$-*distributive*), respectively.

Viewing the derived set $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator conditions (Items i.–iv. of Cor. 3.15 above) as $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*derived operator axioms*, the axiomatic definition of the concept of a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator, then, can be defined as a set-valued map $\mathfrak{g}$-$\mathrm{Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ satisfying a list of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*derived operator axioms*. The axiomatic definition of the concept of a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator in $\mathscr{T}_{\mathfrak{g}}$-spaces follows.

Definition 3.17 (*Axiomatic Definition: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Derived Operator*)**.** A set-valued map of the type $\mathfrak{g}$-$\mathrm{Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ if and only if, for any $\left(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\right) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$ such that $\{\xi\} \subset \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, it satisfies each $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator axiom in $\mathrm{AX}\left[\mathfrak{g}\text{-}\mathrm{DE}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\right] \stackrel{\mathrm{def}}{=} \left\{\mathrm{Ax}_{\mathrm{DE},\nu}\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\right) : \nu \in I_4^*\right\}$, where $\mathrm{Ax}_{\mathrm{DE},\nu} : \mathfrak{g}\text{-}\mathrm{DE}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B} \stackrel{\mathrm{def}}{=} \{0,1\}$, $\nu \in I_4^*$, is defined as thus:

- $\mathrm{Ax}_{\mathrm{DE},1}\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\right) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\emptyset) = \emptyset$
- $\mathrm{Ax}_{\mathrm{DE},2}\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\right) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})\right)$
- $\mathrm{Ax}_{\mathrm{DE},3}\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\right) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$
- $\mathrm{Ax}_{\mathrm{DE},4}\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\right) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$

Thus, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator $\mathfrak{g}$-$\mathrm{Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a $\emptyset$-*grounded* ($\mathrm{Ax}_{\mathrm{DE},1}$), ξ-*invariant* ($\mathrm{Ax}_{\mathrm{DE},2}$), $\mathfrak{g}$-$\mathrm{Cl}_{\mathfrak{g}}$-*intensive* ($\mathrm{Ax}_{\mathrm{DE},3}$) and $\cup$-*additive* ($\mathrm{Ax}_{\mathrm{DE},4}$) $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set-valued set map forming a generalization of the $\mathfrak{T}_{\mathfrak{g}}$-set-valued set map $\mathrm{der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in the strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, provided

$$\begin{gathered}
\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\right) \\
\wedge\left(\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right) \\
\Updownarrow \\
\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})
\end{gathered}$$

holds for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$.

Theorem 3.18. *Let* $\mathrm{AX}\left[\mathfrak{g}\text{-}\mathrm{DE}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\right] \stackrel{\mathrm{def}}{=} \left\{\mathrm{Ax}_{\mathrm{DE},\nu} : \nu \in I_4^*\right\}$ *be the class of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*derived operator axioms in a strong* $\mathscr{T}_{\mathfrak{g}}$-*space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *and, let* $\mathrm{Ax}_{\mathrm{DE,I}}$:

$\mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B}$ *such that, for any* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$,

$$\begin{aligned} \mathrm{Ax}_{\mathrm{DE,I}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) \quad \stackrel{\text{def}}{\longleftrightarrow} \quad & \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \\ & \cup \Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \left(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right) \Bigg) \\ & = \big(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}})\big) \setminus \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset) \end{aligned} \tag{3.13}$$

Then, $\mathrm{Ax}_{\mathrm{DE,I}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) = 1 \longrightarrow \bigwedge_{\nu \in I_4^*} \mathrm{Ax}_{\mathrm{DE},\nu}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) = 1.$

Proof. Let $\mathrm{AX}\big[\mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big] = \big\{\mathrm{Ax}_{\mathrm{DE},\nu} : \ \nu \in I_4^*\big\}$ be the class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator axioms in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, let $\mathrm{Ax}_{\mathrm{DE,I}} : \mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B}$ such that, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$,

$$\begin{aligned} \mathrm{Ax}_{\mathrm{DE,I}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) \quad \stackrel{\text{def}}{\longleftrightarrow} \quad & \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \\ & \cup \Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \left(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right) \Bigg) \\ & = \big(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}})\big) \setminus \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset) \end{aligned}$$

Suppose $\mathrm{Ax}_{\mathrm{DE,I}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) = 1$ holds. Then:

– CASE I. If $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) = (\emptyset, \emptyset)$, then

$$\begin{aligned} \mathrm{Ax}_{\mathrm{DE,I}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) \quad \stackrel{\text{def}}{\longleftrightarrow} \quad & \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\emptyset \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset)\big) \\ & \cup \Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}} = \emptyset, \emptyset} \left(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right) \Bigg) \\ & = \big(\emptyset \cup \emptyset \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset \cup \emptyset)\big) \setminus \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset) \end{aligned}$$

Consequently, $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset)$, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset) = \emptyset$. Therefore, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset) = \emptyset \stackrel{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{DE,1}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right)$ and thus, $\mathrm{Ax}_{\mathrm{DE,I}} \longrightarrow \mathrm{Ax}_{\mathrm{DE,1}}$.

– CASE II. If $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ be arbitrary such that $\mathscr{S}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})$ and $\{\xi\} \subset \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, then

$$\begin{aligned} \mathrm{Ax}_{\mathrm{DE,I}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) \quad \stackrel{\text{def}}{\longleftrightarrow} \quad & \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big((\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big) \\ & \cup \Bigg(\bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})} \left(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\right) \Bigg) \\ & = \big(\mathscr{R}_{\mathfrak{g}} \cup \big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \\ & \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big)\big) \setminus \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset) \end{aligned}$$

Since the relation $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \subseteq \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds, implying $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \cup \big(\mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) = \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, and $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset) = \emptyset$ by virtue of $\mathrm{Ax}_{\mathrm{DE,1}}$, the above expression reduces to

$$\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big)$$

Clearly, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cup\big(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big)\longleftrightarrow\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. Consequently, it results that $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\subseteq\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. But,

$$\begin{aligned}
\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \quad &\subseteq \quad \big\{\xi\in\mathfrak{T}_{\mathfrak{g}}:\ \xi\in\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\}\\
&\longleftrightarrow \quad \big\{\xi\in\mathfrak{T}_{\mathfrak{g}}:\ \xi\in\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\big(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\}\\
&\longleftrightarrow \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)
\end{aligned}$$

Consequently, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\supseteq\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. Therefore, it results that $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})=\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\stackrel{\text{def}}{\longleftrightarrow}\text{Ax}_{\text{DE},2}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big)$ holds and hence, $\text{Ax}_{\text{DE,I}}\longrightarrow\text{Ax}_{\text{DE},2}$.

– Case iii. If $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ such that $\mathscr{S}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}}$ be arbitrary, then

$$\begin{aligned}
\text{Ax}_{\text{DE,I}}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big) \quad &\stackrel{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\\
&\qquad \cup\Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}}\big(\mathscr{U}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)\Big)\\
&\qquad =\big(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{R}_{\mathfrak{g}})\big)\setminus\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset)
\end{aligned}$$

Consequently, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\subseteq\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, since $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset)=\emptyset$ by virtue of $\text{Ax}_{\text{DE},1}$. But, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\supseteq\mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. Therefore, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\subseteq\mathscr{R}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\stackrel{\text{def}}{\longleftrightarrow}\text{Ax}_{\text{DE},3}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big)$ and thus, $\text{Ax}_{\text{DE,I}}\longrightarrow\text{Ax}_{\text{DE},3}$.

– Case iv. If $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\times_{\alpha\in I_2^*}\mathscr{P}(\Omega)$ be arbitrary, then

$$\begin{aligned}
\text{Ax}_{\text{DE,I}}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big) \quad &\stackrel{\text{def}}{\longleftrightarrow} \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\\
&\qquad \cup\Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\big(\mathscr{U}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)\Big)\\
&\qquad =\big(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})\big)\setminus\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset)
\end{aligned}$$

By virtue of the relation $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\subseteq\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ or equivalently, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\cup\big(\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)=\mathscr{S}_{\mathfrak{g}}\cup\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, together with $\text{Ax}_{\text{DE},1}$, $\text{Ax}_{\text{DE,I}}$ reduces to $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})=\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}}\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\stackrel{\text{def}}{\longleftrightarrow}\text{Ax}_{\text{DE},4}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big)$ and hence, $\text{Ax}_{\text{DE,I}}\longrightarrow\text{Ax}_{\text{DE},4}$.

Hence, $\text{Ax}_{\text{DE,I}}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big)=1\ \longrightarrow\ \bigwedge_{\nu\in I_4^*}\text{Ax}_{\text{DE},\nu}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big)=1$ and the proof of the theorem is, therefore, complete. □

The proposition given below contains further properties.

Proposition 11. Let $\text{AX}\big[\mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big]\stackrel{\text{def}}{=}\big\{\text{Ax}_{\text{DE},\nu}:\ \nu\in I_4^*\big\}$ be the class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator axioms in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ and, let $\text{Ax}_{\text{DE,II}}$:

$\mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B}$ such that, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$,

$$\begin{aligned} \mathrm{Ax}_{\mathrm{DE,II}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) \quad \overset{\text{def}}{\longleftrightarrow} \quad & \Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)\Big) \\ & \cup\Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)\Big) \\ & = (\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \end{aligned} \tag{3.14}$$

Then, $\mathrm{Ax}_{\mathrm{DE,II}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) = 1 \longrightarrow \bigwedge_{\nu \in I_4^*} \mathrm{Ax}_{\mathrm{DE},\nu}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) = 1$.

Proof. Let $\mathrm{AX}\big[\mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big] \overset{\text{def}}{=} \{\mathrm{Ax}_{\mathrm{DE},\nu} : \nu \in I_4^*\}$ be the class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator axioms in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, let $\mathrm{Ax}_{\mathrm{DE,II}} : \mathfrak{g}\text{-DE}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B}$ such that, for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$,

$$\begin{aligned} \mathrm{Ax}_{\mathrm{DE,II}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) \quad \overset{\text{def}}{\longleftrightarrow} \quad & \Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)\Big) \\ & \cup\Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)\Big) \\ & = (\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \end{aligned}$$

Suppose $\mathrm{Ax}_{\mathrm{DE,II}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) = 1$ holds. Then:

– Case i. If $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) = (\emptyset, \emptyset)$, then

$$\begin{aligned} \mathrm{Ax}_{\mathrm{DE,II}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) \quad \overset{\text{def}}{\longleftrightarrow} \quad & \Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\emptyset,\emptyset} \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)\Big) \\ & \cup\Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\emptyset,\emptyset} \big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)\Big) \\ & = (\emptyset \cup \emptyset) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset \cup \emptyset) \end{aligned}$$

Consequently, $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset)$. But, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset) \longleftrightarrow \emptyset \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\emptyset) = \emptyset$. Therefore, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset) = \emptyset \overset{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{DE},1}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right)$ and thus, $\mathrm{Ax}_{\mathrm{DE,II}} \longrightarrow \mathrm{Ax}_{\mathrm{DE},1}$.

