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On Quasi-para-Sasakian Structures On 5-Dimensions

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Research Article

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Abstract

In this study, we investigate the existence of quasi-para-Sasakian structures on five dimensional nilpotent Lie algebras. There are six non-abelian nilpotent Lie algebras. We show that quasi-para-Sasakian structures exist only on one of these algebras. Quasi-para-Sasakian structures correspond to the class $\mathbb{G}_5 \oplus \mathbb{G}_8$ in the classification of almost paracontact metric structures. We show that a quasi-para-Sasakian structure on a five dimensional nilpotent Lie algebra is either in $\mathbb{G}_5$ or $\mathbb{G}_8$.

Keywords: Almost Paracontact Metric Structure, 5-dimensional Nilpotent Lie Algebra, Quasi-para-Sasakian Structure

5-Boyutta Kuasi-para-Sasaki Yapılar Üzerine

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Öz

Bu çalışmada 5 boyutlu nilpotent Lie cebirleri üzerinde kuasi-para-Sasaki yapıların varlığı incelenmiştir. Birbirine izomorf olmayan altı tane Abelyen olmayan nilpotent Lie cebri vardır. Kuasi-para-Sasaki yapıların bu Lie cebirlerinden sadece birinde olduğu gösterilmiştir. Kuasi-para-Sasaki yapılar hemen-hemen parakontak metrik yapıların sınıflandırılmasına göre $\mathbb{G}_5 \oplus \mathbb{G}_8$ sınıfına karşılık gelmektedir. 5 boyutlu nilpotent bir Lie cebri üzerinde kuasi-para-Sasaki bir yapının $\mathbb{G}_5$ veya $\mathbb{G}_8$ sınıfından olduğu kanıtlanmıştır.

Anahtar Kelimeler: Hemen-hemen Parakontak Metrik Yapı, 5-boyutlu Nilpotent Lie Cebri, Kuasi-para-Sasaki Yapı

Introduction

Almost paracontact structures on differentiable manifolds were introduced by [1] and after that many authors have made contribution, see for example [2–8] and references therein. Almost paracontact metric manifolds were classified according to symmetry properties of the structure tensor and there are 12 basic classes and thus 2^{12} classes of almost paracontact metric structures. Definitions of each basic class and projections onto each subspace are obtained in [4] and [3]. An almost paracontact metric manifold is called quasi-para-Sasakian if the fundamental 2-form is closed and the structure is normal. It is known that the class of quasi-para-Sasakian structures is $\mathbb{G}_5 \oplus \mathbb{G}_8$ according to the classification in [3]. Our aim is to study the existence of quasi-para-Sasakian structures on non-abelian five dimensional nilpotent Lie algebras classified in [9]. We prove that only one of the non-isomorphic non-abelian nilpotent Lie algebras admits quasi-para-Sasakian structures and a quasi-para-Sasakian structure on a five dimensional nilpotent Lie algebra is either in $\mathbb{G}_5$ or $\mathbb{G}_8$. There is no quasi-para-Sasakian structure which is in $\mathbb{G}_5 \oplus \mathbb{G}_8$ properly. For the existence of some other classes of almost paracontact metric structures on 5-dimensional nilpotent Lie algebras, see [6]. For the almost contact case, see [10, 11].

Preliminaries

In this section we give necessary preliminary information. One may also refer to [3] for definitions of basic concepts. An almost paracontact structure on an odd dimensional differentiable manifold M^{2n+1} is an ordered triple (φ, ξ, η), where φ is an endomorphism, ξ a vector field and η a 1-form such that

$$\varphi^2 = I - \eta \otimes \xi, \qquad \eta(\xi) = 1, \varphi(\xi) = 0, \tag{1}$$

there is a distribution

$$\mathbb{D} : p \in M \longrightarrow \mathbb{D}_p = Ker\eta.$$

M is called an almost paracontact manifold. If an almost paracontact manifold M admits a semi-Riemannian metric g satisfying

$$g(\varphi(u), \varphi(v)) = -g(u, v) + \eta(u)\eta(v) \tag{2}$$

for all $u, v \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the set of smooth vector fields on M, in this case M is called an almost paracontact metric manifold. The 2-form defined by

$$\Phi(u, v) = g(\varphi u, v)$$

for all $u, v \in \mathfrak{X}(M)$, is said to be the fundamental 2-form. We denote the vector fields and tangent vectors by letters u, v, w. Let F be the tensor defined by

$$F(u, v, w) = g((\nabla_u \varphi)(v), w), \tag{3}$$

for all $u, v, w \in T_pM$, where T_pM is the tangent space at p and ∇ is the covariant derivative of g. Then F has properties

$$F(u, v, w) = -F(u, w, v), \tag{4}$$

$$F(u, \varphi v, \varphi w) = F(u, v, w) + \eta(v)F(u, w, \xi) - \eta(w)F(u, v, \xi). \tag{5}$$

