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UNITS IN $F(C_n \times Q_{12})$ AND $F(C_n \times D_{12})$

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Abstract. Let C_n, Q_n and D_n be the cyclic group, the quaternion group and the dihedral group of order n, respectively. Recently, the structures of the unit groups of the finite group algebras of 2-groups that contain a normal cyclic subgroup of index 2 have been studied. The dihedral groups $D_{2n}, n \geq 3$ and the generalized quaternion groups $Q_{4n}, n \geq 2$ also contain a normal cyclic subgroup of index 2. The structures of the unit groups of the finite group algebras FQ_{12}, FD_{12}, $F(C_2 \times Q_{12})$ and $F(C_2 \times D_{12})$ over a finite field F are well known. In this article, we continue this investigation and establish the structures of the unit groups of the group algebras $F(C_n \times Q_{12})$ and $F(C_n \times D_{12})$ over a finite field F of characteristic p containing p^k elements.

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1. Introduction

Let FG be the group algebra of a finite group G over a finite field F of characteristic p having $q = p^k$ elements. Let $U(FG)$ be the unit group of FG and let $J(FG)$ be the Jacobson radical of FG. If $V = 1 + J(FG)$, then $U(FG) \cong V \rtimes U(FG/J(FG))$ [16]. A good description of the structure of $U(FG)$ has applications in various areas like the group ring cryptography [10] and the combinatorial number theory [5], etc. This necessitates finding the explicit structure of $U(FG)$. A comprehensive review of the well-known properties of $U(FG)$ is given in [3].

If K is a normal subgroup of G then the natural group epimorphism $G \to G/K$ can be extended to an F-algebra epimorphism $FG \to F(G/K)$. The kernel of this epimorphism $\omega(K)$, is the ideal of FG generated by $\{k - 1 \mid k \in K\}$. In particular, if $K = G$, then the epimorphism $\epsilon : FG \to F$ given by $\epsilon(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g$ is called the augmentation mapping of FG and the ideal $\omega(G)$ is called the augmentation ideal of FG. Clearly, $FG/\omega(G) \cong F$.

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Let C_n, Q_n and D_n be the cyclic group, the quaternion group and the dihedral group of order n, respectively. The four non-isomorphic nonabelian groups of order 2^n which have a cyclic subgroup of index 2 are the dihedral group D_{2^n}, the generalized quaternion group Q_{2^n}, the semidihedral group SD_{2^n} and the modular group M_{2^n}, see [4]. Certain properties of the group of normalized units of the modular group algebras of these groups were studied in [1,2]. Moreover, the structures of the semisimple group algebras of these groups have been obtained in [17,21]. The groups $D_{2n}, n \geq 3$ and $Q_{4n}, n \geq 2$ also contain a normal cyclic subgroup of index 2 and the unit groups of the group algebras of these groups and their extensions have been extensively studied [6,7,9,11,13,14,16,17,18,21,23,25,26]. In this paper, we aim to contribute in this direction further by describing the structures of $U(F(C_n \times Q_{12}))$ and $U(F(C_n \times D_{12}))$.

There is only one nonabelian group of order $2p$, up to isomorphism, namely D_{2p} for any prime $p \geq 3$. The structure of $U(\mathbb{Z}_2 D_{2p})$ for an odd prime p is described in [11]. This was extended to a field containing 2^k elements in [13]. In [14], the structure of the centre of the maximal p-subgroup of $U(FD_{2p^n})$ for $n \geq 2$ is discussed. Further, by using an established isomorphism between FG and a certain ring of $n \times n$ matrices in conjunction with other techniques, Gildea [6] has obtained the order of $U(FD_{2p^n})$ for an odd prime p as $p^{2k(p^n-1)}(p^k-1)^2$ whereas in [7], he has proved that the centre of the maximal p-subgroup of $U(FD_{2p})$ is $C_p^{k(p+1)/2}$. The three nonabelian groups of order 12 are D_{12}, Q_{12} and A_4, the alternating group on 4 letters. The structures of the unit groups of FQ_{12} and $F(C_2 \times Q_{12})$ have been studied in [9,18,25,26]. Also, the unit groups of FD_{12} and $F(C_2 \times D_{12})$ have been obtained in [9,16,18,23,25], whereas $U(FA_4)$ is given in [8,24].

Throughout the paper, C_n^k is the direct product of k copies of C_n, F_n is the extension field of F of degree n and $GL(n, F)$ is the general linear group of degree n over F. For co-prime integers l and m, $ord_m(l)$ denotes the multiplicative order of l modulo m.

It is well known that, if G and H are groups, then $F(G \times H) \cong (FG)H$, the group ring of H over the ring FG, see [20, Chap 3, Page 134]. This result will be used frequently. Now we state here some of the Lemmas needed for our work.

Lemma 1.1. [15, Theorem 2.1] *Let F be a field of characteristic p having $q = p^k$ elements. If $(n,p) = 1$, where $n \in \mathbb{N}$, then*

$$FC_n \cong F \oplus \Big(\oplus_{l>1,\ l|n} F_{d_l}^{e_l} \Big),$$

where $d_l = ord_l(q)$ and $e_l = \frac{\phi(l)}{d_l}$.