– Case ii. If $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ be arbitrary such that $\mathscr{S}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})$ and $\{\xi\} \subset \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, then

$$\begin{aligned} \mathrm{Ax}_{\mathrm{DE,II}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) \quad \overset{\text{def}}{\longleftrightarrow} \quad & \Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})} \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)\Big) \\ & \cup\Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})} \big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)\Big) \\ & = \big(\mathscr{R}_{\mathfrak{g}} \cup \big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big) \\ & \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big) \end{aligned}$$

Since the relation $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) \subseteq \mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ holds, implying $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) \cup \big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) = \mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ for any $\mathscr{U}_{\mathfrak{g}} \in \big\{\mathscr{R}_{\mathfrak{g}},\ \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big\}$, $\mathrm{Ax}_{\mathrm{DE,II}}$ reduces to

$$\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big)$$

Because $\mathscr{R}_{\mathfrak{g}} \supseteq \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})$ holds, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big) \longleftrightarrow \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. Consequently, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. But,

$$\begin{aligned} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \quad &\subseteq \quad \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\} \\ &\longleftrightarrow \quad \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} :\ \xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)\big\} \\ &\longleftrightarrow \quad \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \end{aligned}$$

implying $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. Therefore, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \stackrel{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{DE},2}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big)$ and hence, $\mathrm{Ax}_{\mathrm{DE,I}} \longrightarrow \mathrm{Ax}_{\mathrm{DE},2}$.

– Case iii. If $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ such that $\mathscr{S}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}$ be arbitrary, then

$$\begin{aligned} \mathrm{Ax}_{\mathrm{DE,II}}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big) \quad \stackrel{\text{def}}{\longleftrightarrow} \quad & \Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)\Big) \\ & \cup\Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}} \mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\Big) \\ & = (\mathscr{R}_{\mathfrak{g}} \cup \mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{R}_{\mathfrak{g}}) \end{aligned}$$

Consequently, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \cup \big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) = \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, implying $\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. But, because the relation $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)$ holds, it results, therefore, that $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{DE},3}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big)$ and thus, $\mathrm{Ax}_{\mathrm{DE,II}} \longrightarrow \mathrm{Ax}_{\mathrm{DE},3}$.

– Case iv. If $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ be arbitrary, then

$$\begin{aligned} \mathrm{Ax}_{\mathrm{DE,II}}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big) \quad \stackrel{\text{def}}{\longleftrightarrow} \quad & \Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)\Big) \\ & \cup\Big(\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\Big) \\ & = (\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) \end{aligned}$$

Since $\bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) \subseteq \bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \big(\mathscr{U}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)$ holds, $\mathrm{Ax}_{\mathrm{DE,II}}$, evidently, reduces to $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{U}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \stackrel{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{DE},4}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big)$ and hence, $\mathrm{Ax}_{\mathrm{DE,II}} \longrightarrow \mathrm{Ax}_{\mathrm{DE},4}$.

Thus, $\mathrm{Ax}_{\mathrm{DE,II}}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big) = 1 \longrightarrow \bigwedge_{\nu\in I_4^*} \mathrm{Ax}_{\mathrm{DE},\nu}\big(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\big) = 1$ and the proof of the proposition is, therefore, complete. □

The corollary stated below is an immediate consequence of the foregoing theorem and proposition.

Corollary 3.19. *If* $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-derived operator on* $\mathscr{P}(\Omega)$ *ranging in* $\mathscr{P}(\Omega)$ *in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then it satisfies the following* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-derived operator axiomatic diagram:*

$$\begin{array}{ccc} \mathrm{Ax}_{\mathrm{DE,I}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) = 1 & \longrightarrow & \bigwedge_{\nu \in I_4^*} \mathrm{Ax}_{\mathrm{DE},\nu}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) = 1 \\ & \nwarrow\!\!\searrow & \uparrow \\ & & \mathrm{Ax}_{\mathrm{DE,II}}\left(\mathfrak{g}\text{-Der}_{\mathfrak{g}}\right) = 1 \end{array} \tag{3.15}$$

Likewise, viewing the derived set $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operator conditions (Items i.–iv. of Cor. 3.16 above) as $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-coderived operator axioms*, the axiomatic definition of the concept of a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operator, then, can be defined as a set-valued map $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ satisfying a list of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-coderived operator axioms*. The axiomatic definition of the concept of a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operator in $\mathscr{T}_{\mathfrak{g}}$-spaces follows.

Definition 3.20 (*Axiomatic Definition:* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-Coderived Operator*)**.** A one-valued map of the type $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is called a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operator on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ if and only if, for any $\left(\{\zeta\}, \mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\right) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$, it satisfies each $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operator axiom in $\mathrm{AX}\big[\mathfrak{g}\text{-CD}[\mathfrak{T}_{\mathfrak{g}}];\mathbb{B}\big] \stackrel{\mathrm{def}}{=} \left\{\mathrm{Ax}_{\mathrm{CD},\nu}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right) :\ \nu \in I_4^*\right\}$, where $\mathrm{Ax}_{\mathrm{CD},\nu} : \mathfrak{g}\text{-CD}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathbb{B} \stackrel{\mathrm{def}}{=} \{0,1\}$, $\nu \in I_4^*$, is defined as thus:

- $\mathrm{Ax}_{\mathrm{CD},1}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\Omega) = \Omega$
- $\mathrm{Ax}_{\mathrm{CD},2}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}\right)$
- $\mathrm{Ax}_{\mathrm{CD},3}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$
- $\mathrm{Ax}_{\mathrm{CD},4}\left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right) \stackrel{\mathrm{def}}{\longleftrightarrow} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})$

Hence, a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operator $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a Ω-*grounded* ($\mathrm{Ax}_{\mathrm{CD},1}$), ζ-*invariant* ($\mathrm{Ax}_{\mathrm{CD},2}$), $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$-*extensive* ($\mathrm{Ax}_{\mathrm{CD},3}$) and $\cap$-*additive* ($\mathrm{Ax}_{\mathrm{CD},4}$) $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-set-valued set map forming a generalization of the $\mathfrak{T}_{\mathfrak{g}}$-set-valued set map $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, provided

$$\begin{array}{c} \left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right) \\ \wedge\left(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\left(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right) \\ \Updownarrow \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \end{array}$$

holds for any $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$.

As above, having introduced an alternative definition defining the notion of a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-coderived operator in a $\mathscr{T}_{\mathfrak{g}}$-space axiomatically, it may not be without interest to prove some further propositions based on such axiomatic definition. The theorem follows.

Theorem 3.21. *Let* $\mathrm{AX}\big[\mathfrak{g}\text{-}\mathrm{CD}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\big] \stackrel{\text{def}}{=} \big\{\mathrm{Ax}_{\mathrm{CD},\nu}:\ \nu \in I_4^*\big\}$ *be the class of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operator axioms in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *and, let* $\mathrm{Ax}_{\mathrm{CD,I}}: \mathfrak{g}\text{-}\mathrm{CD}\left[\mathfrak{T}_{\mathfrak{g}}\right] \longrightarrow \mathbb{B}$ *such that, for any* $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$,

$$\begin{aligned}\mathrm{Ax}_{\mathrm{CD,I}}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) \quad \stackrel{\text{def}}{\longleftrightarrow} \quad & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) \\ & \cap\Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} \left(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\right)\Bigg) \\ & = \big(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}})\big) \setminus \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega)\end{aligned} \tag{3.16}$$

Then, $\mathrm{Ax}_{\mathrm{CD,I}}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) = 1 \ \longrightarrow\ \bigwedge_{\nu\in I_4^*} \mathrm{Ax}_{\mathrm{CD},\nu}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) = 1.$

Proof. Let $\mathrm{AX}\big[\mathfrak{g}\text{-}\mathrm{CD}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\big] = \big\{\mathrm{Ax}_{\mathrm{CD},\nu}:\ \nu \in I_4^*\big\}$ be the class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator axioms in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, let $\mathrm{Ax}_{\mathrm{CD,I}}: \mathfrak{g}\text{-}\mathrm{CD}\left[\mathfrak{T}_{\mathfrak{g}}\right] \longrightarrow \mathbb{B}$ such that, for any $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$,

$$\begin{aligned}\mathrm{Ax}_{\mathrm{CD,I}}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) \quad \stackrel{\text{def}}{\longleftrightarrow} \quad & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) \\ & \cap\Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} \left(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\right)\Bigg) \\ & = \big(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}})\big) \setminus \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega)\end{aligned}$$

Suppose $\mathrm{Ax}_{\mathrm{CD,I}}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) = 1$ holds. Then:

– CASE I. If $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) = (\Omega, \Omega)$, then

$$\begin{aligned}\mathrm{Ax}_{\mathrm{CD,I}}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) \quad \stackrel{\text{def}}{\longleftrightarrow} \quad & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\Omega \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega)\big) \\ & \cap\Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\Omega,\Omega} \left(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\right)\Bigg) \\ & = \big(\Omega \cap \Omega \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega \cap \Omega)\big) \setminus \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega)\end{aligned}$$

Consequently, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega)$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega)$. Thus, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) = \Omega \stackrel{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{CD},1}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right)$ and hence, $\mathrm{Ax}_{\mathrm{CD,I}} \longrightarrow \mathrm{Ax}_{\mathrm{CD},1}$.

– CASE II. If $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ be arbitrary such that $\mathscr{V}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}$ and $\{\zeta\} \subset \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$, then

$$\begin{aligned}\mathrm{Ax}_{\mathrm{CD,I}}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) \quad \stackrel{\text{def}}{\longleftrightarrow} \quad & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) \\ & \cap\Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}} \left(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\right)\Bigg) \\ & = \big(\mathscr{U}_{\mathfrak{g}} \cap (\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}) \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap (\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\})\big)\big) \\ & \setminus \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega)\end{aligned}$$

Since the relation $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) \supseteq \mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ holds, implying $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) \cap \big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) = \mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$, and $\mathrm{Ax}_{\mathrm{CD},1}$ implies $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) = \emptyset$, $\mathrm{Ax}_{\mathrm{CD,I}}$ reduces to

$$\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}\big) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap (\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\})\big)$$

Clearly, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap (\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\})\big) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. Consequently, it results that $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}\big) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. But,

$$\begin{aligned}
\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \quad &\supseteq \quad \big\{\zeta \in \mathfrak{T}_{\mathfrak{g}} :\ \zeta \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}\big)\big\} \\
&\longleftrightarrow \quad \big\{\zeta \in \mathfrak{T}_{\mathfrak{g}} :\ \zeta \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big((\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}) \cup \{\zeta\}\big)\big\} \\
&\longleftrightarrow \quad \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\})
\end{aligned}$$

Consequently, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}\big) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. Therefore, it follows that the relation $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}\big) \overset{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{CD},2}\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big)$ holds and thus, $\mathrm{Ax}_{\mathrm{CD,I}} \longrightarrow \mathrm{Ax}_{\mathrm{CD},2}$.