The Lee forms associated with F are

$$\theta(u) = g^{ij}F(e_i, e_j, u), \qquad \theta^*(u) = g^{ij}F(e_i, \varphi e_j, u), \qquad \omega(u) = F(\xi, \xi, u),$$

where $u \in T_pM$, $\{e_i, \xi\}$ is a basis for T_pM and g^{ij} is the inverse of the matrix g_{ij}. Let $\mathcal{F}$ be the set of $(0,3)$ tensors over T_pM which satisfy (4), (5). $\mathcal{F}$ is the direct sum of twelve subspaces $\mathbb{G}_i$, $i = 1, \ldots, 12$, see [3,4]. We give definitions of classes we use.

$$\begin{aligned}
&\mathbb{G}_5 : F(u,v,w) = \frac{\theta_F(\xi)}{2n}\{g(\varphi u, \varphi w)\eta(v) - g(\varphi u, \varphi v)\eta(w)\} \\
&\mathbb{G}_8 : F(u,v,w) = -\eta(v)F(u,w,\xi) + \eta(w)F(u,v,\xi), \\
&\qquad F(u,v,\xi) = F(v,u,\xi) = -F(\varphi u, \varphi v, \xi), \;\; \theta_F(\xi) = 0
\end{aligned} \tag{6}$$

An almost paracontact metric manifold is in the class $\mathbb{G}_i \oplus \mathbb{G}_j$ if the tensor F is in the class $\mathbb{G}_i \oplus \mathbb{G}_j$ over T_pM for all $p \in M$. An almost paracontact metric manifold is normal if [3,12]

$$\varphi(\nabla_u \varphi)v - (\nabla_{\varphi u}\varphi)v + (\nabla_u \eta)(v)\xi = 0,$$

or equivalently,

$$F(u,v,\varphi w) + F(\varphi u, v, w) + F(u, \varphi v, \eta(w)\xi) = 0.$$

An almost paracontact metric manifold is said to be quasi-para-Sasakian if the fundamental 2-form is closed, that is,

$$d\Phi(u,v,w) = F(u,v,w) + F(v,w,u) + F(w,u,v) = 0 \tag{7}$$

and the structure is normal. In this case, the characteristic vector field ξ is Killing and the class of quasi-para-Sasakian manifolds is $\mathbb{G}_5 \oplus \mathbb{G}_8$ [3]. In addition, for a quasi-para-Sasakian manifold it is known that

$$(\nabla_u \varphi)(v) = -g(\nabla_u \xi, \varphi v)\xi - \eta(v)\varphi(\nabla_u \xi), \tag{8}$$

or equivalently,

$$F(u,v,w) = -g(\nabla_u \xi, \varphi v)\eta(w) + \eta(v) g(\nabla_u \xi, \varphi w), \tag{9}$$

see [7].

Quasi-para-Sasakian Structures on $\mathfrak{g}_i$

Each left invariant almost paracontact metric structure (φ, ξ, η, g) on a connected odd dimensional Lie group G induces an almost paracontact metric structure on the Lie algebra $\mathfrak{g}$ of G. We use the same notation for the structure on the Lie algebra. According to the classification of 5 dimensional nilpotent Lie algebras in [9], there are six non-abelian non-isomorphic nilpotent algebras $\mathfrak{g}_i$ with basis $\{e_1, \ldots, e_5\}$

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and non-zero brackets:

$\mathfrak{g}_1 : [e_1, e_2] = e_5, [e_3, e_4] = e_5$

$\mathfrak{g}_2 : [e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_4] = e_5$

$\mathfrak{g}_3 : [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5$

$\mathfrak{g}_4 : [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5$

$\mathfrak{g}_5 : [e_1, e_2] = e_4, [e_1, e_3] = e_5$

$\mathfrak{g}_6 : [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_2, e_3] = e_5.$

First note the following.

Proposition 1. Let $(M, \varphi, \xi, \eta, g)$ be a quasi-para-Sasakian manifold, that is $M \in \mathbb{G}_5 \oplus \mathbb{G}_8$. If $\theta_F(\xi) = 0$, then M is in $\mathbb{G}_8$.

Proof. Let $(M, \varphi, \xi, \eta, g)$ be quasi-para-Sasakian. Then (9) implies

$$\begin{aligned} -\eta(v)F(u, w, \xi) + \eta(w)F(u, v, \xi) &= -\eta(v)\{-g(\nabla_u\xi, \varphi(w))\} + \eta(w)\{-g(\nabla_u\xi, \varphi(v))\} \\ &= F(u, v, w). \end{aligned} \tag{10}$$

Since ξ is Killing, by (9)

$$F(\varphi u, \varphi v, \xi) = -g(\nabla_{\varphi u}\xi, \varphi^2 v) = -g(\nabla_{\varphi u}\xi, v) = g(\nabla_v\xi, \varphi u) = -F(v, u, \xi), \tag{11}$$

since $\varphi(\nabla_u\xi) = \nabla_{\varphi u}\xi$ [7],

$$F(u, v, \xi) = -g(\nabla_u\xi, \varphi v) = g(\varphi(\nabla_u\xi), v) = g(\nabla_{\varphi u}\xi, v) = -g(\nabla_v\xi, \varphi u) = F(v, u, \xi). \tag{12}$$

If $\theta_F(\xi) = 0$, then defining relations (6) hold and as a consequence the quasi-para-Sasakian manifold is in $\mathbb{G}_8$.