Lemma 1.2. [25, Lemma 3.3] *Let F be a finite field of characteristic p with $|F| = q = p^k$. If $p \neq 2$, then*

$$FC_4 \cong \begin{cases} F^4, & \text{if } p \equiv 1 \bmod 4 \text{ or } n \text{ is even;} \\ F^2 \oplus F_2, & \text{if } p \equiv -1 \bmod 4 \text{ and } n \text{ is odd.} \end{cases}$$

Lemma 1.3. [15, Lemma 2.3] *Let F be a finite field of characteristic p with $|F| = q = p^k$. Then*

$$U(FC_{p^n}) \cong \begin{cases} C_p^{(p-1)k} \times C_{p^k-1}, & \text{if } n = 1; \\ \prod_{s=1}^{n} C_{p^s}^{h_s} \times C_{p^k-1}, & \text{otherwise,} \end{cases}$$

where $h_n = k(p-1)$ and $h_s = kp^{n-s-1}(p-1)^2$, for all s, $1 \leq s < n$.

Lemma 1.4. [22, Lemma 3.2] *Let F be a finite field of characteristic p with $|F| = q = p^k$. If $p \neq 2$, then*

$$U(FC_2^n) \cong C_{q-1}^{2^n}.$$

2. Units in $F(C_n \times Q_{12})$

The quaternion group $Q_{12} = \langle x, y \mid x^6 = 1, x^3 = y^2, xy = yx^5 \rangle$. The structures of the unit groups of the finite group algebras FQ_{12} and $F(C_2 \times Q_{12})$ have been studied in [9,18,25,26]. In this section, we establish the structure of the unit group of $F(C_n \times Q_{12})$. We shall use the following presentation of $C_n \times Q_{12}$:

$$C_n \times Q_{12} = \langle x, y, z \mid x^3 = y^4 = z^n = 1, xy = yx^2, xz = zx, yz = zy \rangle.$$

Theorem 2.1. *Let F be a finite field of characteristic 2 containing $q = 2^k$ elements and let $G = C_n \times Q_{12}$. If n is odd, then*

$$U(FG) \cong (C_2^{5nk} \times C_4^{nk}) \rtimes \Bigg(\Big(C_{q-1} \times GL(2,F) \Big) \times \Big(\prod_{l>1,\ l|n} \big(C_{q^{d_l}-1} \times GL(2, F_{d_l}) \big)^{e_l} \Big) \Bigg).$$

Proof. Since n is odd, FC_n is semisimple. Thus by Lemma 1.1,

$$\begin{aligned} FG &\cong (FC_n)Q_{12}, \\ &\cong \Big(F \oplus \big(\oplus_{l>1,\ l|n} F_{d_l}^{e_l} \big) \Big) Q_{12}, \\ &\cong FQ_{12} \oplus \Big(\oplus_{l>1,\ l|n} (F_{d_l} Q_{12})^{e_l} \Big). \end{aligned}$$

Now by [26, Theorem 3.2], $U(FQ_{12}) \cong (C_2^{5k} \times C_4^k) \rtimes \big(C_{q-1} \times GL(2,F) \big)$ and so

$$U(F_{d_l} Q_{12})^{e_l} \cong (C_2^{5\phi(l)k} \times C_4^{\phi(l)k}) \rtimes \big(C_{q^{d_l}-1} \times GL(2, F_{d_l}) \big)^{e_l}.$$

As $\sum_{l|n} \phi(l) = n$, so

$$U(FG) \cong (C_2^{5nk} \times C_4^{nk}) \rtimes \left(\left(C_{q-1} \times GL(2,F)\right) \times \left(\prod_{l>1,\ l|n} \left(C_{q^{d_l}-1} \times GL(2,F_{d_l})\right)^{e_l}\right)\right).$$

□

Theorem 2.2. *Let F be a finite field of characteristic 3 containing $q = 3^k$ elements and let $G = C_n \times Q_{12}$. Then*

$$U(FG) \cong (C_3^{6nk} \rtimes C_3^{2nk}) \rtimes U\big(F(C_n \times C_4)\big).$$

Proof. Let $K = \langle x \rangle$. Then $G/K \cong H = \langle y, z \rangle = C_n \times C_4$. Thus from the ring epimorphism $FG \to FH$ given by

$$\sum_{l=0}^{n-1} \sum_{j=0}^{3} \sum_{i=0}^{2} a_{i+3j+12l} x^i y^j z^l \mapsto \sum_{l=0}^{n-1} \sum_{j=0}^{3} \sum_{i=0}^{2} a_{i+3j+12l} y^j z^l,$$

we get a group epimorphism $\theta : U(FG) \to U(FH)$.

Further, from the inclusion map $FH \to FG$, we have $i : U(FH) \to U(FG)$ such that $\theta i = 1_{U(FH)}$. Therefore $U(FG)$ is a split extension of $U(FH)$ by $V = ker(\theta) = 1 + \omega(K)$. Hence

$$U(FG) \cong V \rtimes U(FH).$$

Now, let $u = \sum_{l=0}^{n-1} \sum_{j=0}^{3} \sum_{i=0}^{2} a_{i+3j+12l} x^i y^j z^l \in U(FG)$, then $u \in V$ if and only if $\sum_{i=0}^{2} a_i = 1$ and $\sum_{i=0}^{2} a_{i+3j} = 0$ for $j = 1, 2, \ldots, (4n-1)$. Therefore

$$V = \left\{1 + \sum_{l=0}^{n-1} \sum_{j=0}^{3} \sum_{i=1}^{2} (x^i - 1) b_{i+2j+8l} y^j z^l \mid b_i \in F\right\}$$

and $|V| = 3^{8nk}$. Since $\omega(K)^3 = 0$, $V^3 = 1$. We now study the structure of V in the following steps:

Step 1: $C_V(x) = \{v \in V \mid vx = xv\} \cong C_3^{6nk}$.