– Case iii. If $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ such that $\mathscr{V}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}}$ be arbitrary, then

$$\begin{aligned}
\mathrm{Ax}_{\mathrm{CD,I}}\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big) \quad \overset{\text{def}}{\longleftrightarrow} \quad & \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) \\
& \cap \Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}}, \mathscr{U}_{\mathfrak{g}}} \big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\big) \Bigg) \\
& = \big(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g}})\big) \setminus \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\Omega)
\end{aligned}$$

Consequently, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) \supseteq \mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$, since $\mathrm{Ax}_{\mathrm{CD},1}$ implies $\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\Omega) = \emptyset$. But, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. Therefore, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{CD},3}\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big)$ and thus, $\mathrm{Ax}_{\mathrm{CD,I}} \longrightarrow \mathrm{Ax}_{\mathrm{CD},3}$.

– Case iv. If $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ be arbitrary, then

$$\begin{aligned}
\mathrm{Ax}_{\mathrm{CD,I}}\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big) \quad \overset{\text{def}}{\longleftrightarrow} \quad & \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) \\
& \cap \Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}} \big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\big) \Bigg) \\
& = \big(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}})\big) \setminus \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\Omega)
\end{aligned}$$

By virtue of the relation $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) \supseteq \mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$ or equivalently, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) \cap \big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) = \mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$, together with $\mathrm{Ax}_{\mathrm{CD},1}$, $\mathrm{Ax}_{\mathrm{CD,I}}$ reduces to $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) \overset{\text{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{CD},4}\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big)$ and hence, $\mathrm{Ax}_{\mathrm{CD,I}} \longrightarrow \mathrm{Ax}_{\mathrm{CD},4}$.

Thus, $\mathrm{Ax}_{\mathrm{CD,I}}\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big) = 1 \ \longrightarrow\ \bigwedge_{\nu \in I_4^*} \mathrm{Ax}_{\mathrm{CD},\nu}\big(\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big) = 1$ and the proof of the theorem is, therefore, complete. □

The proposition given below contains further properties.

Proposition 12. Let $\mathrm{AX}\big[\mathfrak{g}\text{-CD}[\mathfrak{T}_{\mathfrak{g}}]; \mathbb{B}\big] \overset{\text{def}}{=} \big\{\mathrm{Ax}_{\mathrm{CD},\nu} :\ \nu \in I_4^*\big\}$ be the class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator axioms in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and, let $\mathrm{Ax}_{\mathrm{CD,II}}$:

$\mathfrak{g}\text{-CD}\left[\mathfrak{T}_{\mathfrak{g}}\right] \longrightarrow \mathbb{B}$ such that, for any $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$,

$$\begin{aligned}
\mathrm{Ax}_{\mathrm{CD,II}}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) \quad &\stackrel{\mathrm{def}}{\longleftrightarrow} \quad \Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} \left(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\right)\right)\Bigg) \\
&\qquad \cap \Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\right)\big)\Bigg) \\
&\qquad = \left(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}\right) \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}\right)
\end{aligned} \tag{3.17}$$

Then, $\mathrm{Ax}_{\mathrm{CD,II}}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) = 1 \;\longrightarrow\; \bigwedge_{\nu\in I_4^*} \mathrm{Ax}_{\mathrm{CD},\nu}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) = 1$.

Proof. Let $\mathrm{AX}\big[\mathfrak{g}\text{-CD}\left[\mathfrak{T}_{\mathfrak{g}}\right];\mathbb{B}\big] \stackrel{\mathrm{def}}{=} \left\{\mathrm{Ax}_{\mathrm{CD},\nu} : \ \nu \in I_4^*\right\}$ be the class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator axioms in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and further, let $\mathrm{Ax}_{\mathrm{CD,II}} : \mathfrak{g}\text{-CD}\left[\mathfrak{T}_{\mathfrak{g}}\right] \longrightarrow \mathbb{B}$ such that, for any $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$,

$$\begin{aligned}
\mathrm{Ax}_{\mathrm{CD,II}}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) \quad &\stackrel{\mathrm{def}}{\longleftrightarrow} \quad \Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} \left(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\right)\right)\Bigg) \\
&\qquad \cap \Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\right)\big)\Bigg) \\
&\qquad = \left(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}\right) \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}\right)
\end{aligned}$$

Suppose $\mathrm{Ax}_{\mathrm{CD,II}}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) = 1$ holds. Then:

– Case i. If $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) = (\Omega, \Omega)$, then

$$\begin{aligned}
\mathrm{Ax}_{\mathrm{CD,II}}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) \quad &\stackrel{\mathrm{def}}{\longleftrightarrow} \quad \Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\Omega,\Omega} \left(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\right)\right)\Bigg) \\
&\qquad \cap \Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\Omega,\Omega} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\right)\big)\Bigg) \\
&\qquad = \left(\Omega \cap \Omega\right) \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\Omega \cap \Omega\right)
\end{aligned}$$

Consequently, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega)$. But, the relation $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) \longleftrightarrow \Omega \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\Omega) = \Omega$ holds. Therefore, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) = \Omega \stackrel{\mathrm{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{CD},1}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right)$ and hence, $\mathrm{Ax}_{\mathrm{CD,II}} \longrightarrow \mathrm{Ax}_{\mathrm{CD},1}$.

– Case ii. If $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ be arbitrary such that $\mathscr{V}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}$ and $\{\zeta\} \subset \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$, then

$$\begin{aligned}
\mathrm{Ax}_{\mathrm{CD,II}}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) \quad &\stackrel{\mathrm{def}}{\longleftrightarrow} \quad \Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}} \left(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\right)\right)\Bigg) \\
&\qquad \cap \Bigg(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\left(\mathscr{W}_{\mathfrak{g}}\right)\big)\Bigg) \\
&\qquad = \big(\mathscr{U}_{\mathfrak{g}} \cap \left(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}\right)\big) \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap \left(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}\right)\big)
\end{aligned}$$

Since the relation $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\big) \supseteq \mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})$ holds, implying $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\big) \cap \big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\big) = \mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})$ for any $\mathscr{W}_{\mathfrak{g}} \in$

$\{\mathscr{U}_{\mathfrak{g}},\ \mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}\}$, $\mathrm{Ax}_{\mathrm{CD,II}}$ reduces to

$$\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{U}_{\mathfrak{g}}\cup\{\zeta\}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap (\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\})\big)$$

Because $\mathscr{U}_{\mathfrak{g}} \subseteq \mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}$ holds, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap (\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\})\big) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. Consequently, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}\big) \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. But,

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \quad &\supseteq \quad \big\{\zeta \in \mathfrak{T}_{\mathfrak{g}} :\ \zeta \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}\big)\big\} \\ &\longleftrightarrow \quad \big\{\zeta \in \mathfrak{T}_{\mathfrak{g}} :\ \zeta \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big((\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}) \cup \{\zeta\}\big)\big\} \\ &\longleftrightarrow \quad \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}) \end{aligned}$$

implying the relation $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}\big) \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. Therefore, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}\big) \overset{\mathrm{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{CD},2}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big)$ and hence, $\mathrm{Ax}_{\mathrm{CD,I}} \longrightarrow \mathrm{Ax}_{\mathrm{CD},2}$.

– CASE III. If $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ such that $\mathscr{V}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}}$ be arbitrary, then

$$\begin{aligned} \mathrm{Ax}_{\mathrm{CD,II}}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big) \quad \overset{\mathrm{def}}{\longleftrightarrow} \quad & \Big(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{U}_{\mathfrak{g}}} (\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}))\Big) \\ & \cap \Big(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{U}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\big)\Big) \\ & = (\mathscr{U}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{U}_{\mathfrak{g}}) \end{aligned}$$

Consequently, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) \cap \big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) = \mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$, implying $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big) \supseteq \mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$. But, because the relation $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})\big)$ holds, it results, therefore, that $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \overset{\mathrm{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{CD},3}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big)$ and thus, $\mathrm{Ax}_{\mathrm{CD,II}} \longrightarrow \mathrm{Ax}_{\mathrm{CD},3}$.

– CASE IV. If $(\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}) \in \times_{\alpha\in I_2^*} \mathscr{P}(\Omega)$ be arbitrary, then

$$\begin{aligned} \mathrm{Ax}_{\mathrm{CD,II}}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big) \quad \overset{\mathrm{def}}{\longleftrightarrow} \quad & \Big(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} (\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}))\Big) \\ & \cap \Big(\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\big)\Big) \\ & = (\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) \end{aligned}$$

Since $\bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\big) \supseteq \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} \big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})\big)$ holds, $\mathrm{Ax}_{\mathrm{CD,II}}$, evidently, reduces to $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}}=\mathscr{U}_{\mathfrak{g}},\mathscr{V}_{\mathfrak{g}}} \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) \overset{\mathrm{def}}{\longleftrightarrow} \mathrm{Ax}_{\mathrm{CD},4}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big)$ and hence, $\mathrm{Ax}_{\mathrm{CD,II}} \longrightarrow \mathrm{Ax}_{\mathrm{CD},4}$.

Thus, $\mathrm{Ax}_{\mathrm{CD,II}}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big) = 1 \longrightarrow \bigwedge_{\nu\in I_4^*} \mathrm{Ax}_{\mathrm{CD},\nu}\big(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\big) = 1$ and the proof of the proposition is, therefore, complete. □

The corollary stated below is an immediate consequence of the foregoing theorem and proposition.

Corollary 3.22. *If* $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operator on* $\mathscr{P}(\Omega)$ *ranging in* $\mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then it satisfies the following* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operator axiomatic diagram:*

$$\begin{array}{ccc} \mathrm{Ax}_{\mathrm{CD,I}}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) = 1 & \longrightarrow & \bigwedge_{\nu \in I_4^*} \mathrm{Ax}_{\mathrm{CD},\nu}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) = 1 \\ & \nwarrow\!\!\searrow & \uparrow \\ & & \mathrm{Ax}_{\mathrm{CD,II}}\left(\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right) = 1 \end{array} \tag{3.18}$$

The proven lemma presented below will be helpful in proving the theorem following it.