We study the existence of quasi-para-Sasakian structures on 5 dimensional nilpotent Lie algebras and deduce the result below.

Theorem 1. A five dimensional nilpotent Lie algebra has a quasi-para-Sasakian structure if and only if its Lie algebra is isomorphic to $\mathfrak{g}_1$. Moreover, a quasi-para-Sasakian structure on $\mathfrak{g}_1$ is either in the class $\mathbb{G}_5$ or $\mathbb{G}_8$.

Examples of quasi-para-Sasakian structure on $\mathfrak{g}_1$ are given in the proof of Theorem 1.

Proof. For the proof of Theorem 1, we investigate each Lie algebra separately. Assume that (φ, ξ, η, g) is a left invariant quasi-para-Sasakian structure on a connected Lie group G_i with corresponding Lie algebra $\mathfrak{g}_i$, $i = 1, \ldots, 6$ and g is the metric such that the basis $\{e_1, \ldots, e_5\}$ is g-orthonormal and $g(e_i, e_i) = \epsilon_i = \pm 1$. Denote the corresponding quasi-para-Sasakian structure on $\mathfrak{g}_i$ by the same quadruple. The Levi-Civita covariant derivatives and also the subspaces of Killing vector fields are evaluated in [6].

The algebra $\mathfrak{g}_1$: Since ξ is Killing, $\xi = \xi_5 e_5$ [6] and $g(\xi,\xi) = \xi_5^2\epsilon_5 = 1$ implies $\epsilon_5 = 1$, $\xi_5^2 = 1$. Let the endomorphism φ of the quasi-para-Sasakian structure be given by

$$\varphi(e_1) = a_1e_1 + \ldots + a_5e_5, \;\; \varphi(e_2) = b_1e_1 + \ldots + b_5e_5,$$
$$\varphi(e_3) = c_1e_1 + \ldots + c_5e_5, \;\; \varphi(e_4) = d_1e_1 + \ldots + d_5e_5, \;\; \varphi(e_5) = 0.$$

Since Φ is a 2-form, $\Phi(e_i, e_i) = g(\varphi(e_i), e_i) = 0$, and we have $a_1 = b_2 = c_3 = d_4 = 0$. Also $g(\varphi(e_i), e_5) = -g(e_i, \varphi(e_5)) = 0$ implies $a_5 = b_5 = c_5 = d_5 = 0$. We evaluate $F(e_i, e_j, e_k)$ both from (9) and (3) by using the Levi-Civita covariant derivatives in [6] and find the possible nonzero structure constants:

$$F(e_1,e_1,e_5) = \frac{1}{2}a_2 = -F(e_1,e_5,e_1), \;\; F(e_1,e_3,e_5) = \frac{1}{2}c_2 = -F(e_1,e_5,e_3),$$
$$F(e_1,e_4,e_5) = \frac{1}{2}d_2 = -F(e_1,e_5,e_4), \;\; F(e_2,e_2,e_5) = -\frac{1}{2}b_1 = -F(e_2,e_5,e_2),$$
$$F(e_2,e_3,e_5) = -\frac{1}{2}c_1 = -F(e_2,e_5,e_3), \;\; F(e_2,e_4,e_5) = -\frac{1}{2}d_1 = -F(e_2,e_5,e_4),$$
$$F(e_3,e_1,e_5) = \frac{1}{2}a_4 = -F(e_3,e_5,e_1), \;\; F(e_3,e_2,e_5) = \frac{1}{2}b_4 = -F(e_3,e_5,e_2),$$
$$F(e_3,e_3,e_5) = \frac{1}{2}c_4 = -F(e_3,e_5,e_3), \;\; F(e_4,e_1,e_5) = -\frac{1}{2}a_3 = -F(e_4,e_5,e_1),$$
$$F(e_4,e_2,e_5) = -\frac{1}{2}b_3 = -F(e_4,e_5,e_2), \;\; F(e_4,e_4,e_5) = -\frac{1}{2}d_3 = -F(e_4,e_5,e_4).$$

From (9), it is obvious that $F(e_5, e_j, e_k) = 0$. On the other hand evaluating $F(e_5, e_j, e_k)$ from (3) and comparing with (9) implies

$$a_2 + \epsilon_1\epsilon_2 b_1 = 0, \tag{13}$$
$$a_4 + \epsilon_2\epsilon_3 b_3 = 0, \tag{14}$$
$$\epsilon_1\epsilon_4 a_4 + b_3 = 0, \tag{15}$$
$$-a_3 + \epsilon_2\epsilon_4 b_4 = 0, \tag{16}$$
$$-\epsilon_1\epsilon_3 a_3 + b_4 = 0, \tag{17}$$
$$and$$
$$c_4 + \epsilon_3\epsilon_4 d_3 = 0, \tag{18}$$
$$-c_1 + \epsilon_2\epsilon_4 d_2 = 0, \tag{19}$$
$$-\epsilon_1\epsilon_3 c_1 + d_2 = 0, \tag{20}$$
$$c_2 + \epsilon_1\epsilon_4 d_1 = 0, \tag{21}$$
$$\epsilon_2\epsilon_3 c_2 + d_1 = 0. \tag{22}$$