If $v = 1 + \sum_{l=0}^{n-1} \sum_{j=0}^{3} \sum_{i=1}^{2} (x^i - 1) b_{i+2j+8l} y^j z^l \in C_V(x) = \{v \in V \mid vx = xv\}$, then $vx - xv = \widehat{x} \sum_{l=0}^{n-1} \big((b_{3+8l} - b_{4+8l}) y + (b_{7+8l} - b_{8+8l}) y^3\big) z^i$. Thus $v \in C_V(x)$ if and only if $b_{j+8l} = b_{1+j+8l}$ for $j = 3, 7$ and $l = 0, 1, \ldots, n-1$. Hence

$$C_V(x) = \Big\{1 + \sum_{l=0}^{n-1} \sum_{j=0}^{1} \sum_{i=1}^{2} (x^i - 1) c_{i+2j+4l} y^{2j} z^l + \widehat{x} \sum_{l=0}^{n-1} \sum_{j=0}^{1} c_{4n+nj+l+1} y^{2j+1} z^l \mid c_i \in F\Big\}.$$

So $C_V(x)$ is an abelian subgroup of V and $|C_V(x)| = 3^{6nk}$. Therefore $C_V(x) \cong C_3^{6nk}$.

Step 2: Let T be the subset of V consisting of elements of the form

$$1 + \sum_{j=0}^{n-1} \Big(\widehat{x}(t_{j1} + t_{j2}y^2) + (x + 2x^2)(t_{j3}y + t_{j4}y^3)\Big)z^j,$$

where $t_{j_i} \in F$. Then T is an abelian subgroup of V and $T \cong C_3^{4nk}$.

Let

$$t_1 = 1 + \sum_{j=0}^{n-1} \Big(\widehat{x}(r_{j1} + r_{j2}y^2) + (x + 2x^2)(r_{j3}y + r_{j4}y^3)\Big)z^j \in T$$

and

$$t_2 = 1 + \sum_{j=0}^{n-1} \Big(\widehat{x}(s_{j1} + s_{j2}y^2) + (x + 2x^2)(s_{j3}y + s_{j4}y^3)\Big)z^j \in T.$$

Then

$$\begin{aligned} t_1t_2 =& 1 + \sum_{j=0}^{n-1} \Big(\widehat{x}\big((r_{j1} + s_{j1} + \gamma_1) + (r_{j2} + s_{j2} + \gamma_2)y^2\big) \\ &+ (x + 2x^2)\big((r_{j3} + s_{j3})y + (r_{j4} + s_{j4})y^3\big)\Big)z^j \in T, \end{aligned}$$

where

$$\gamma_1 = 2\sum_{i=0}^{n-1}(r_{j3}s_{i4} + r_{j4}s_{i3})z^i,$$
$$\gamma_2 = 2\sum_{i=0}^{n-1}(r_{j3}s_{i3} + r_{j4}s_{i4})z^i.$$

So T is an abelian subgroup of V and $|T| = 3^{4nk}$. Therefore $T \cong C_3^{4nk}$.

Now, let

$$\begin{aligned} c = 1 &+ \sum_{l=0}^{n-1}\sum_{j=0}^{1}\sum_{i=1}^{2}(x^i - 1)c_{i+2j+4l}y^{2j}z^l \\ &+ \widehat{x}\sum_{l=0}^{n-1}\sum_{j=0}^{1} c_{4n+nj+l+1}y^{2j+1}z^l \in C_V(x) \end{aligned}$$

and

$$t = 1 + \sum_{j=0}^{n-1} \Big(\widehat{x}(t_{j1} + t_{j2}y^2) + (x + 2x^2)(t_{j3}y + t_{j4}y^3)\Big)z^j \in T.$$

Then

$$t^{-1} = 1 + 2\sum_{j=0}^{n-1}\Big(\widehat{x}(t_{j1} + t_{j2}y^2) + (x + 2x^2)(t_{j3}y + t_{j4}y^3)\Big)z^j$$
$$+ 2\sum_{j=0}^{n-1}\widehat{x}\Big((t_{j3}^2 + t_{j4}^2)y^2 + 2t_{j3}t_{j4}\Big)z^{2j}$$

and

$$c^t = c + \widehat{x}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\Big((c_{1+4i} - c_{2+4i})t_{j3} + (c_{3+4i} - c_{4+4i})t_{j4}\Big)yz^{i+j}$$
$$+ \widehat{x}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\Big((c_{1+4i} - c_{2+4i})t_{j4} + (c_{3+4i} - c_{4+4i})t_{j3}\Big)yz^{i+j}.$$

Clearly, $c^t \in C_V(x)$. Thus T normalizes $C_V(x)$. Now, if $U = C_V(x) \cap T$, then

$$U = \Big\{1 + \widehat{x}\sum_{j=0}^{n-1}(t_{j1} + t_{j2}y^2)z^j \mid t_{ji} \in F\Big\} \cong C_3^{2nk}.$$

So for some subgroup $W \cong C_3^{2nk}$ of T, we have $T = U \times W$, $C_V(x) \cap W = 1$ and $|C_V(x)W| = |V| = 3^{8nk}$. Hence $V \cong C_V(x) \rtimes W \cong C_3^{6nk} \rtimes C_3^{2nk}$. □

For $U\big(F(C_n \times C_4)\big)$, we prove the following:

Theorem 2.3. *Let F be a finite field of characteristic 3 containing $q = 3^k$ elements and let $H = C_n \times C_4$, where $n = 3^r s$ such that $r \geq 0$ and $(3, s) = 1$. Then $U(FH)$ is isomorphic to*

