Advances in the Theory of Nonlinear Analysis and its Applications 7 (2023) No. 2, 387–404.
https://doi.org/10.31197/atnaa.1299905
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications
ISSN: 2587-2648
Peer-Reviewed Scientific Journal

Mean convergence theorems to generalized acute points for generalized pseudocontractions

Toshiharu Kawasaki[a]

[a]Faculty of Engineering, Tamagawa University, Tokyo 194–8610, Japan; Faculty of Science, Chiba University and Faculty of Law, Politics and Economics, Chiba University, Chiba 263–8522, Japan.

Abstract

Convergence theorems required more assumptions on parameters than fixed point theorems. In this paper we generalize the concept of acute point and we introduce some convergence theorems that holds under the same assumptions on parameters as fixed point theorems.

Keywords: Mean convergence theorem Generalized acute point Generalized skew-acute point Generalized pseudocontraction Banach space .
2010 MSC: 47H10.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping T from C into H is said to be generalized hybrid [22] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for any $x, y \in C$. Such a mapping is said to be (α, β)-generalized hybrid. The class of all generalized hybrid mappings is a new class of nonlinear mappings including nonexpansive mappings, nonspreading mappings [24] and hybrid mappings [26]. A mapping T from C into H is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

Email address: toshiharu.kawasaki@nifty.ne.jp (Toshiharu Kawasaki)

Received May 20, 2023; Accepted: June 11, 2023; Online: June 14, 2023.

for any $x, y \in C$; nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for any $x, y \in C$; hybrid if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for any $x, y \in C$. Any nonexpansive mapping is $(1, 0)$-generalized hybrid; any nonspreading mapping is $(2, 1)$-generalized hybrid; any hybrid mapping is $\left(\frac{3}{2}, \frac{1}{2}\right)$-generalized hybrid.

Motivated these mappings, in [19] Kawasaki and Takahashi introduced a new very wider class of mappings, called widely more generalized hybrid mappings, than the class of all generalized hybrid mappings. A mapping T from C into H is widely more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned}&\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &\quad + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0\end{aligned}$$

for any $x, y \in C$. Such a mapping is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid. This class includes the class of all generalized hybrid mappings and also the class of all k-pseudocontractions [3] for $k \in [0, 1]$. A mapping T from C into H is called a k-pseudocontraction if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2$$

for any $x, y \in C$. Any (α, β)-generalized hybrid mapping is $(\alpha, 1-\alpha, -\beta, \beta-1, 0, 0, 0)$-widely more generalized hybrid; any k-pseudocontraction is $(1, 0, 0, -1, 0, 0, -k)$-widely more generalized hybrid. Furthermore they proved some fixed point theorems [7–12, 18–21] and some ergodic theorems [7, 8, 18–20].

There are some studies on Banach space related to these results. In [28] Takahashi, Wong and Yao introduced the generalized nonspreading mapping and the skew-generalized nonspreading mapping in a Banach space. Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping T from C into E is said to be generalized nonspreading if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned}&\alpha\phi(Tx, Ty) + \beta\phi(x, Ty) + \gamma\phi(Tx, y) + \delta\phi(x, y) \\ &\quad \leq \varepsilon(\phi(Ty, Tx) - \phi(Ty, x)) + \zeta(\phi(y, Tx) - \phi(y, x))\end{aligned}$$

for any $x, y \in C$, where J is the duality mapping on E and

$$\phi(u, v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2.$$

Such a mapping is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-generalized nonspreading. A mapping T from C into E is said to be skew-generalized nonspreading if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned}&\alpha\phi(Tx, Ty) + \beta\phi(x, Ty) + \gamma\phi(Tx, y) + \delta\phi(x, y) \\ &\quad \leq \varepsilon(\phi(Ty, Tx) - \phi(y, Tx)) + \zeta(\phi(Ty, x) - \phi(y, x))\end{aligned}$$

for any $x, y \in C$. Such a mapping is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-skew-generalized nonspreading. These classes include the class of generalized hybrid mappings in a Hilbert space, however, it does not include the class of widely more generalized hybrid mappings.

Motivated these results, we introduced a new class of mappings [13–16] on Banach space corresponding to the class of all widely more generalized hybrid mappings on Hilbert space. Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping T from C into E is called a generalized pseudocontraction if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2 \in \mathbb{R}$ such that

$$\begin{aligned}&\alpha_1\phi(Tx, Ty) + \alpha_2\phi(Ty, Tx) + \beta_1\phi(x, Ty) + \beta_2\phi(Ty, x) \\ &\qquad + \gamma_1\phi(Tx, y) + \gamma_2\phi(y, Tx) + \delta_1\phi(x, y) + \delta_2\phi(y, x) \\ &\qquad + \varepsilon_1\phi(Tx, x) + \varepsilon_2\phi(x, Tx) + \zeta_1\phi(y, Ty) + \zeta_2\phi(Ty, y) \\ &\quad \leq 0\end{aligned}$$