In addition calculating (7) for basis elements gives

$$c_2 = a_4 = a, \;\; d_2 = -a_3 = b, \;\; c_1 = -b_4 = c, \;\; d_1 = b_3 = d. \tag{23}$$

The normality condition is satisfied, it does not give any restriction on structure constants. Now we show that a quasi-para-Sasakian structure in $\mathfrak{g}_1$ is either in the class $\mathbb{G}_5$ or $\mathbb{G}_8$. There is no quasi-para-Sasakian structure which is strictly in $\mathbb{G}_5 \oplus \mathbb{G}_8$, that is

$$\mathbb{G}_5 \oplus \mathbb{G}_8 = \mathbb{G}_5 \cup \mathbb{G}_8$$

for quasi-para-Sasakian structures in $\mathfrak{g}_1$. By direct calculation,

$$\theta_F(\xi) = g^{ij} F(e_i, e_j, \xi) = \frac{\xi_5}{2}\{\epsilon_1 a_2 - \epsilon_2 b_1 + \epsilon_3 c_4 - \epsilon_4 d_3\}, \tag{24}$$

where $\{e_1, e_2, \ldots, \xi = \xi_5 e_5\}$ is the g-orthonormal basis. Comparing (14) and (15), we get $a_4 = -\epsilon_2\epsilon_3 b_3 = -\epsilon_1\epsilon_4 b_3$, thus $\epsilon_2\epsilon_3 = \epsilon_1\epsilon_4$. Multiply both sides by $\epsilon_2\epsilon_4$. Then we also have $\epsilon_1\epsilon_2 = \epsilon_3\epsilon_4$. By (13) and (18), if $\epsilon_1\epsilon_2 = \epsilon_3\epsilon_4 = 1$, then $a_2 = -b_1$, $c_4 = -d_3$ and the Lee form (24) is

$$\theta_F(\xi) = \frac{\xi_5}{2}\{(\epsilon_1 + \epsilon_2)a_2 + (\epsilon_3 + \epsilon_4)c_4\}. \tag{25}$$

If $\epsilon_1\epsilon_2 = \epsilon_3\epsilon_4 = -1$, then $a_2 = b_1$, $c_4 = d_3$ and (24) becomes

$$\theta_F(\xi) = \frac{\xi_5}{2}\{(\epsilon_1 - \epsilon_2)a_2 + (\epsilon_3 - \epsilon_4)c_4\}. \tag{26}$$

We know that $\epsilon_5 = 1$. For other ϵ_i there are six cases: Since $\epsilon_1\epsilon_2 = \epsilon_3\epsilon_4 = \pm 1$ and the signature is $(3, 2)$,

1. $\epsilon_1 = 1$, $\epsilon_2 = 1$, $\epsilon_3 = -1$, $\epsilon_4 = -1$. In this case, $\theta_F(\xi) = \xi_5(a_2 - c_4)$. By (13–23), we have $a_2 = -b_1$, $c_4 = -d_3$, $c_2 = a_4 = d_1 = b_3 = a$ and $-a_3 = d_2 = b_4 = -c_1 = b$ and thus

   $$\varphi(e_1) = a_2 e_2 - b e_3 + a e_4, \;\; \varphi(e_2) = -a_2 e_1 + a e_3 + b e_4,$$
   $$\varphi(e_3) = -b e_1 + a e_2 + c_4 e_4, \;\; \varphi(e_4) = a e_1 + b e_2 - c_4 e_3, \;\; \varphi(e_5) = 0.$$

   From (1), $\varphi^2(e_1) = e_1$ and this implies

   $$\begin{aligned} &-a_2^2 + b^2 + a^2 = 1, \\ &a(a_2 - c_4) = 0, \\ &b(a_2 - c_4) = 0. \end{aligned} \tag{27}$$

   If $a_2 - c_4 = 0$, then $\theta_F(\xi) = \xi_5(a_2 - c_4) = 0$ and by Proposition 1, a quasi-para-Sasakian structure is in $\mathbb{G}_8$. If $a_2 - c_4 \neq 0$, then equations (27) imply $a = b = 0$ and $-a_2^2 = 1$, which can not hold. So for these ϵ_i, a quasi-para Sasakian structure can not be in the class $\mathbb{G}_5$. In this case a quasi-para Sasakian structure has the property that $\theta_F(\xi) = 0$ and is in $\mathbb{G}_8$.

2. $\epsilon_1 = -1$, $\epsilon_2 = -1$, $\epsilon_3 = 1$, $\epsilon_4 = 1$ Similar to case 1, a quasi-para Sasakian structure is in the class $\mathbb{G}_8$.