(1) *If $3 \nmid n$, then*
 (a) $C_{q-1}^4 \times \big(\prod_{l>1,\ l|n} C_{q^{d_l}-1}^{4e_l}\big)$, *if $q \equiv 1$ mod 4;*
 (b) $C_{q-1}^2 \times C_{q^2-1} \times \big(\prod_{l>1,\ l|n}(C_{q^{d_l}-1}^{2e_l} \times C_{q^{d'_l}-1}^{e'_l})\big)$, *if $q \equiv -1$ mod 4.*

(2) *If $3|n$, then*
 (a) $C_{q-1}^4 \times \big(\prod_{l>1,\ l|s} C_{q^{d_l}-1}^{4e_l}\big) \times \big(\prod_{t=1}^{r} C_{3^t}^{4sn_t}\big)$, *if $q \equiv 1$ mod 4;*
 (b) $C_{q-1}^2 \times C_{q^2-1} \times \big(\prod_{l>1,\ l|s}(C_{q^{d_l}-1}^{2e_l} \times C_{q^{d'_l}-1}^{e'_l})\big) \times \big(\prod_{t=1}^{r}(C_{3^t}^{2sn_t} \times C_{3^t}^{sn'_t})\big)$, *if $q \equiv -1$ mod 4;*
 where $d'_l = ord_l(q^2)$, $e'_l = \frac{\phi(l)}{d'_l}$, $n_r = 2k$, $n_t = 4.3^{r-t-1}k$, for all $1 \leq t < r$ and $n'_t = 2n_t$, for all $1 \leq t \leq r$.

Proof. As $FH \cong (FC_4)C_n$, so using Lemma 1.2, we have

$$FH \cong \begin{cases} (FC_n)^4, & \text{if } q \equiv 1 \text{ mod } 4;\\ (FC_n)^2 \oplus F_2C_n, & \text{if } q \equiv -1 \text{ mod } 4.\end{cases}$$

(1) If $3 \nmid n$, then by Lemma 1.1,

$$FC_n \cong F \oplus \Big(\oplus_{l>1,\ l|n} F_{d_l}^{e_l} \Big)$$

and so

$$F_2C_n \cong F_2 \oplus \Big(\oplus_{l>1,\ l|n} F_{d_l'}^{e_l'} \Big),$$

where $d_l' = ord_l(q^2)$ and $e_l' = \frac{\phi(l)}{d_l'}$. Hence

$$FH \cong \begin{cases} F^4 \oplus (\oplus_{l>1,\ l|n} F_{d_l}^{4e_l}), & \text{if } q \equiv 1 \bmod 4; \\ F^2 \oplus F_2 \oplus \big(\oplus_{l>1,\ l|n} (F_{d_l}^{2e_l} \oplus F_{d_l'}^{e_l'}) \big), & \text{if } q \equiv -1 \bmod 4. \end{cases}$$

It is obvious that

$$d_l' = \begin{cases} d_l/2, & \text{if } d_l \text{ is even;} \\ d_l, & \text{if } d_l \text{ is odd.} \end{cases}$$

Also

$$e_l' = \begin{cases} 2e_l, & \text{if } d_l \text{ is even;} \\ e_l, & \text{if } d_l \text{ is odd.} \end{cases}$$

(2) If $3|n$, then by Lemma 1.1,

$$\begin{aligned} FC_n &\cong (FC_s)C_{3^r}, \\ &\cong \big(F \oplus (\oplus_{l>1,\ l|s} F_{d_l}^{e_l})\big)C_{3^r}, \\ &\cong FC_{3^r} \oplus \big(\oplus_{l>1,\ l|s} (F_{d_l}C_{3^r})^{e_l} \big). \end{aligned}$$

By Lemma 1.3,

$$U(FC_{3^r}) \cong C_{3^k-1} \times \Big(\prod_{t=1}^{r} C_{3^t}^{n_t} \Big)$$

where $n_r = 2k$, $n_t = 4.3^{r-t-1}k$. Thus

$$U(F_{d_l}C_{3^r})^{e_l} \cong C_{3^{d_l k}-1}^{e_l} \times \Big(\prod_{t=1}^{r} C_{3^t}^{\phi(l) n_t} \Big).$$

Since $\sum_{l|s} \phi(l) = s$,

$$U(FC_n) \cong C_{3^k-1} \times \Big(\prod_{l>1,\ l|s} C_{3^{d_l k}-1}^{e_l} \Big) \times \Big(\prod_{t=1}^{r} C_{3^t}^{sn_t} \Big)$$

and

$$U(F_2C_n) \cong C_{3^{2k}-1} \times \Big(\prod_{l>1,\ l|s} C_{3^{d_l' k}-1}^{e_l'} \Big) \times \Big(\prod_{t=1}^{r} C_{3^t}^{sn_t'} \Big),$$

where $n_t' = 2n_t$, for all $1 \le t \le r$. Hence the claim holds.