Lemma 3.23. *Let* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$*,* $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. Then:*

– I. $\mathscr{T}_{\mathfrak{g},\mathrm{Der}}(\Omega) \stackrel{\mathrm{def}}{=} \left\{\mathscr{K}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \ \mathscr{K}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\right\}$ *satisfies the* $\mathscr{T}_{\mathfrak{g}}$*-closed set axioms for the strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}$*,*

– II. $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}}(\Omega) \stackrel{\mathrm{def}}{=} \left\{\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \ \mathscr{O}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\right\}$ *satisfies the* $\mathscr{T}_{\mathfrak{g}}$*-open set axioms for the strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}$*.*

Proof. Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Since $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, it satisfies the $\mathscr{T}_{\mathfrak{g}}$-open set axioms $\mathscr{T}_{\mathfrak{g}}(\emptyset) = \emptyset$, $\mathscr{T}_{\mathfrak{g}}(\Omega) = \Omega$, $\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathscr{O}_{\mathfrak{g}}$, and $\mathscr{T}_{\mathfrak{g}}\big(\bigcup_{\nu \in I_\infty^*} \mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu \in I_\infty^*} \mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$, and it also satisfies the $\mathscr{T}_{\mathfrak{g}}$-closed set axioms $\neg\mathscr{T}_{\mathfrak{g}}(\Omega) = \Omega$, $\neg\mathscr{T}_{\mathfrak{g}}(\emptyset) = \emptyset$, $\neg\mathscr{T}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}}) \supseteq \mathscr{K}_{\mathfrak{g}}$, and $\neg\mathscr{T}_{\mathfrak{g}}\big(\bigcap_{\nu \in I_\infty^*} \mathscr{K}_{\mathfrak{g},\nu}\big) = \bigcap_{\nu \in I_\infty^*} \neg\mathscr{T}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\nu})$. Therefore, to prove ITEM I. and ITEM II., it suffices to show that $\mathscr{T}_{\mathfrak{g},\mathrm{Der}} \longleftrightarrow \neg\mathscr{T}_{\mathfrak{g}}$ and $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}} \longleftrightarrow \mathscr{T}_{\mathfrak{g}}$, respectively. Then:

– I. By the definition of $\mathscr{T}_{\mathfrak{g},\mathrm{Der}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, $\Omega \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\Omega)$. Thus, $\mathscr{T}_{\mathfrak{g},\mathrm{Der}}(\Omega) = \Omega$. By virtue of $\mathrm{Ax}_{\mathrm{DE},1}$, $\emptyset = \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\emptyset) \longleftrightarrow \emptyset \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\emptyset)$. Hence, $\mathscr{T}_{\mathfrak{g},\mathrm{Der}}(\emptyset) = \emptyset$. Since $\mathscr{T}_{\mathfrak{g},\mathrm{Der}}(\Omega) \supseteq \left\{\mathscr{K}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \ \mathscr{K}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\right\}$, it results that, for every $\left(\mathscr{K}_{\mathfrak{g}}, \mathscr{T}_{\mathfrak{g},\mathrm{Der}}(\mathscr{K}_{\mathfrak{g}})\right) \in \mathscr{P}(\Omega) \times \mathscr{T}_{\mathfrak{g},\mathrm{Der}}(\Omega)$, the relation $\mathscr{T}_{\mathfrak{g},\mathrm{Der}}(\mathscr{K}_{\mathfrak{g}}) \supseteq \mathscr{K}_{\mathfrak{g}}$ holds. Suppose $(\mathscr{K}_{\mathfrak{g},\nu}, \mathscr{K}_{\mathfrak{g},\mu}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ such that, for each $\eta \in \{\nu, \mu\}$, $\mathscr{K}_{\mathfrak{g},\mu} \supseteq \mathscr{K}_{\mathfrak{g},\nu}$ and, for all $\sigma \in I_\infty^*$, the relation $\mathscr{K}_{\mathfrak{g},\sigma} \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\sigma})$ holds. Then, $\mathscr{K}_{\mathfrak{g},\mu} \supseteq \mathscr{K}_{\mathfrak{g},\nu}$ implies $\mathscr{K}_{\mathfrak{g},\mu} \longleftrightarrow \mathscr{K}_{\mathfrak{g},\mu} \cup \mathscr{K}_{\mathfrak{g},\nu} \longleftrightarrow \mathscr{K}_{\mathfrak{g},\mu} \cup (\mathscr{K}_{\mathfrak{g},\mu} \cap \mathscr{K}_{\mathfrak{g},\nu})$. By virtue of $\mathrm{Ax}_{\mathrm{DE},4}$, it follows that the relation $\bigcap_{\eta=\nu,\mu} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\eta}) \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\nu} \cap \mathscr{K}_{\mathfrak{g},\mu}) \cap \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\nu}) \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\nu} \cap \mathscr{K}_{\mathfrak{g},\mu})$ holds, implying $\bigcap_{\eta=\nu,\mu} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\eta}) \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\nu} \cap \mathscr{K}_{\mathfrak{g},\mu})$. But $\mathscr{K}_{\mathfrak{g},\eta} \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\eta})$ holds for each $\eta \in \{\nu, \mu\}$ implies $\bigcap_{\eta=\nu,\mu} \mathscr{K}_{\mathfrak{g},\eta} \supseteq \bigcap_{\eta=\nu,\mu} \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\eta})$. Thus, $\bigcap_{\eta=\nu,\mu} \mathscr{K}_{\mathfrak{g},\eta} \supseteq \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g},\nu} \cap \mathscr{K}_{\mathfrak{g},\mu})$. The condition $\mathscr{T}_{\mathfrak{g},\mathrm{Der}} \longleftrightarrow \neg\mathscr{T}_{\mathfrak{g}}$ is proved and hence, $\mathscr{T}_{\mathfrak{g},\mathrm{Der}}\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ satisfies the $\mathscr{T}_{\mathfrak{g}}$-closed set axioms for the strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

– II. By virtue of $\mathrm{Ax}_{\mathrm{CD},1}$, $\Omega = \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega) \longleftrightarrow \Omega \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\Omega)$. Thus, $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}}(\Omega) = \Omega$. By the definition of $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, $\emptyset \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\emptyset)$. Hence, $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}}(\emptyset) = \emptyset$. Since $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}}(\Omega) \subseteq \left\{\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \ \mathscr{O}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\right\}$, it follows that, for every $\left(\mathscr{O}_{\mathfrak{g}}, \mathscr{T}_{\mathfrak{g},\mathrm{Cod}}(\mathscr{O}_{\mathfrak{g}})\right) \in \mathscr{P}(\Omega) \times \mathscr{T}_{\mathfrak{g},\mathrm{Cod}}(\Omega)$, the relation $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathscr{O}_{\mathfrak{g}}$ holds. Let $(\mathscr{O}_{\mathfrak{g},\nu}, \mathscr{O}_{\mathfrak{g},\mu}) \in \times_{\alpha \in I_2^*} \mathscr{P}(\Omega)$ such that, for each $\eta \in \{\nu, \mu\}$, $\mathscr{O}_{\mathfrak{g},\mu} \subseteq \mathscr{O}_{\mathfrak{g},\nu}$ and, for all $\sigma \in I_\infty^*$, the relation $\mathscr{O}_{\mathfrak{g},\sigma} \subseteq \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\sigma})$ holds. Then, $\mathscr{O}_{\mathfrak{g},\nu} \subseteq \mathscr{O}_{\mathfrak{g},\nu}$ implies $\mathscr{O}_{\mathfrak{g},\mu} \longleftrightarrow \mathscr{O}_{\mathfrak{g},\mu} \cap \mathscr{O}_{\mathfrak{g},\nu} \longleftrightarrow (\mathscr{O}_{\mathfrak{g},\mu} \cup \mathscr{O}_{\mathfrak{g},\nu}) \cap$

$\mathscr{O}_{\mathfrak{g},\nu}$. By virtue of $\mathrm{Ax}_{\mathrm{CD},4}$, it results that the relation $\bigcup_{\eta=\nu,\mu}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\eta})\subseteq\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}\cup\mathscr{O}_{\mathfrak{g},\mu})\cup\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}\cup\mathscr{O}_{\mathfrak{g},\mu})$ holds which, in turn, implies $\bigcup_{\eta=\nu,\mu}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\eta})\subseteq\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}\cup\mathscr{O}_{\mathfrak{g},\mu})$. But the relation $\mathscr{O}_{\mathfrak{g},\eta}\subseteq\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\eta})$ holding true for each $\eta\in\{\nu,\mu\}$ implies, in turn, $\bigcup_{\eta=\nu,\mu}\mathscr{O}_{\mathfrak{g},\eta}\subseteq\bigcap_{\eta=\nu,\mu}\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\eta})$. Thus, $\bigcup_{\eta=\nu,\mu}\mathscr{O}_{\mathfrak{g},\eta}\subseteq\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}\cup\mathscr{O}_{\mathfrak{g},\mu})$. The condition $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}}\longleftrightarrow\mathscr{T}_{\mathfrak{g}}$ is proved and hence, $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}}\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ satisfies the $\mathscr{T}_{\mathfrak{g}}$-open set axioms for the strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. The proof of the lemma is, therefore, complete. □

The theorem is now stated and proved by the aid of the above lemma.