3. $\epsilon_1 = 1$, $\epsilon_2 = -1$, $\epsilon_3 = 1$, $\epsilon_4 = -1$ $\theta_F(\xi) = \xi_5(a_2 + c_4)$. Equations (13–23) yield $a_2 = b_1$,

$c_4 = d_3$, $c_2 = a_4 = d_1 = b_3 = a$, $-a_3 = d_2 = -b_4 = c_1 = b$ and endomorphism φ is of the form

$$\varphi(e_1) = a_2e_2 - be_3 + ae_4, \;\; \varphi(e_2) = a_2e_1 + ae_3 - be_4,$$
$$\varphi(e_3) = be_1 + ae_2 + c_4e_4, \;\; \varphi(e_4) = ae_1 + be_2 + c_4e_3, \;\; \varphi(e_5) = 0.$$

Then $\varphi^2(e_1) = e_1$ implies

$$a_2^2 - b^2 + a^2 = 1, \quad a(a_2 + c_4) = 0, \quad b(a_2 + c_4) = 0. \tag{28}$$

If $a_2+c_4 = 0$, then by Proposition 1, the quasi-para-Sasakian structure is in $\mathbb{G}_8$. If $a_2+c_4 \neq 0$, then by (28), $a = b = 0$ and $a_2^2 = 1$. Since $\varphi^2(e_3) = e_3$, we also have $c_4^2 = 1$. Since $a_2 + c_4 \neq 0$, a_2 and c_4 are both equal to 1 or both equal to -1. Then the quasi-para-Sasakian structure (φ, ξ, η, g) such that $\epsilon_1 = 1$, $\epsilon_2 = -1$, $\epsilon_3 = 1$, $\epsilon_4 = -1$, $\epsilon_5 = 1$, $\xi = e_5$, $\eta = e^5$, $\varphi(e_1) = e_2$, $\varphi(e_2) = e_1$, $\varphi(e_3) = e_4$, $\varphi(e_4) = e_3$, $\varphi(e_5) = 0$ satisfies the defining relation of the class $\mathbb{G}_5$. Also for $a_2 = c_4 = -1$, structure (φ, ξ, η, g), where $\epsilon_1 = 1$, $\epsilon_2 = -1$, $\epsilon_3 = 1$, $\epsilon_4 = -1$, $\epsilon_5 = 1$, $\xi = e_5$, $\eta = e^5$, $\varphi(e_1) = -e_2$, $\varphi(e_2) = -e_1$, $\varphi(e_3) = -e_4$, $\varphi(e_4) = -e_3$, $\varphi(e_5) = 0$ is in $\mathbb{G}_5$. Similarly for cases below, a quasi-para Sasakian structure is either in the class $\mathbb{G}_8$ or $\mathbb{G}_5$.

4. $\epsilon_1 = -1$, $\epsilon_2 = 1$, $\epsilon_3 = 1$, $\epsilon_4 = -1$

5. $\epsilon_1 = 1$, $\epsilon_2 = -1$, $\epsilon_3 = -1$, $\epsilon_4 = 1$

6. $\epsilon_1 = -1$, $\epsilon_2 = 1$, $\epsilon_3 = -1$, $\epsilon_4 = 1$ Now we give an example of a quasi-para-Sasakian structure in $\mathbb{G}_8$. The quasi-para Sasakian structure (φ, ξ, η, g) satisfying $\epsilon_1 = 1$, $\epsilon_2 = 1$, $\epsilon_3 = -1$, $\epsilon_4 = -1$, $\epsilon_5 = 1$, $\xi = e_5$, $\eta = e^5$, $\varphi(e_1) = e_4$, $\varphi(e_2) = e_3$, $\varphi(e_3) = e_2$, $\varphi(e_4) = e_1$, $\varphi(e_5) = 0$ satisfies the defining relation of the class $\mathbb{G}_8$, so in $\mathfrak{g}_1$, there are quasi-para Sasakian structures of type $\mathbb{G}_5$ or $\mathbb{G}_8$, however there are no quasi-para Sasakian structures which contain parts from both $\mathbb{G}_5$ and $\mathbb{G}_8$.

The algebra $\mathfrak{g}_2$: Since ξ is Killing, $\xi = \xi_5e_5$ [6] and $g(\xi, \xi) = \xi_5^2\epsilon_5 = 1$ implies $\epsilon_5 = 1$, $\xi_5^2 = 1$. Endomorphism φ of the quasi-para-Sasakian structure is of the form

$$\varphi(e_1) = a_1e_1 + \ldots + a_5e_5, \;\; \varphi(e_2) = b_1e_1 + \ldots + b_5e_5, \;\; \varphi(e_3) = c_1e_1 + \ldots + c_5e_5,$$
$$\varphi(e_4) = d_1e_1 + \ldots + d_5e_5, \;\; \varphi(e_5) = 0$$

and $a_1 = b_2 = c_3 = d_4 = 0$ since $g(\varphi(e_i), e_i) = 0$. In addition $g(\varphi(e_i), e_5) = -g(e_i, \varphi(e_5)) = 0$ gives $a_5 = b_5 = c_5 = d_5 = 0$. We evaluate the possible nonzero structure constants $F(e_i, e_j, e_k)$ of the tensor