□

Theorem 2.4. *Let F be a finite field of characteristic $p > 3$ containing $q = p^k$ elements and let $G = C_n \times Q_{12}$ where $n = p^r s$, $r \geq 0$ such that $(p, s) = 1$. If $V = 1 + J(FG)$, then $U(FG)/V$ is isomorphic to*

(1) $C_{q-1}^4 \times GL(2,F)^2 \times \Big(\prod_{l>1,\ l|s} \big(C_{q^{d_l}-1}^4 \times GL(2,F_{d_l})^2\big)^{e_l}\Big)$, *if $q \equiv 1, 5$ mod 12;*

(2) $C_{q-1}^2 \times C_{q^2-1} \times GL(2,F)^2 \times \Big(\prod_{l>1,\ l|s} \big(C_{q^{d_l}-1}^2 \times C_{q^{2d_l}-1} \times GL(2,F_{d_l})^2\big)^{e_l}\Big)$, *if $q \equiv -1, -5$ mod 12;*

where V is a group of exponent p^r and order $p^{12sk(p^r-1)}$.

Proof. Let $K = \langle z^s \rangle$. Then $G/K \cong H = C_s \times Q_{12}$. If $\theta : FG \to FH$ is the canonical ring epimorphism, then $J(FG) = ker(\theta)$, $FG/J(FG) \cong FH$ and $dim_F(J(FG)) = 12s(p^r - 1)$. Hence $U(FG) \cong V \rtimes U(FH)$, where $V = 1 + J(FG)$. Clearly, exponent of $V = p^r$ and $|V| = p^{12sk(p^r-1)}$.

By Lemma 1.1,

$$\begin{aligned} FH &\cong (FC_s)Q_{12}, \\ &\cong \big(F \oplus (\oplus_{l>1,\ l|s} F_{d_l}^{e_l})\big)Q_{12}, \\ &\cong FQ_{12} \oplus \big(\oplus_{l>1,\ l|s} (F_{d_l}Q_{12})^{e_l}\big). \end{aligned}$$

Now, by [25, Theorem 4.2],

$$FQ_{12} \cong \begin{cases} F^4 \oplus M(2,F)^2, & \text{if } q \equiv 1,5 \text{ mod } 12; \\ F^2 \oplus F_2 \oplus M(2,F)^2, & \text{if } q \equiv -1,-5 \text{ mod } 12, \end{cases}$$

and so

$$(F_{d_l}Q_{12})^{e_l} \cong \begin{cases} F_{d_l}^{4e_l} \oplus M(2,F_{d_l})^{2e_l}, & \text{if } q \equiv 1,5 \text{ mod } 12; \\ F_{d_l}^{2e_l} \oplus F_{2d_l}^{e_l} \oplus M(2,F_{d_l})^{2e_l}, & \text{if } q \equiv -1,-5 \text{ mod } 12. \end{cases}$$

□

In the above theorem, if $r = 0$, then we have the unit group of the semisimple group algebra FG given by

(1) $U(FG) \cong C_{q-1}^4 \times GL(2,F)^2 \times \Big(\prod_{l>1,\ l|n} \big(C_{q^{d_l}-1}^4 \times GL(2,F_{d_l})^2\big)^{e_l}\Big)$, if $q \equiv 1, 5$ mod 12.

(2) $U(FG) \cong C_{q-1}^2 \times C_{q^2-1} \times GL(2,F)^2 \times \Big(\prod_{l>1,\ l|n} \big(C_{q^{d_l}-1}^2 \times C_{q^{2d_l}-1} \times GL(2,F_{d_l})^2\big)^{e_l}\Big)$, if $q \equiv -1, -5$ mod 12.

3. Units in $F(C_n \times D_{12})$

The dihedral group $D_{12} = \langle x, y \mid x^6 = y^2 = 1, yx = x^5y\rangle$. The structure of the unit group of the finite group algebra FD_{12} has been studied in [9,16,25] whereas the structure of unit group of $F(C_2 \times D_{12})$ is described in [12,18,23]. In this section, we establish the structure of the unit group of $F(C_n \times D_{12})$. We shall use the following presentation of $C_n \times D_{12}$:

$$C_n \times D_{12} = \langle x, y, z \mid x^6 = y^2 = z^n = 1, yx = x^5y, xz = zx, yz = zy\rangle.$$

Theorem 3.1. *Let F be a finite field of characteristic 2 containing $q = 2^k$ elements and let $G = C_n \times D_{12}$. If n is odd, then*

$$U(FG) \cong C_2^{7nk} \rtimes \left(C_{q-1} \times GL(2,F) \times \Big(\prod_{l>1,\ l|n} \big(C_{q^{d_l}-1} \times GL(2,F_{d_l})\big)^{e_l}\Big)\right).$$

Proof. Since n is odd, FC_n is semisimple. Thus by Lemma 1.1,

$$\begin{aligned} FG &\cong (FC_n)D_{12}, \\ &\cong \big(F \oplus (\oplus_{l>1,\ l|n} F_{d_l}^{e_l})\big)D_{12}, \\ &\cong FD_{12} \oplus \big(\oplus_{l>1,\ l|n} (F_{d_l}D_{12})^{e_l}\big). \end{aligned}$$

Now by [16, Theorem 2.6],

$$U(FD_{12}) \cong C_2^{7k} \rtimes \Big(C_{q-1} \times GL(2,F)\Big)$$

and so

$$U(F_{d_l}D_{12})^{e_l} \cong C_2^{7\phi(l)k} \rtimes \Big(C_{q^{d_l}-1}^{e_l} \times GL(2,F_{d_l})^{e_l}\Big).$$

Since $\sum_{l|n} \phi(l) = n$,

$$U(FG) \cong C_2^{7nk} \rtimes \left(C_{q-1} \times GL(2,F) \times \Big(\prod_{l>1,\ l|n} \big(C_{q^{d_l}-1} \times GL(2,F_{d_l})\big)^{e_l}\Big)\right). \quad \square$$