Theorem 3.24. *Let* $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-derived and a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-coderived operators, respectively, in a unique strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. *Then:*

- I. $\mathscr{T}_{\mathfrak{g},\mathrm{Der}}(\Omega)\stackrel{\mathrm{def}}{=}\{\mathscr{K}_{\mathfrak{g}}\in\mathscr{P}(\Omega):\ \mathscr{K}_{\mathfrak{g}}\supseteq\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\}$ *forms the* $\mathscr{T}_{\mathfrak{g}}$*-closed sets for the unique strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}$*,*
- II. $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}}(\Omega)\stackrel{\mathrm{def}}{=}\{\mathscr{O}_{\mathfrak{g}}\in\mathscr{P}(\Omega):\ \mathscr{O}_{\mathfrak{g}}\subseteq\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\}$ *forms the* $\mathscr{T}_{\mathfrak{g}}$*-open set axioms for the unique strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}$*.*

Proof. Let $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators, respectively, in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then:

— I. $\mathscr{T}_{\mathfrak{g},\mathrm{Der}}(\Omega)\stackrel{\mathrm{def}}{=}\{\mathscr{K}_{\mathfrak{g}}\in\mathscr{P}(\Omega):\ \mathscr{K}_{\mathfrak{g}}\supseteq\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\}$ forms the collection of $\mathscr{T}_{\mathfrak{g}}$-closed set in the strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. Suppose $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mathrm{ind}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ be the induced $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator, then to show uniqueness it only suffices to prove that $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}})\longleftrightarrow\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ holds true for any $\mathscr{R}_{\mathfrak{g}}\subseteq\mathfrak{T}_{\mathfrak{g}}$. Let $\mathscr{R}_{\mathfrak{g}}\subseteq\mathfrak{T}_{\mathfrak{g}}$ be arbitrary and by hypothesis, let $(\xi\in\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}))\wedge(\xi\notin\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}}))$ hold true. Then, uniqueness is shown by proving that such hypothesis is a contradiction. Thus, the following cases present themselves:

– Case I. Suppose $\xi\notin\mathscr{R}_{\mathfrak{g}}$. Then, $(\xi\notin\mathscr{R}_{\mathfrak{g}})\wedge(\xi\notin\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}}))$ by virtue of the supposition and the hypothesis. Consequently, $\xi\notin\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\}))$. Therefore, a $\mathscr{T}_{\mathfrak{g}}$-open set $\mathscr{O}_{\mathfrak{g}}\in\mathscr{T}_{\mathfrak{g}}$ can be found, satisfying $\xi\in\mathscr{O}_{\mathfrak{g}}$, such that $\mathscr{O}_{\mathfrak{g}}\cap(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\}))=\mathscr{O}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}}=\emptyset$. Clearly, $\mathscr{K}_{\mathfrak{g}}=\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})$ is a $\mathscr{T}_{\mathfrak{g}}$-closed set and therefore, it satisfies $\mathscr{K}_{\mathfrak{g}}\supseteq\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})$, implying $(\mathscr{K}_{\mathfrak{g}}\in\mathscr{T}_{\mathfrak{g},\mathrm{Der}})\wedge(\mathscr{K}_{\mathfrak{g}}\supseteq\mathscr{R}_{\mathfrak{g}})$ holds true. Consequently, it follows that $\mathscr{K}_{\mathfrak{g}}\supseteq\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$. But, $\xi\in\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ by hypothesis. Hence, $(\xi\in\mathscr{O}_{\mathfrak{g}})\wedge(\xi\in\mathscr{K}_{\mathfrak{g}}=\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}))$, a contradiction. The hypothesis is therefore a contradiction.

– Case II. Suppose $\xi\in\mathscr{R}_{\mathfrak{g}}$. Then, $(\xi\in\mathscr{R}_{\mathfrak{g}})\wedge(\xi\in\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}))$ by virtue of the supposition and the hypothesis. Consequently, $\xi\in\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\}))$. Therefore, a $\mathscr{T}_{\mathfrak{g}}$-closed set $\mathscr{K}_{\mathfrak{g}}\in\neg\mathscr{T}_{\mathfrak{g}}$ can be found, satisfying $\xi\in\mathscr{K}_{\mathfrak{g}}$, such that $\mathscr{K}_{\mathfrak{g}}\cap(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\}))=\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})\neq\emptyset$. Then, $\mathscr{K}_{\mathfrak{g}}\supseteq\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\})$ and consequently, $\mathscr{K}_{\mathfrak{g}}\supseteq\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{K}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi\}))=\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}})$. But, $\xi\notin\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}})$ by hypothesis. Thus, $(\xi\in\mathscr{K}_{\mathfrak{g}})\wedge(\xi\notin\mathscr{K}_{\mathfrak{g}})$, a contradiction. The hypothesis is therefore a contradiction. Hence, $\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}})$.

The relation $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}})$ is now proved. By hypothesis, let $\big(\xi \notin \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \wedge \big(\xi \in \mathfrak{g}\text{-Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}})\big)$ hold true. Then, uniqueness is again shown by proving that such hypothesis is a contradiction. Thus, the following cases present themselves:

– Case i. Suppose $\xi \notin \mathscr{R}_{\mathfrak{g}}$. Clearly, a $\mathscr{T}_{\mathfrak{g}}$-closed set $\mathscr{K}_{\mathfrak{g}} \in \neg\mathscr{T}_{\mathfrak{g}}$ can be found such that $\mathscr{K}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, and consequently, $\mathscr{K}_{\mathfrak{g}} \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}})$. But, by virtue of the supposition and the hypothesis, $\big(\xi \notin \mathscr{R}_{\mathfrak{g}}\big) \wedge \big(\xi \notin \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)$, implying $\xi \notin \mathfrak{g}\text{-Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}})$, a contradiction. The hypothesis is therefore a contradiction.

– Case ii. Suppose $\xi \in \mathscr{R}_{\mathfrak{g}}$. Then, $\big(\xi \in \mathscr{R}_{\mathfrak{g}}\big) \wedge \big(\xi \in \mathfrak{g}\text{-Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}})\big)$ by virtue of the supposition and the hypothesis. Consequently, $\xi \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)$. Since $\xi \in \mathfrak{g}\text{-Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}})$ is equivalent to $\xi \in \mathfrak{g}\text{-Der}_{\mathfrak{g},\mathrm{ind}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)$ and, on the other hand, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)$ is equivalent to $\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \cup \mathfrak{g}\text{-Der}_{\mathfrak{g},\mathrm{ind}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) = \mathscr{K}_{\mathfrak{g}}$ for some $\mathscr{T}_{\mathfrak{g}}$-closed set $\mathscr{K}_{\mathfrak{g}} \in \neg\mathscr{T}_{\mathfrak{g}}$, it follows that $\xi \in \mathfrak{g}\text{-Der}_{\mathfrak{g},\mathrm{ind}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}})$, implying $\xi \in \mathfrak{g}\text{-Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}})$. But, by virtue of the supposition and the hypothesis, $\big(\xi \in \mathscr{R}_{\mathfrak{g}}\big) \wedge \big(\xi \in \mathfrak{g}\text{-Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}})\big)$, implying $\xi \in \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, a contradiction. The hypothesis is therefore a contradiction. Thus, $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},\mathrm{ind}}(\mathscr{R}_{\mathfrak{g}})$

— II. $\mathscr{T}_{\mathfrak{g},\mathrm{Cod}}(\Omega) \overset{\mathrm{def}}{=} \big\{\mathscr{O}_{\mathfrak{g}} \in \mathscr{P}(\Omega) : \ \mathscr{O}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\big\}$ forms the collection of $\mathscr{T}_{\mathfrak{g}}$-open set in the strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. Suppose $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be the induced $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator, then to show uniqueness it only suffices to prove that $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds true for any $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$. Let $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ be arbitrary and by hypothesis, let $\big(\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \wedge \big(\zeta \notin \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}(\mathscr{S}_{\mathfrak{g}})\big)$ hold true. Then, uniqueness is shown by proving that such hypothesis is a contradiction. Thus, the following cases present themselves:

– Case i. Suppose $\zeta \notin \mathscr{S}_{\mathfrak{g}}$. By virtue of the supposition and the hypothesis, the relation $\big(\zeta \notin \mathscr{S}_{\mathfrak{g}}\big) \wedge \big(\zeta \notin \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}(\mathscr{S}_{\mathfrak{g}})\big)$, then, holds true. Consequently, $\zeta \notin \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big)$, implying $\zeta \in \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big)$. Therefore, a $\mathscr{T}_{\mathfrak{g}}$-closed set $\mathscr{K}_{\mathfrak{g}} \in \neg\mathscr{T}_{\mathfrak{g}}$ can be found, satisfying $\zeta \in \mathscr{K}_{\mathfrak{g}}$, such that $\mathscr{K}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big)$. Clearly, $\mathscr{O}_{\mathfrak{g}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})$ is a $\mathscr{T}_{\mathfrak{g}}$-open set and therefore, it satisfies $\mathscr{O}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})$, implying $\big(\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\mathrm{Cod}}\big) \wedge \big(\mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}\big)$ holds true. Consequently, it follows that $\mathscr{O}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But, $\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ by hypothesis. Hence, $\big(\zeta \notin \mathscr{O}_{\mathfrak{g}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\big) \wedge \big(\zeta \notin \mathscr{K}_{\mathfrak{g}}\big)$, a contradiction. The hypothesis is therefore a contradiction.

– Case ii. Suppose $\zeta \in \mathscr{S}_{\mathfrak{g}}$. By virtue of the supposition and the hypothesis, the relation $\big(\zeta \in \mathscr{S}_{\mathfrak{g}}\big) \wedge \big(\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$, then, holds true. Consequently, $\zeta \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big)$, implying $\zeta \notin \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big)$. Therefore, a $\mathscr{T}_{\mathfrak{g}}$-open set $\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g}}$ can be found, satisfying $\zeta \in \mathscr{O}_{\mathfrak{g}}$, such that $\mathscr{O}_{\mathfrak{g}} \cup (\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}) = \mathscr{S}_{\mathfrak{g}} \cup \{\xi\}$. Then, $\mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}} \cup \{\xi\}$ and consequently, $\mathscr{O}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}(\mathscr{O}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big) = \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}(\mathscr{S}_{\mathfrak{g}})$. But, $\zeta \notin \mathfrak{g}\text{-Cod}_{\mathfrak{g},\mathrm{ind}}(\mathscr{S}_{\mathfrak{g}})$ by hypothesis. Thus, $\big(\zeta \notin \mathscr{O}_{\mathfrak{g}}\big) \wedge \big(\zeta \notin \mathscr{O}_{\mathfrak{g}}\big)$, a contradiction. The hypothesis is therefore a contradiction. Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},\text{ind}}(\mathscr{S}_{\mathfrak{g}})$.

The relation $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},\text{ind}}(\mathscr{S}_{\mathfrak{g}})$ is now proved. By hypothesis, let $\big(\xi \notin \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \wedge \big(\xi \in \mathfrak{g}\text{-Cod}_{\mathfrak{g},\text{ind}}(\mathscr{S}_{\mathfrak{g}})\big)$ hold true. Then, uniqueness is again shown by proving that such hypothesis is a contradiction. Thus, the following cases present themselves:

– Case i. Suppose $\zeta \notin \mathscr{S}_{\mathfrak{g}}$. Clearly, a $\mathscr{T}_{\mathfrak{g}}$-open set $\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g}}$ can be found such that $\mathscr{O}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, and consequently, $\mathscr{O}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},\text{ind}}(\mathscr{S}_{\mathfrak{g}})$. But, by virtue of the supposition and the hypothesis, $\big(\zeta \notin \mathscr{S}_{\mathfrak{g}}\big) \wedge \big(\zeta \notin \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$, implying $\zeta \notin \mathfrak{g}\text{-Cod}_{\mathfrak{g},\text{ind}}(\mathscr{S}_{\mathfrak{g}})$, a contradiction. The hypothesis is therefore a contradiction.