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F by (9):

$$F(e_1,e_1,e_5)=\frac{1}{2}a_3=-F(e_1,e_5,e_1),\ \ F(e_1,e_2,e_5)=\frac{1}{2}b_3=-F(e_1,e_5,e_2),$$
$$F(e_1,e_4,e_5)=\frac{1}{2}d_3=-F(e_1,e_5,e_4),\ \ F(e_2,e_1,e_5)=\frac{1}{2}a_4=-F(e_2,e_5,e_1),$$
$$F(e_2,e_2,e_5)=\frac{1}{2}b_4=-F(e_2,e_5,e_2),\ \ F(e_2,e_3,e_5)=\frac{1}{2}c_4=-F(e_2,e_5,e_3),$$
$$F(e_3,e_2,e_5)=-\frac{1}{2}b_1=-F(e_3,e_5,e_2),\ \ F(e_3,e_3,e_5)=-\frac{1}{2}c_1=-F(e_3,e_5,e_3),$$
$$F(e_3,e_4,e_5)=-\frac{1}{2}d_1=-F(e_3,e_5,e_4),\ \ F(e_4,e_1,e_5)=-\frac{1}{2}a_2=-F(e_4,e_5,e_1),$$
$$F(e_4,e_3,e_5)=-\frac{1}{2}c_2=-F(e_4,e_5,e_3),\ \ F(e_4,e_4,e_5)=-\frac{1}{2}d_2=-F(e_4,e_5,e_4).$$

Now from (7) we get

$$0=F(e_1,e_2,e_5)+F(e_2,e_5,e_1)+F(e_5,e_1,e_2)=\frac{1}{2}\{b_3-a_4\},$$
$$0=F(e_1,e_4,e_5)+F(e_4,e_5,e_1)+F(e_5,e_1,e_4)=\frac{1}{2}\{d_3+a_2\},$$
$$0=F(e_2,e_3,e_5)+F(e_3,e_5,e_2)+F(e_5,e_2,e_3)=\frac{1}{2}\{c_4+b_1\},$$
$$0=F(e_3,e_4,e_5)+F(e_4,e_5,e_3)+F(e_5,e_3,e_4)=\frac{1}{2}\{-d_1+c_2\},$$

thus $b_3=a_4$, $d_3=-a_2$, $c_4=-b_1$ and $c_2=d_1$. Set $b_3=a_4=a$, $d_3=-a_2=b$, $c_4=-b_1=c$, $c_2=d_1=d$. Then

$$\varphi(e_1)=-be_2+a_3e_3+ae_4,\ \ \varphi(e_2)=-ce_1+ae_3+b_4e_4,$$
$$\varphi(e_3)=c_1e_1+de_2+ce_4,\ \ \varphi(e_4)=de_1+d_2e_2+be_3,\ \ \varphi(e_5)=0.$$

We evaluate $F(e_i,e_j,e_k)$ from (3) and compare with (9):

$$0=F(e_1,e_2,e_4)=g((\nabla_{e_1}\varphi)(e_2),e_4)=-\frac{c}{2}\epsilon_4,$$
$$0=F(e_1,e_2,e_1)=g((\nabla_{e_1}\varphi)(e_2),e_1)=-\frac{c_1}{2}\epsilon_1$$

imply $c=0$, $c_1=0$. Similarly, since $F(e_1,e_1,e_2)=0$, $F(e_1,e_1,e_3)=0$, $F(e_1,e_3,e_4)=0$, $F(e_1,e_4,e_3)=0$, $F(e_2,e_1,e_2)=0$, we get $a_3=0$, $b=0$, $b_4=0$, $d_2=0$, $d=0$ respectively. Thus $\varphi(e_4)=0$ and (2) does not hold for $u=v=e_4$.

$$0=g(\varphi(e_4),\varphi(e_4))=-g(e_4,e_4)+\eta(e_4)\eta(e_4)=-\epsilon_4\neq 0.$$

Thus there is no quasi-para Sasakian structure on $\mathfrak{g}_2$.

The algebra $\mathfrak{g}_3$: In this Lie algebra if ξ is Killing, $\xi=\xi_5e_5$ [6] and $g(\xi,\xi)=\xi_5^2\epsilon_5=1$ implies ξ_5^2,

$\epsilon_5 = 1$. Similar to $\mathfrak{g}_1$ and $\mathfrak{g}_2$, φ is of the form

$$\varphi(e_1) = a_2e_2 + a_3e_3 + a_4e_4,\ \ \varphi(e_2) = b_1e_1 + b_3e_3 + b_4e_4,$$
$$\varphi(e_3) = c_1e_1 + c_2e_2 + c_4e_4,\ \ \varphi(e_4) = d_1e_1 + d_2e_2 + d_3e_3,\ \ \varphi(e_5) = 0.$$