Theorem 3.2. *Let F be a finite field of characteristic 3 containing $q = 3^k$ elements and let $G = C_n \times D_{12}$. Then*

$$U(FG) \cong (C_3^{6nk} \rtimes C_3^{2nk}) \rtimes U\big(F(C_n \times C_2^2)\big).$$

Proof. Let $K = \langle x^2\rangle$. Then $G/K \cong H = \langle x^3, y, z\rangle = C_2 \times C_2 \times C_n$. Thus from the ring epimorphism $FG \to FH$ given by

$$\sum_{l=0}^{1}\sum_{j=0}^{n-1}\sum_{i=0}^{2} x^{2i}(a_{i+6(j+nl)} + x^3 a_{i+6(j+nl)+3})y^l z^j \mapsto$$
$$\sum_{l=0}^{1}\sum_{j=0}^{n-1}\sum_{i=0}^{2} (a_{i+6(j+nl)} + x^3 a_{i+6(j+nl)+3})y^l z^j$$

we get a group epimorphism $\theta : U(FG) \to U(FH)$.

Further, from the inclusion map $i : FH \to FG$, we have $i : U(FH) \to U(FG)$ such that $\theta i = 1_{U(FH)}$. Therefore $U(FG)$ is a split extension of $U(FH)$ by $V = ker(\theta) = 1 + \omega(K)$. Hence

$$U(FG) \cong V \rtimes U(FH).$$

Let $u = \sum_{l=0}^{1}\sum_{j=0}^{n-1}\sum_{i=0}^{2} x^{2i}(a_{i+6(j+nl)} + x^3 a_{i+6(j+nl)+3})y^l z^j \in U(FG)$, then $u \in V$ if and only if $\sum_{i=0}^{2} a_i = 1$ and $\sum_{i=0}^{2} a_{i+3j} = 0$ for $j = 1, 2, \ldots, (4n-1)$. Therefore

$$V = \Big\{1 + \sum_{l=0}^{1}\sum_{j=0}^{n-1}\sum_{i=1}^{2}(x^{2i}-1)(b_{i+4(j+nl)} + x^3 b_{i+4(j+nl)+2})y^l z^j \mid b_i \in F\Big\}$$

and $|V| = 3^{8nk}$. Since $\omega(K)^3 = 0$, $V^3 = 1$. We now study the structure of V in the following steps:

Step 1: $C_V(x^2) = \{v \in V \mid vx^2 = x^2 v\} \cong C_3^{6nk}$.

If $v = 1+\sum_{l=0}^{1}\sum_{j=0}^{n-1}\sum_{i=1}^{2}(x^{2i}-1)(b_{i+4(j+nl)}+x^3 b_{i+4(j+nl)+2})y^l z^j \in C_V(x^2) = \{v \in V \mid vx^2 = x^2 v\}$, then $vx^2 - x^2 v = \widehat{x^2}\sum_{j=0}^{n-1}\big((b_{1+4(j+n)} - b_{2+4(j+n)}) + x^3(b_{3+4(j+n)} - b_{4+4(j+n)})\big)yz^j$. Thus $v \in C_V(x^2)$ if and only if $b_{i+4(j+n)} = b_{1+i+4(j+n)}$ for $j = 0, 1, \ldots, n-1$ and $i = 1, 3$. Hence

$$\begin{aligned} C_V(x^2) = &\Big\{1 + \sum_{j=0}^{n-1}\sum_{i=1}^{2}(x^{2i}-1)(c_{i+4j} + x^3 c_{i+4j+2})z^j \\ &+ \widehat{x^2}\sum_{j=0}^{n-1}\sum_{i=0}^{1} c_{n(i+4)+j+1} x^{3i} y z^j \mid c_i \in F\Big\}. \end{aligned}$$

So $C_V(x^2)$ is an abelian subgroup of V and $|C_V(x^2)| = 3^{6nk}$. Therefore $C_V(x^2) \cong C_3^{6nk}$.

Step 2: Let S be the subset of V consisting of elements of the form

$$1 + \sum_{j=0}^{n-1} x^2(1-x^2)(s_{j_1} + s_{j_2}x^3)(1+y)z^j,$$

where $s_{j_1}, s_{j_2} \in F$. Then S is an abelian subgroup of V and $S \cong C_3^{2nk}$.

Let

$$s_1 = 1 + \sum_{j=0}^{n-1} x^2(1-x^2)(r_{j_1} + r_{j_2}x^3)(1+y)z^j \in S$$

and

$$s_2 = 1 + \sum_{j=0}^{n-1} x^2(1-x^2)(t_{j_1} + t_{j_2}x^3)(1+y)z^j \in S.$$

Then

$$s_1 s_2 = 1 + \sum_{j=0}^{n-1} x^2(1-x^2)\Big((r_{j_1} + t_{j_1}) + (r_{j_2} + t_{J_2})x^3\Big)(1+y)z^j \in S.$$

So S is an abelian subgroup of V and $|S| = 3^{2nk}$. Therefore $S \cong C_3^{2nk}$.