– Case ii. Suppose $\zeta \in \mathscr{S}_{\mathfrak{g}}$. Then, $\big(\zeta \in \mathscr{S}_{\mathfrak{g}}\big) \wedge \big(\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g},\text{ind}}(\mathscr{S}_{\mathfrak{g}})\big)$ by virtue of the supposition and the hypothesis. Consequently, $\zeta \in \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big)$. Since $\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g},\text{ind}}(\mathscr{S}_{\mathfrak{g}})$ is equivalent to $\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g},\text{ind}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big)$ and, on the other hand, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big)$ is equivalent to $\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big) \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g},\text{ind}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\zeta\}\big) = \mathscr{O}_{\mathfrak{g}}$ for some $\mathscr{T}_{\mathfrak{g}}$-open set $\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g}}$, it follows that $\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g},\text{ind}}\big(\mathscr{S}_{\mathfrak{g}} \cup \{\xi\}\big) = \mathfrak{g}\text{-Cod}_{\mathfrak{g},\text{ind}}(\mathscr{S}_{\mathfrak{g}})$, implying $\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g},\text{ind}}(\mathscr{S}_{\mathfrak{g}})$. But, by virtue of the supposition and the hypothesis, $\big(\zeta \in \mathscr{S}_{\mathfrak{g}}\big) \wedge \big(\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g},\text{ind}}(\mathscr{S}_{\mathfrak{g}})\big)$, implying $\zeta \in \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, a contradiction. The hypothesis is therefore a contradiction. Hence, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},\text{ind}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the theorem is, therefore, complete. □

On the essential properties of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces, the discussion of the present section terminates here.

4. Discussion

4.1. Categorical Classifications. Having classified the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operators in terms of their categories, namely $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-derived and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators in $\mathscr{T}_{\mathfrak{a}}$-spaces, $(\nu, \mathfrak{a}) \in I_3^0 \times \{\mathfrak{o}, \mathfrak{g}\}$, it is proposed here to establish the various relationships amongst the classes of $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived and $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, and to illustrate such relationships through diagrams.

Of the lists of notations $\mathfrak{T}_{\mathfrak{o}} = (\Omega, \mathscr{T}_{\mathfrak{o}})$, $\text{int}_{\mathfrak{o}}$, $\mathfrak{g}\text{-Int}_{\mathfrak{o}}$, $\text{cl}_{\mathfrak{o}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{o}}$, …, $\text{der}_{\mathfrak{o}}$, $\mathfrak{g}\text{-Der}_{\mathfrak{o}}$, $\text{cod}_{\mathfrak{o}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{o}}$, … and $\mathfrak{T} = (\Omega, \mathscr{T})$, int, $\mathfrak{g}$-Int, cl, $\mathfrak{g}$-Cl, …, der, $\mathfrak{g}$-Der, cod, $\mathfrak{g}$-Cod, …, respectively, either the first will be used instead of the second, or both will be used interchangeably.

In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$, $\mathfrak{g}\text{-Int}_{\mathfrak{a},0}(\mathscr{S}_{\mathfrak{a}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{a},1}(\mathscr{S}_{\mathfrak{a}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{a},3}(\mathscr{S}_{\mathfrak{a}}) \supseteq \mathfrak{g}\text{-Int}_{\mathfrak{a},2}(\mathscr{S}_{\mathfrak{a}})$ holds for any $\mathscr{S}_{\mathfrak{a}} \in \mathscr{P}(\Omega)$. Moreover, the relation $\mathfrak{g}\text{-Int}_{\nu}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$ also holds true for any $(\nu, \mathscr{S}_{\mathfrak{g}}) \in I_3^0 \times \mathfrak{T}_{\mathfrak{g}}$. But, for every $(\nu, \mathscr{S}_{\mathfrak{g}}) \in I_3^0 \times \mathfrak{T}_{\mathfrak{g}}$, the relations $\mathfrak{g}\text{-Int}_{\nu}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\nu}(\mathscr{S}_{\mathfrak{g}})$, $\mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})$ and $\big(\mathfrak{g}\text{-Cod}_{\nu}(\mathscr{S}_{\mathfrak{g}}), \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}})\big) \supseteq \big(\text{cod}(\mathscr{S}_{\mathfrak{g}}), \text{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$ hold. Thus, the following diagram, which is to be read horizontally, from left to right and vertically, from top to bottom, presents itself:

$$\begin{array}{ccccccc}
\mathrm{cod}\left(\mathscr{S}_{\mathfrak{g}}\right) & \subseteq & \mathrm{cod}\left(\mathscr{S}_{\mathfrak{g}}\right) & \subseteq & \mathrm{cod}\left(\mathscr{S}_{\mathfrak{g}}\right) & \supseteq & \mathrm{cod}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
\cap\!\!| & & \cap\!\!| & & \cap\!\!| & & \cap\!\!| \\
\mathfrak{g}\text{-}\mathrm{Cod}_0\left(\mathscr{S}_{\mathfrak{g}}\right) & \subseteq & \mathfrak{g}\text{-}\mathrm{Cod}_1\left(\mathscr{S}_{\mathfrak{g}}\right) & \subseteq & \mathfrak{g}\text{-}\mathrm{Cod}_3\left(\mathscr{S}_{\mathfrak{g}}\right) & \supseteq & \mathfrak{g}\text{-}\mathrm{Cod}_2\left(\mathscr{S}_{\mathfrak{g}}\right) \\
\cap\!\!| & & \cap\!\!| & & \cap\!\!| & & \cap\!\!| \\
\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},0}\left(\mathscr{S}_{\mathfrak{g}}\right) & \subseteq & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},1}\left(\mathscr{S}_{\mathfrak{g}}\right) & \subseteq & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},3}\left(\mathscr{S}_{\mathfrak{g}}\right) & \supseteq & \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},2}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
\cup\!\!| & & \cup\!\!| & & \cup\!\!| & & \cup\!\!| \\
\mathrm{cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) & \subseteq & \mathrm{cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) & \subseteq & \mathrm{cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) & \supseteq & \mathrm{cod}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)
\end{array} \tag{4.1}$$

In FIG. 1, we present the relationships between the elements of the collections $\left\{\mathfrak{g}\text{-}\mathrm{Cod}_{\nu} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-}\mathrm{Cod}_{\nu}\left(\mathscr{S}_{\mathfrak{g}}\right) : \ \nu \in I_3^0\right\}$ in the $\mathscr{T}$-space $\mathfrak{T} \subset \mathfrak{T}_{\mathfrak{g}}$ and $\left\{\mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g},\nu}\left(\mathscr{S}_{\mathfrak{g}}\right) : \ \nu \in I_3^0\right\}$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} \supset \mathfrak{T}$; FIG. 1 may well be called a $\left(\mathfrak{g}\text{-}\mathrm{Cod}, \mathfrak{g}\text{-}\mathrm{Cod}_{\mathfrak{g}}\right)$-*valued diagram*.

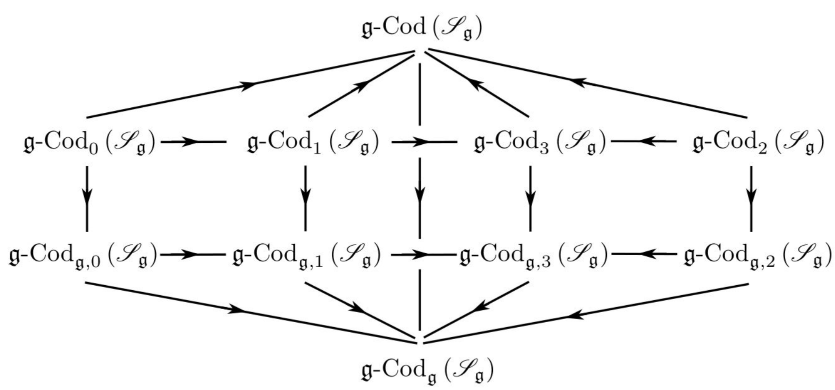

FIGURE 1. Relationships: $\mathfrak{g}$-$\mathfrak{T}$-coderived operators in $\mathscr{T}$-spaces and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

In a $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T})$, the relation $\mathfrak{g}\text{-}\mathrm{Cl}_0\left(\mathscr{S}_{\mathfrak{g}}\right) \supseteq \mathfrak{g}\text{-}\mathrm{Cl}_1\left(\mathscr{S}_{\mathfrak{g}}\right) \supseteq \mathfrak{g}\text{-}\mathrm{Cl}_3\left(\mathscr{S}_{\mathfrak{g}}\right) \subseteq \mathfrak{g}\text{-}\mathrm{Cl}_2\left(\mathscr{S}_{\mathfrak{g}}\right)$ holds for any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. Likewise, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, the relation $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},0}\left(\mathscr{S}_{\mathfrak{g}}\right) \supseteq \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},1}\left(\mathscr{S}_{\mathfrak{g}}\right) \supseteq \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},3}\left(\mathscr{S}_{\mathfrak{g}}\right) \subseteq \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},2}\left(\mathscr{S}_{\mathfrak{g}}\right)$ holds for any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. Moreover, the relation $\mathfrak{g}\text{-}\mathrm{Cl}_{\nu}\left(\mathscr{S}_{\mathfrak{g}}\right) \supseteq \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu}\left(\mathscr{S}_{\mathfrak{g}}\right)$ also holds true for any $(\nu, \mathscr{S}_{\mathfrak{g}}) \in I_3^0 \times \mathfrak{T}_{\mathfrak{g}}$. But, for every $(\nu, \mathscr{S}_{\mathfrak{g}}) \in I_3^0 \times \mathfrak{T}_{\mathfrak{g}}$, the relations $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu}\left(\mathscr{S}_{\mathfrak{g}}\right) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}\left(\mathscr{S}_{\mathfrak{g}}\right)$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu}\left(\mathscr{S}_{\mathfrak{g}}\right) \longleftrightarrow \mathscr{S}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}\left(\mathscr{S}_{\mathfrak{g}}\right)$ and $\left(\mathfrak{g}\text{-}\mathrm{Der}_{\nu}\left(\mathscr{S}_{\mathfrak{g}}\right), \mathfrak{g}\text{-}\mathrm{Der}_{\mathfrak{g},\nu}\left(\mathscr{S}_{\mathfrak{g}}\right)\right) \subseteq \left(\mathrm{der}\left(\mathscr{S}_{\mathfrak{g}}\right), \mathrm{der}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)$ hold true. Hence, the following diagram, which is to be read horizontally, from left to right and vertically, from top to bottom, presents itself:

$$\begin{array}{ccccccc}
\operatorname{der}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \operatorname{der}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \operatorname{der}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \operatorname{der}(\mathscr{S}_{\mathfrak{g}}) \\
\cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid \\
\mathfrak{g}\text{-Der}_0(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-Der}_1(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-Der}_3(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-Der}_2(\mathscr{S}_{\mathfrak{g}}) \\
\cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid \\
\mathfrak{g}\text{-Der}_{\mathfrak{g},0}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-Der}_{\mathfrak{g},1}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \mathfrak{g}\text{-Der}_{\mathfrak{g},3}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \mathfrak{g}\text{-Der}_{\mathfrak{g},2}(\mathscr{S}_{\mathfrak{g}}) \\
\cap\!\shortmid & & \cap\!\shortmid & & \cap\!\shortmid & & \cap\!\shortmid \\
\operatorname{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \operatorname{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \supseteq & \operatorname{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) & \subseteq & \operatorname{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})
\end{array} \tag{4.2}$$