By (9), nonzero structure constants $F(e_i, e_j, e_k)$ of the tensor F are

$$F(e_1, e_1, e_5) = \frac{1}{2}a_4 = -F(e_1, e_5, e_1),\ \ F(e_1, e_2, e_5) = \frac{1}{2}b_4 = -F(e_1, e_5, e_2),$$
$$F(e_1, e_3, e_5) = \frac{1}{2}c_4 = -F(e_1, e_5, e_4),\ \ F(e_2, e_1, e_5) = \frac{1}{2}a_3 = -F(e_2, e_5, e_1),$$
$$F(e_2, e_2, e_5) = \frac{1}{2}b_3 = -F(e_2, e_5, e_2),\ \ F(e_2, e_4, e_5) = \frac{1}{2}d_3 = -F(e_2, e_5, e_4),$$
$$F(e_2, e_5, e_2) = -\frac{1}{2}b_3 = -F(e_2, e_2, e_5),\ \ F(e_3, e_1, e_5) = -\frac{1}{2}a_2 = -F(e_3, e_5, e_1),$$
$$F(e_3, e_3, e_5) = -\frac{1}{2}c_2 = -F(e_3, e_5, e_3),\ \ F(e_3, e_4, e_5) = -\frac{1}{2}d_2 = -F(e_3, e_5, e_4),$$
$$F(e_4, e_2, e_5) = -\frac{1}{2}b_1 = -F(e_4, e_5, e_2),\ \ F(e_4, e_3, e_5) = -\frac{1}{2}c_1 = -F(e_4, e_5, e_3),$$
$$F(e_4, e_4, e_5) = -\frac{1}{2}d_1 = -F(e_4, e_5, e_4).$$

Then from (7) we get

$$0 = F(e_1, e_2, e_5) + F(e_2, e_5, e_1) + F(e_5, e_1, e_2) = \frac{1}{2}\{b_4 - a_3\},$$
$$0 = F(e_1, e_3, e_5) + F(e_3, e_5, e_1) + F(e_5, e_1, e_3) = \frac{1}{2}\{d_3 + a_2\},$$
$$0 = F(e_2, e_4, e_5) + F(e_4, e_5, e_2) + F(e_5, e_2, e_4) = \frac{1}{2}\{d_3 + b_1\},$$
$$0 = F(e_3, e_4, e_5) + F(e_4, e_5, e_3) + F(e_5, e_3, e_4) = \frac{1}{2}\{-d_2 + c_1\},$$

thus $b_3 = a_4$, $c_4 = -a_2$, $d_3 = -b_1$ and $d_2 = c_1$. Let $b_4 = a_3 = a$, $c_4 = -a_2 = b$, $d_3 = -b_1 = c$, $c_1 = d_2 = d$. Then

$$\varphi(e_1) = -be_2 + ae_3 + a_4e_4,\ \ \varphi(e_2) = -ce_1 + b_3e_3 + ae_4,$$
$$\varphi(e_3) = de_1 + c_2e_2 + be_4,\ \ \varphi(e_4) = d_1e_1 + de_2 + ce_3,\ \ \varphi(e_5) = 0.$$

By comparing (3) with (9) for basis elements, we have $a = a_4 = d = c_2 = b = 0$. Then $\varphi(e_1) = 0$ and (2) does not hold. As a result there is no quasi-para Sasakian structure on $\mathfrak{g}_3$.

The algebra $\mathfrak{g}_4$: The Killing characteristic vector field ξ is of the form $\xi = \xi_5e_5$ and $g(\xi, \xi) = \xi_5^2\epsilon_5 = 1$ gives ξ_5^2, $\epsilon_5 = 1$.

$$\varphi(e_1) = a_2e_2 + a_3e_3 + a_4e_4,\ \ \varphi(e_2) = b_1e_1 + b_3e_3 + b_4e_4,$$
$$\varphi(e_3) = c_1e_1 + c_2e_2 + c_4e_4,\ \ \varphi(e_4) = d_1e_1 + d_2e_2 + d_3e_3,\ \ \varphi(e_5) = 0.$$

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By (9),

$$
\begin{aligned}
&F(e_1,e_5,e_1)=-\frac{1}{2}a_4=-F(e_1,e_1,e_5),\;\; F(e_1,e_5,e_3)=-\frac{1}{2}c_4=-F(e_1,e_3,e_5),\\
&F(e_4,e_5,e_2)=\frac{1}{2}b_1=-F(e_4,e_2,e_5),\;\; F(e_4,e_5,e_3)=\frac{1}{2}c_1=-F(e_4,e_3,e_5),\\
&F(e_4,e_5,e_4)=\frac{1}{2}d_1=-F(e_4,e_4,e_5),\;\; F(e_1,e_2,e_5)=\frac{1}{2}b_4=-F(e_1,e_5,e_2).
\end{aligned}
$$

From (7),

$$
\begin{aligned}
&0=F(e_4,e_5,e_2)+F(e_5,e_2,e_4)+F(e_2,e_4,e_5)=\frac{1}{2}b_1,\\
&0=F(e_1,e_2,e_5)+F(e_2,e_5,e_1)+F(e_5,e_1,e_2)=\frac{1}{2}b_4,
\end{aligned}
$$

thus $b_1=0$, $b_4=0$ and $\varphi(e_2)=b_3e_3$. Also

$$
\begin{aligned}
&0=F(e_1,e_5,e_3)+F(e_5,e_3,e_1)+F(e_3,e_1,e_5)=-\frac{1}{2}c_4,\\
&0=F(e_4,e_5,e_3)+F(e_5,e_3,e_4)+F(e_3,e_4,e_5)=\frac{1}{2}c_1,
\end{aligned}
$$

thus $c_4=0$, $c_1=0$ and $\varphi(e_3)=c_2e_2$. (3) and (9) yield

$$
0=F(e_1,e_2,e_4)=g((\nabla_{e_1}\varphi)(e_2),e_4)=\frac{b_3}{2}\epsilon_4,
$$

thus $b_3=0$ and $\varphi(e_2)=0$. Then (2) is not satisfied and there is no quasi-para Sasakian structure on $\mathfrak{g}_4$.
The algebra $\mathfrak{g}_5$: Since ξ is Killing, $\xi=\xi_4e_4+\xi_5e_5$.