Now, let

$$\begin{aligned} c =& 1 + \sum_{j=0}^{n-1}\sum_{i=1}^{2}(x^{2i}-1)(c_{i+4j} + x^3 c_{i+4j+2})z^j \\ &+ \widehat{x^2}\sum_{j=0}^{n-1}\sum_{i=0}^{1} c_{n(i+4)+j+1}x^{3i}yz^j \in C_V(x^2) \end{aligned}$$

and

$$s = 1 + \sum_{j=0}^{n-1} x^2(1-x^2)(s_{j_1} + s_{j_2}x^3)(1+y)z^j \in S.$$

Then

$$c^s = c + \widehat{x^2}(\gamma_1 + \gamma_2 x^3)y,$$

where

$$\begin{aligned} \gamma_1 &= \sum_{j=0}^{n-1}\sum_{i=0}^{n-1}\Big(s_{j_1}(c_{1+4i} - c_{2+4i}) + s_{j_2}(c_{3+4i} - c_{4+4i})\Big)z^{i+j}, \\ \gamma_2 &= \sum_{j=0}^{n-1}\sum_{i=0}^{n-1}\Big(s_{j_1}(c_{3+4i} - c_{4+4i}) + s_{j_2}(c_{1+4i} - c_{2+4i})\Big)z^{i+j}. \end{aligned}$$

Clearly, $c^s \in C_V(x^2)$. Thus S normalizes $C_V(x^2)$. Since $C_V(x^2) \cap S = 1$, $|C_V(x^2)S| = 3^{8nk} = |V|$. Therefore

$$V = C_V(x^2)S \cong C_V(x^2) \rtimes S \cong C_3^{6nk} \rtimes C_3^{2nk}.$$

Hence the claim holds. □

For $U\big(F(C_n \times C_2^2)\big)$, we prove the following:

Theorem 3.3. *Let F be a finite field of characteristic 3 containing $q = 3^k$ elements and let $H = C_n \times C_2^2$, where $n = 3^r s$ such that $r \geq 0$ and $(3, s) = 1$. Then $U(FH)$ is isomorphic to*

(1) $C_{q-1}^4 \times \big(\prod_{l>1,\ l|n} C_{q^{d_l}-1}^{4e_l}\big)$, *if* $3 \nmid n$;

(2) $C_{q-1}^4 \times \left(\prod_{l>1,\ l|s} C_{q^{d_l}-1}^{4e_l}\right) \times \left(\prod_{t=1}^{r} C_{3^t}^{4sn_t}\right)$, *if* $3|n$;
where $n_r = 2k$ *and* $n_t = 4k3^{r-t-1}$, *for all* t, $1 \leq t < r$.

Proof. By Lemma 1.4, we have

$$FH \cong (FC_2^2)C_n \cong (FC_n)^4.$$

(1) If $3 \nmid n$, i.e., if $r = 0$, then by Lemma 1.1,

$$FC_n \cong F \oplus \Big(\oplus_{l>1,\ l|n} F_{d_l}^{e_l} \Big).$$

Hence

$$U(FH) \cong C_{3^k-1}^4 \times \Big(\prod_{l>1,\ l|n} C_{3^{d_l k}-1}^{4e_l} \Big).$$

(2) If $3|n$, i.e., if $r > 0$, then by Lemma 1.1,

$$\begin{aligned} FC_n &\cong (FC_s)C_{3^r}, \\ &\cong \big(F \oplus (\oplus_{l>1,\ l|s} F_{d_l}^{e_l})\big)C_{3^r}, \\ &\cong FC_{3^r} \oplus \big(\oplus_{l>1,\ l|s} (F_{d_l}C_{3^r})^{e_l} \big). \end{aligned}$$

By Lemma 1.3,

$$U(FC_{3^r}) \cong C_{3^k-1} \times \Big(\prod_{t=1}^{r} C_{3^t}^{n_t} \Big),$$

where $n_r = 2k$ and $n_t = 4k3^{r-t-1}$, for all t, $1 \leq t < r$ and

$$U(F_{d_l}C_{3^r})^{e_l} \cong C_{3^{d_l k}-1}^{e_l} \times \Big(\prod_{t=1}^{r} C_{3^t}^{\phi(l)n_t} \Big).$$

Since $\sum_{l|s} \phi(l) = s$,

$$U(FC_n) \cong C_{3^k-1} \times \Big(\prod_{l>1,\ l|s} C_{3^{d_l k}-1}^{e_l} \Big) \times \Big(\prod_{t=1}^{r} C_{3^t}^{sn_t} \Big)$$

and hence

$$U(FH) \cong C_{3^k-1}^4 \times \Big(\prod_{l>1,\ l|s} C_{3^{d_l k}-1}^{4e_l} \Big) \times \Big(\prod_{t=1}^{r} C_{3^t}^{4sn_t} \Big). \qquad \square$$

Theorem 3.4. *Let F be a finite field of characteristic $p > 3$ containing $q = p^k$ elements and let $G = C_n \times D_{12}$, where $n = p^r s$, $r \geq 0$ such that $(p, s) = 1$. If $V = 1 + J(FG)$, then*

$$U(FG)/V \cong C_{q-1}^4 \times GL(2, F)^2 \times \Big(\prod_{l>1,\ l|s} \big(C_{q^{d_l}-1}^4 \times GL(2, F_{d_l})^2\big)^{e_l} \Big),$$

where V is a group of exponent p^r and order $p^{12sk(p^r-1)}$.