In FIG. 2, we present the relationships between the elements of the collections $\{\mathfrak{g}\text{-Der}_{\nu} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Der}_{\nu}(\mathscr{S}_{\mathfrak{g}}) : \ \nu \in I_3^0\}$ in the $\mathscr{T}$-space $\mathfrak{T} \subset \mathfrak{T}_{\mathfrak{g}}$ and $\{\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{g}}) : \ \nu \in I_3^0\}$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} \supset \mathfrak{T}$; FIG. 2 may well be called a $(\mathfrak{g}\text{-Der}, \mathfrak{g}\text{-Der}_{\mathfrak{g}})$-*valued diagram*.

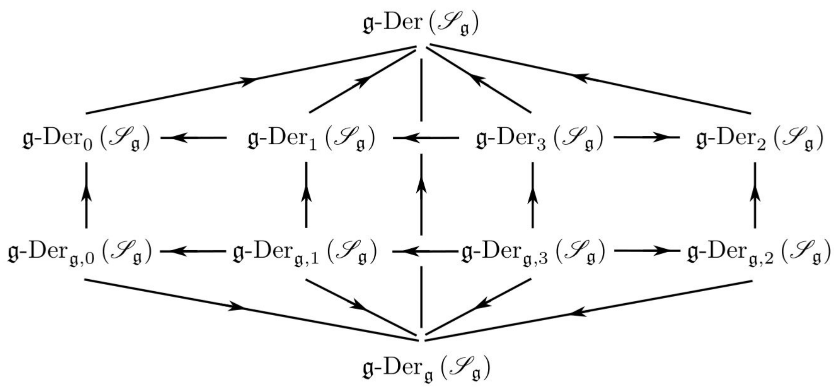

FIGURE 2. Relationships: $\mathfrak{g}$-$\mathfrak{T}$-derived operators in $\mathscr{T}$-spaces and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

As in our previous works [33, 34, 35, 36, 37], the manner we have positioned the arrows is solely to stress that, in general, the implications in FIGS 1, 2 and EQS (4.1), (4.2) are irreversible. The various relationships amongst the classes of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators in the $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}}$ are therefore established.

4.2. **A Nice Application.** In this section, we present a nice application in an attempt to shed lights on some essential properties of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in a $\mathscr{T}_{\mathfrak{g}}$-space.

Let the 7-point set $\Omega = \{\xi_{\nu} : \ \nu \in I_7^*\}$ denotes the underlying set and consider the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, where Ω is 4-element topologized by the choice:

$$\begin{aligned} \mathscr{T}_{\mathfrak{g}}(\Omega) &= \{\emptyset, \ \{\xi_1\}, \ \{\xi_1, \xi_3, \xi_5\}, \ \{\xi_1, \xi_3, \xi_4, \xi_5, \xi_7\}\} \\ &= \{\mathscr{O}_{\mathfrak{g},1}, \ \mathscr{O}_{\mathfrak{g},2}, \ \mathscr{O}_{\mathfrak{g},3}, \ \mathscr{O}_{\mathfrak{g},4}\}; \end{aligned} \tag{4.3}$$

$$\begin{aligned} \neg\mathscr{T}_{\mathfrak{g}}(\Omega) &= \{\Omega, \ \{\xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7\}, \ \{\xi_2, \xi_4, \xi_6, \xi_7\}, \ \{\xi_2, \xi_6\}\} \\ &= \{\mathscr{K}_{\mathfrak{g},1}, \ \mathscr{K}_{\mathfrak{g},2}, \ \mathscr{K}_{\mathfrak{g},3}, \ \mathscr{K}_{\mathfrak{g},4}\}. \end{aligned} \tag{4.4}$$

Evidently, $\mathscr{T}_{\mathfrak{g}},\ \neg\mathscr{T}_{\mathfrak{g}} :\ \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ establish the classes of $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets, respectively. Since conditions $\mathscr{T}_{\mathfrak{g}}(\emptyset) = \emptyset$, $\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) \subseteq \mathscr{O}_{\mathfrak{g},\nu}$ for every $\nu \in I_4^*$, and $\mathscr{T}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_4^*}\mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu\in I_4^*}\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$ are satisfied, then $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is a $\mathfrak{g}$-topology and hence, $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a $\mathscr{T}_{\mathfrak{g}}$-space. Moreover, it is easily checked that $\big(\mathscr{O}_{\mathfrak{g},\mu}, \mathscr{K}_{\mathfrak{g},\mu}\big) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}\big] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}\big]$ for each $(\nu,\mu) \in I_3^0 \times I_4^*$. Thus, the $\mathscr{T}_{\mathfrak{g}}$-open sets forming the $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and the $\mathscr{T}_{\mathfrak{g}}$-closed sets forming the complement $\mathfrak{g}$-topology $\neg\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ of the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ are, respectively, $\mathfrak{g}$-$\mathfrak{T}$-open and $\mathfrak{g}$-$\mathfrak{T}$-closed sets relative to the $\mathscr{T}$-space $\mathfrak{T} = (\Omega, \mathscr{T}) = \big(\Omega, \mathscr{T}_{\mathfrak{g}} \cup \{\Omega\}\big)$.

After calculations, the classes $\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ and $\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, respectively, of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of categories $\nu \in \big\{0,2\big\}$ then take the following forms:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] &= \big\{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ \mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{O}_{\mathfrak{g},4}\big\}; \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] &= \big\{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ \mathscr{S}_{\mathfrak{g}} \supseteq \mathscr{K}_{\mathfrak{g},4}\big\} \quad \forall \nu \in \{0,2\}. \end{aligned} \tag{4.5}$$

On the other hand, those of categories $\nu \in \big\{1,3\big\}$ take the following forms:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] &= \big\{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ \mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{K}_{\mathfrak{g},1}\big\}; \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] &= \big\{\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ \mathscr{S}_{\mathfrak{g}} \supseteq \mathscr{O}_{\mathfrak{g},1}\big\} \quad \forall \nu \in \{1,3\}. \end{aligned} \tag{4.6}$$

Based on the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathfrak{g}$-0-O$[\mathfrak{T}_{\mathfrak{g}}]$, $\mathfrak{g}$-0-K$[\mathfrak{T}_{\mathfrak{g}}]$, …, $\mathfrak{g}$-3-O$[\mathfrak{T}_{\mathfrak{g}}]$, $\mathfrak{g}$-3-K$[\mathfrak{T}_{\mathfrak{g}}]$, introduce the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}} = \big\{\xi_1, \xi_2, \xi_4\big\}$, $\mathscr{S}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}} \cup \big\{\xi_7\big\}$, $\mathscr{U}_{\mathfrak{g}} = \big\{\xi_3, \xi_5, \xi_6, \xi_7\big\}$, and $\mathscr{V}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}} \setminus \big\{\xi_3\big\}$; thus, $\big(\mathscr{S}_{\mathfrak{g}}, \mathscr{U}_{\mathfrak{g}}\big) \supseteq \big(\mathscr{R}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\big)$. Then, for each $\mathscr{W}_{\mathfrak{g}} \in \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}$ and $\mathscr{Y}_{\mathfrak{g}} \in \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}$, the following results present themselves:

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu}\big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi_\mu\})\big) &= \big(\mathscr{W}_{\mathfrak{g}} \setminus \{\xi_\mu\}\big) \cup \mathscr{K}_{\mathfrak{g},4} \quad \forall (\mu,\nu) \in I_7^* \times \{0,2\}, \\ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu}\big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi_\mu\})\big) &= \mathscr{W}_{\mathfrak{g}} \setminus \{\xi_\mu\} \quad \forall (\mu,\nu) \in I_7^* \times \{1,3\}, \\ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu}\big(\mathscr{Y}_{\mathfrak{g}} \cup \{\xi_\mu\}\big) &= \big(\mathscr{Y}_{\mathfrak{g}} \cup \{\xi_\mu\}\big) \setminus \mathscr{K}_{\mathfrak{g},4} \quad \forall (\mu,\nu) \in I_7^* \times \{0,2\}, \\ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu}\big(\mathscr{Y}_{\mathfrak{g}} \cup \{\xi_\mu\}\big) &= \mathscr{Y}_{\mathfrak{g}} \cup \{\xi_\mu\} \quad \forall (\mu,\nu) \in I_7^* \times \{1,3\}. \end{aligned} \tag{4.7}$$

For each $\mathscr{W}_{\mathfrak{g}} \in \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}$ and $\mathscr{Y}_{\mathfrak{g}} \in \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}$, the following results also present themselves:

$$\begin{aligned} \mathrm{cl}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi_\mu\})\big) &= \mathscr{K}_{\mathfrak{g},3} \quad \forall (\mu,\mathscr{W}_{\mathfrak{g}}) \in I_1^* \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}, \\ \mathrm{cl}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi_\mu\})\big) &= \mathscr{K}_{\mathfrak{g},1} \quad \forall (\mu,\mathscr{W}_{\mathfrak{g}}) \in (I_7^* \setminus I_1^*) \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}, \\ \mathrm{int}_{\mathfrak{g}}\big(\mathscr{Y}_{\mathfrak{g}} \cup \{\xi_\mu\}\big) &= \mathscr{O}_{\mathfrak{g},2} \setminus \{\xi_\mu\} \quad \forall (\mu,\mathscr{W}_{\mathfrak{g}}) \in I_7^* \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}. \end{aligned} \tag{4.8}$$