$$
\begin{aligned}
&\varphi(e_1)=a_1e_1+\ldots+a_5e_5,\;\; \varphi(e_2)=b_1e_1+\ldots+b_5e_5,\\
&\varphi(e_3)=c_1e_1+\ldots+c_5e_5,\;\; \varphi(e_4)=d_1e_1+\ldots+d_5e_5,\;\; \varphi(e_5)=f_1e_1+\ldots+f_5e_5.
\end{aligned}
$$

Since $g(\varphi(e_i),e_i)=0$, we have $a_1=b_2=c_3=d_4=f_5=0$. In addition, $0=g(\varphi(e_4),\xi)$ gives $d_5=0$ and $0=g(\varphi(e_5),\xi)$ implies $f_4=0$. We calculate $F(e_i,e_j,e_k)$ by (9) and by (3). By (3), we have

$$
F(e_2,e_1,e_i)=g(\epsilon_1\epsilon_4\frac{a_4}{2}e_1+\frac{1}{2}\{d_1e_1+d_2e_2+d_3e_3\},e_i).
$$

By (9), $F(e_2,e_1,e_i)=0$ for $i=1,2,3$. Comparing these two equations we get $a_4=\pm d_1$, $d_2=0$ and $d_3=0$ from $i=1,2,3$ respectively. Also by (9), $F(e_1,e_1,e_2)=F(e_1,e_1,e_3)=0$ and by (3), $F(e_1,e_1,e_2)=-\frac{a_4}{2}\epsilon_4$ and $F(e_1,e_1,e_3)=-\frac{a_5}{2}\epsilon_5$ and thus $a_4=a_5=0$. In addition $a_4=\pm d_1=0$. Thus $\varphi(e_4)=0$. The equation (2) for $u=v=e_4$ implies $0=-\epsilon_4+\xi_4^2$ and thus $\xi_4^2=\epsilon_4=1$. Now since $g(\xi,\xi)=\xi_4^2\epsilon_4+\xi_5^2\epsilon_5=1$, we have $\xi_5=0$. Since $\xi=\xi_4e_4$ and $0=g(\varphi(e_i),\xi)$, we get $b_4=c_4=0$. The equation (9) implies $F(e_3,e_1,e_1)=F(e_3,e_1,e_2)=F(e_3,e_1,e_3)=0$. Comparing with (3), we get $f_1=f_2=f_3=0$ and thus $\varphi(e_5)=0$. Thus (2) is not satisfied for $u=v=e_5$. So there is no quasi-para Sasakian structure on $\mathfrak{g}_5$.

The algebra $\mathfrak{g}_6$: Since ξ is Killing, $\xi = \xi_4 e_4 + \xi_5 e_5$.

$$\varphi(e_1) = a_1 e_1 + \ldots + a_5 e_5, \ \ \varphi(e_2) = b_1 e_1 + \ldots + b_5 e_5,$$
$$\varphi(e_3) = c_1 e_1 + \ldots + c_5 e_5, \ \ \varphi(e_4) = d_1 e_1 + \ldots + d_5 e_5, \ \ \ \varphi(e_5) = f_1 e_1 + \ldots + f_5 e_5.$$

Since $g(\varphi(e_i), e_i) = 0$, we have $a_1 = b_2 = c_3 = d_4 = f_5 = 0$. Also since $0 = g(\varphi(e_4), \xi)$, we get $d_5 = 0$ and $0 = g(\varphi(e_5), \xi)$ implies $f_4 = 0$. We calculate $F(e_i, e_j, e_k)$ by (9) and by (3). From (9), $F(e_1, e_2, e_1) = 0$ and comparing this with (3) implies $c_1 = 0$. Similarly, $F(e_1, e_2, e_3) = 0$ yields $b_4 = 0$. In addition,

$$\begin{aligned} F(e_2, e_1, e_1) &= F(e_2, e_1, e_2) = F(e_2, e_1, e_3) = F(e_1, e_3, e_1) = F(e_3, e_1, e_2) = F(e_2, e_3, e_1) \\ &= F(e_3, e_2, e_3) = F(e_4, e_2, e_1) = F(e_1, e_2, e_5) = F(e_5, e_1, e_3) = F(e_5, e_2, e_4) = 0 \end{aligned}$$

imply $a_3 = c_2 = a_5 = d_1 = d_2 = f_1 = f_3 = b_3 = c_5 = a_2 = c_4 = 0$ respectively. Then $\varphi(e_3) = 0$ and (2) does not hold. Thus there is no quasi-para Sasakian structure on $\mathfrak{g}_6$.

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