Proof. Let $K = \langle z^s \rangle$. Then $G/K \cong H = C_s \times D_{12}$. If $\theta : FG \to FH$ is the canonical ring epimorphism, then by [19, Theorem 7.2.7 and Lemma 8.1.17], $J(FG) = ker(\theta)$, $FG/J(FG) \cong FH$ and $dim_F(J(FG)) = 12s(p^r - 1)$. Hence $U(FG) \cong V \rtimes U(FH)$. Clearly, exponent of $V = p^r$ and $|V| = p^{12sk(p^r-1)}$. Now by Lemma 1.1,

$$\begin{aligned} FH &\cong (FC_s)D_{12}, \\ &\cong \big(F \oplus (\oplus_{l>1,\ l|s} F_{d_l}^{e_l})\big)D_{12}, \\ &\cong FD_{12} \oplus \big(\oplus_{l>1,\ l|s} (F_{d_l}D_{12})^{e_l}\big). \end{aligned}$$

Now, by [25, Theorem 4.3], $FD_{12} \cong F^4 \oplus M(2,F)^2$. Hence

$$U(FH) \cong C_{q-1}^4 \times GL(2,F)^2 \times \Big(\prod_{l>1,\ l|s} \big(C_{q^{d_l}-1}^4 \times GL(2,F_{d_l})^2\big)^{e_l} \Big). \qquad \square$$

In the above theorem, if $r = 0$, then we have the unit group of the semisimple group algebra FG given by

$$U(FG) \cong C_{q-1}^4 \times GL(2,F)^2 \times \Big(\prod_{l>1,\ l|n} \big(C_{q^{d_l}-1}^4 \times GL(2,F_{d_l})^2\big)^{e_l} \Big).$$

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References

[1] Zs. Balogh and A. Bovdi, *On units of group algebras of 2-groups of maximal class*, Comm. Algebra, 32(8) (2004), 3227-3245.

[2] Zs. Balogh, *The structure of the unit group of some group algebras*, Miskolc Math. Notes, 21(2) (2020), 615-620.

[3] A. Bovdi, *The group of units of a group algebra of characteristic p*, Publ. Math. Debrecen, 52(1-2) (1998), 193-244.

[4] W. Burnside, Theory of Groups of Finite Order, 2nd ed., Dover Publication, Inc., New York, 1955.

[5] W. D. Gao, A. Geroldinger and F. Halter-Koch, *Group algebras of finite abelian groups and their applications to combinatorial problems*, Rocky Mountain J. Math., 39(3) (2009), 805-823.

[6] J. Gildea, *On the order of* $U(F_{p^k}D_{2p^m})$, Int. J. Pure Appl. Math., 46(2) (2008), 267-272.

[7] J. Gildea, *The centre of the maximal p-subgroup of* $U(F_{p^k}D_{2p^m})$, Glasg. Math. J., 51(3) (2009), 651-657.

[8] J. Gildea, *The structure of the unit group of the group algebra of* $F_{2^k}A_4$, Czechoslovak Math. J., 61(136) (2011), 531-539.

[9] J. Gildea and F. Monaghan, *Units of some group algebras of groups of order* 12 *over any finite field of characteristic* 3, Algebra Discrete Math., 11(1) (2011), 46-58.

[10] P. Hurley and T. Hurley, *Codes from zero-divisors and units in group rings*, Int. J. Inf. Coding Theory, 1(1) (2009), 57-87.

[11] K. Kaur and M. Khan, *Units in* F_2D_{2p}, J. Algebra Appl., 13(2) (2014), 1350090 (9 pp).

[12] S. Maheshwari, Finitely Presented Groups and Units in Group Ring, Ph.D. Thesis, IIT Delhi, (2016).

[13] N. Makhijani, R. K. Sharma and J. B. Srivastava, *Units in* $F_{2^k}D_{2n}$, Int. J. Group Theory, 3(3) (2014), 25-34.

[14] N. Makhijani, R. K. Sharma and J. B. Srivastava, *A note on units in* $F_{p^m}D_{2p^n}$, Acta Math. Acad. Paedagog. Nyházi., 30(1) (2014), 17-25.

[15] N. Makhijani, R. K. Sharma and J. B. Srivastava, *The unit group of algebra of circulant matrices*, Int. J. Group Theory, 3(4) (2014), 13-16.

[16] N. Makhijani, R. K. Sharma and J. B. Srivastava, *Units in finite dihedral and quaternion group algebras*, J. Egyptian Math. Soc., 24(1) (2016), 5-7.

[17] N. Makhijani, R. K. Sharma and J. B. Srivastava, *The unit group of some special semi-simple group algebras*, Quaest. Math., 39(1) (2016), 9-28.

[18] F. Monaghan, *Units of some group algebras of non-abelian groups of order* 24 *over any finite field of characteristic* 3, Int. Electron. J. Algebra, 12 (2012), 133-161.

[19] D. S. Passman, The Algebraic Structure of Group Rings, Wiley Interscience, New York, 1977.

[20] C. Polcino Milies and S. K. Sehgal, An Introduction to Group Rings, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2002.

[21] M. Sahai and S. F. Ansari, *Unit groups of the finite group algebras of generalized quaternion groups*, J. Algebra Appl., 19(6) (2020), 2050112 (5 pp).

[22] M. Sahai and S. F. Ansari, *Unit groups of finite group algebras of abelian groups of order at most* 16, Asian-Eur. J. Math., 14(3) (2021), 2150030 (17 pp).

[23] M. Sahai and S. F. Ansari, *Group of units of finite group algebras of groups of order* 24, Ukrainian Math. J., 75(2) (2023), 215-229.

[24] R. K. Sharma, J. B. Srivastava and M. Khan, *The unit group of* FA_4, Publ. Math. Debrecen, 71(1-2) (2007), 21-26.

[25] G. Tang and Y. Gao, *The unit groups of FG of groups with order* 12, Int. J. Pure Appl. Math., 73(2) (2011), 143-158.

[26] G. Tang, Y. Wei and Y. Li, *Unit groups of group algebras of some small groups*, Czechoslovak Math. J., 64(139) (2014), 149-157.

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