Thus, for each $\mathscr{W}_{\mathfrak{g}} \in \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}$ and $\mathscr{Y}_{\mathfrak{g}} \in \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}$, it follows that:

$$\xi_\mu \in \begin{cases} \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu}\big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi_\mu\})\big) & \forall (\mu,\nu) \in \{2,6\} \times \{0,2\}, \\ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu}\big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi_\mu\})\big) & \forall (\mu,\nu) \in I_7^* \times \{1,3\}, \\ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu}\big(\mathscr{Y}_{\mathfrak{g}} \cup \{\xi_\mu\}\big) & \forall (\mu,\nu) \in I_7^* \times \{1,3\}, \\ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu}\big(\mathscr{Y}_{\mathfrak{g}} \cup \{\xi_\mu\}\big) & \forall (\mu,\nu) \in \{2,6\} \times \{0,2\}. \end{cases} \tag{4.9}$$

On the other hand, it also follows that:

$$\begin{cases} \xi_\mu \in \mathrm{cl}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g}} \cap \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\{\xi_\mu\})\big) & \forall (\mu,\mathscr{W}_{\mathfrak{g}}) \in (I_7^* \setminus I_1^*) \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}, \\ \xi_\mu \notin \mathrm{int}_{\mathfrak{g}}\big(\mathscr{Y}_{\mathfrak{g}} \cup \{\xi_\mu\}\big) & \forall (\mu,\mathscr{W}_{\mathfrak{g}}) \in (I_7^* \setminus I_1^*) \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}. \end{cases} \tag{4.10}$$

Taking the above results into account, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operation of $\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, and the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operation of $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{U}_{\mathfrak{g}}$, $\mathscr{V}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, for all $\nu \in I_3^0$, then, produce the following results:

$$\begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{W}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g},4} & \forall (\nu, \mathscr{W}_{\mathfrak{g}}) \in \{0,2\} \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}, \\ \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{W}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g},1} & \forall (\nu, \mathscr{W}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}, \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{Y}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g},4} & \forall (\nu, \mathscr{Y}_{\mathfrak{g}}) \in \{0,2\} \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}, \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{Y}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g},1} & \forall (\nu, \mathscr{Y}_{\mathfrak{g}}) \in \{1,3\} \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}. \end{cases} \tag{4.11}$$

Likewise, taking the above results into account, the $\mathfrak{T}_{\mathfrak{g}}$-derived operation of $\text{der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, and the $\mathfrak{T}_{\mathfrak{g}}$-coderived operation of $\text{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ on the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{U}_{\mathfrak{g}}$, $\mathscr{V}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, then, also produce the following results:

$$\begin{cases} \text{der}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) = \mathscr{K}_{\mathfrak{g},2} & \forall \mathscr{W}_{\mathfrak{g}} \in \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}, \\ \text{cod}_{\mathfrak{g}}(\mathscr{Y}_{\mathfrak{g}}) = \mathscr{O}_{\mathfrak{g},2} & \forall \mathscr{Y}_{\mathfrak{g}} \in \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}. \end{cases} \tag{4.12}$$

Hence, for each $\mathscr{W}_{\mathfrak{g}} \in \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}$ and $\mathscr{Y}_{\mathfrak{g}} \in \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}$, it results that:

$$\begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{g},0}(\mathscr{W}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},1}(\mathscr{W}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},3}(\mathscr{W}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Der}_{\mathfrak{g},2}(\mathscr{W}_{\mathfrak{g}}), \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},0}(\mathscr{Y}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},1}(\mathscr{Y}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},3}(\mathscr{Y}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cod}_{\mathfrak{g},2}(\mathscr{Y}_{\mathfrak{g}}). \end{cases} \tag{4.13}$$

The $(\precsim, \succsim)$-relations $\mathfrak{g}\text{-Der}_{\mathfrak{g},0} \succsim \mathfrak{g}\text{-Der}_{\mathfrak{g},1} \succsim \mathfrak{g}\text{-Der}_{\mathfrak{g},3} \precsim \mathfrak{g}\text{-Der}_{\mathfrak{g},2}$ and $\mathfrak{g}\text{-Cod}_{\mathfrak{g},0} \precsim \mathfrak{g}\text{-Cod}_{\mathfrak{g},1} \precsim \mathfrak{g}\text{-Cod}_{\mathfrak{g},3} \succsim \mathfrak{g}\text{-Cod}_{\mathfrak{g},2}$ are thus verified. Clearly, the following results also hold true:

$$\begin{cases} \mathfrak{g}\text{-Der}_{\mathfrak{g},\nu}(\mathscr{W}_{\mathfrak{g}}) \subseteq \text{der}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) & \forall (\nu, \mathscr{W}_{\mathfrak{g}}) \in I_3^0 \times \{\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\}, \\ \mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu}(\mathscr{Y}_{\mathfrak{g}}) \supseteq \text{cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}}) & \forall (\nu, \mathscr{Y}_{\mathfrak{g}}) \in I_3^0 \times \{\mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\}. \end{cases} \tag{4.14}$$

Thus, the $(\precsim, \succsim)$-relations $\mathfrak{g}\text{-Der}_{\mathfrak{g},\nu} \precsim \text{der}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-Cod}_{\mathfrak{g},\nu} \succsim \text{cod}_{\mathfrak{g}}$, for all $\nu \in I_3^0$, are also verified.

The application in which are presented some essential properties of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces are therefore accomplished and ends here.

If this nice application be explored a step further, other interesting conclusions can be drawn from the study of the essential properties of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

5. Conclusion

In this paper, we have introduced and studied the essential properties of a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

Concisely, the definitions of the concepts of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators were presented in as general and unified a manner as possible such that the passage from these concepts to $\mathfrak{g}$-$\mathfrak{T}$-derived and $\mathfrak{g}$-$\mathfrak{T}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces, and also to $\mathfrak{T}$-derived and $\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}$-spaces, is not impossible. The essential properties of such novel types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces were discussed in such a manner as to show that much of the fundamental structure of $\mathscr{T}_{\mathfrak{g}}$-spaces is better considered for $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators $\mathfrak{g}\text{-Der}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ than for the $\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{T}_{\mathfrak{g}}$-coderived operators $\mathrm{der}_{\mathfrak{g}}, \mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively. The axiomatic definitions of the concepts of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators in $\mathscr{T}_{\mathfrak{g}}$-spaces were then presented from a purely mathematical or abstract point of view.

Precisely, the outstanding facts on $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators in $\mathscr{T}_{\mathfrak{a}}$-spaces, $\mathfrak{a} \in \{\mathfrak{o}, \mathfrak{g}\}$, are:

— I. If the definitions of $\mathfrak{g}\text{-Der}_{\mathfrak{a}}, \mathfrak{g}\text{-Cod}_{\mathfrak{a}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are based on $\mathrm{cl}_{\mathfrak{a}}, \mathrm{int}_{\mathfrak{a}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ instead of $\mathfrak{g}\text{-Cl}_{\mathfrak{a}}, \mathfrak{g}\text{-Int}_{\mathfrak{a}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, then $\left(\mathfrak{g}\text{-Der}_{\mathfrak{a}}, \mathfrak{g}\text{-Cod}_{\mathfrak{a}}\right) \stackrel{\text{def}}{=} \left(\mathrm{der}_{\mathfrak{a}}, \mathrm{cod}_{\mathfrak{a}}\right)$, and therefore, $\mathrm{der}_{\mathfrak{a}}, \mathrm{cod}_{\mathfrak{a}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are called, respectively, a $\mathfrak{T}_{\mathfrak{a}}$-derived and a $\mathfrak{T}_{\mathfrak{a}}$-coderived operators in a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$.

— II. If $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \precsim \mathrm{der}_{\mathfrak{g}}$ means $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{der}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \succsim \mathrm{cod}_{\mathfrak{g}}$ means $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathrm{cod}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, then: $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathrm{der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathrm{der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\mathrm{cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

— III. A necessary and sufficient condition for the set-valued map $\mathfrak{g}\text{-Der}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ to be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived operator in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is that, for every $\left(\{\xi\}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\right) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$ such that $\{\xi\} \subset \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$, it satisfies:

- I. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\emptyset) = \emptyset$,
- II. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Der}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\{\xi\})\big)$,
- III. $\mathfrak{g}\text{-Der}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
- IV. $\mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \bigcup_{\mathscr{U}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}} \mathfrak{g}\text{-Der}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$.

— IV. A necessary and sufficient condition for the set-valued map $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ to be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is that, for each $\left(\{\zeta\}, \mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}\right) \in \times_{\alpha \in I_3^*} \mathscr{P}(\Omega)$ such that $\{\zeta\} \subset \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$, it satisfies:

- I. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\Omega) = \Omega$,
- II. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\big(\mathscr{U}_{\mathfrak{g}} \cup \{\zeta\}\big)$,
- III. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}}) \supseteq \mathscr{U}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}})$,
- IV. $\mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{U}_{\mathfrak{g}} \cap \mathscr{V}_{\mathfrak{g}}) = \bigcap_{\mathscr{W}_{\mathfrak{g}} = \mathscr{U}_{\mathfrak{g}}, \mathscr{V}_{\mathfrak{g}}} \mathfrak{g}\text{-Cod}_{\mathfrak{g}}(\mathscr{W}_{\mathfrak{g}})$.

Hence, this study has several advantages. Indeed, the study offers very nice features for the passage from the essential properties of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-coderived operators to the essential properties of $\mathfrak{T}_{\mathfrak{a}}$-derived and $\mathfrak{T}_{\mathfrak{a}}$-coderived operators, respectively, in $\mathscr{T}_{\mathfrak{a}}$-spaces. Moreover, the study offers $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived structures as $\left(\Omega, \mathfrak{g}\text{-Der}_{\mathfrak{g}}\right)$ which are coarser than $\mathfrak{T}_{\mathfrak{g}}$-derived structures as $(\Omega, \mathrm{der}_{\mathfrak{g}})$ and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived structures as $\left(\Omega, \mathfrak{g}\text{-Cod}_{\mathfrak{g}}\right)$ which are finer than $\mathfrak{T}_{\mathfrak{g}}$-coderived structures as $(\Omega, \mathrm{cod}_{\mathfrak{g}})$. Hence, such $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-structures can be considered as a means of handling certain problems in Functional Analysis. Accordingly, our study offers $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-(derived, coderived) structures from which many other novel propositions can be deduced by means of these conditions by purely logical processes. Thus, the construction of a purely deductive theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-coderived operators a step further is made possible, and the discussion of this paper ends here.

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