for any $x, y \in C$. Such a mapping is called an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction. Let E^* be the topological dual space of a strictly convex, reflexive and smooth Banach space E and let C^* be a nonempty subset of E^*. A mapping T^* from C^* into E^* is called a *-generalized pseudocontraction if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_1\phi_*(T^*x^*, T^*y^*) + \alpha_2\phi_*(T^*y^*, T^*x^*) + \beta_1\phi_*(x^*, T^*y^*) + \beta_2\phi_*(T^*y^*, x^*) \\ &\quad +\gamma_1\phi_*(T^*x^*, y^*) + \gamma_2\phi_*(y^*, T^*x^*) + \delta_1\phi_*(x^*, y^*) + \delta_2\phi_*(y^*, x^*) \\ &\quad +\varepsilon_1\phi_*(T^*x^*, x^*) + \varepsilon_2\phi_*(x^*, T^*x^*) + \zeta_1\phi_*(y^*, T^*y^*) + \zeta_2\phi_*(T^*y^*, y^*) \\ &\leq 0 \end{aligned}$$

for any $x^*, y^* \in C^*$, where

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2$$

for any $x^*, y^* \in E^*$. Such a mapping is called an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-*-generalized pseudocontraction.

On the other hand, in [27] Takahashi and Takeuchi introduced a concept of attractive point in a Hilbert space. Let H be a real Hilbert space, let C be a nonempty subset of H and let T be a mapping from C into H. $x \in H$ is called an attractive point of T if

$$\|x - Ty\| \leq \|x - y\|$$

for any $y \in C$. Let

$$A(T) = \{x \in H \mid \|x - Ty\| \leq \|x - y\| \text{ for any } y \in C\}.$$

Furthermore they proved that the Baillon type ergodic theorem [2] for generalized hybrid mappings without convexity of C.

In [28] Takahashi, Wong and Yao introduced some extensions of attractive point and proved some attractive point theorems on Banach spaces. $x \in E$ is an attractive point of T if

$$\phi(x, Ty) \leq \phi(x, y)$$

for any $y \in C$; $x \in E$ is a skew-attractive point of T if

$$\phi(Ty, x) \leq \phi(y, x)$$

for any $y \in C$. Let

$$\begin{aligned} A(T) &= \{x \in E \mid \phi(x, Ty) \leq \phi(x, y) \text{ for any } y \in C\}; \\ B(T) &= \{x \in E \mid \phi(Ty, x) \leq \phi(y, x) \text{ for any } y \in C\}. \end{aligned}$$

In [1] Atsushiba, Iemoto, Kubota and Takeuchi introduced a concept of acute point as an extension of attractive point in a Hilbert space. Let H be a real Hilbert space, let C be a nonempty subset of H and let T be a mapping from C into H and $k \in [0, 1]$. $x \in H$ is called a k-acute point of T if

$$\|x - Ty\|^2 \leq \|x - y\|^2 + k\|y - Ty\|^2$$

for any $y \in C$. Let

$$\mathscr{A}_k(T) = \{x \in H \mid \|x - Ty\|^2 \leq \|x - y\|^2 + k\|y - Ty\|^2 \text{ for any } y \in C\}.$$

Furthermore, using a concept of acute point, they proved convergence theorems without convexity of C.

We introduced some extensions of acute point [13–16]. Let E be a smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k, \ell \in \mathbb{R}$. $x \in E$ is called a (k, ℓ)-acute point of T if

$$\phi(x, Ty) \leq \phi(x, y) + k\phi(y, Ty) + \ell\phi(Ty, y)$$

for any $y \in C$. $x \in E$ is called a (k, ℓ)-skew-acute point of T if

$$\phi(Ty, x) \leq \phi(y, x) + k\phi(y, Ty) + \ell\phi(Ty, y)$$

for any $y \in C$. Let

$$\begin{aligned}&\mathscr{A}_{k,\ell}(T)\\&= \{x \in E \mid \phi(x, Ty) \leq \phi(x, y) + k\phi(y, Ty) + \ell\phi(Ty, y) \text{ for any } y \in C\};\\&\mathscr{B}_{k,\ell}(T)\\&= \{x \in E \mid \phi(Ty, x) \leq \phi(y, x) + k\phi(y, Ty) + \ell\phi(Ty, y) \text{ for any } y \in C\}.\end{aligned}$$

Furthermore we proved some fixed point and acute point theorems [13,15], and some convergence theorems [14,16]. However, convergence theorems require more assumptions on parameters than fixed point theorems.

In this paper we generalize the concept of acute point and we introduce some convergence theorems that holds under the same assumptions on parameters as fixed point theorems.

2. Preliminaries

We know that the following hold; for instance, see [4,5,25].

Condition 2.1. *000000*

(T1) Let E be a Banach space, let E^* be the topological dual space of E and let J be the duality mapping on E defined by

$$J(x) = \{x^* \in E^* \mid \|x\|^2 = \langle x, x^* \rangle = \|x^*\|^2\}$$

for any $x \in E$. Then E is strictly convex if and only if J is injective, that is, $x \neq y$ implies $J(x) \cap J(y) = \emptyset$.

(T2) Let E be a Banach space, let E^* be the topological dual space of E and let J be the duality mapping on E. Then E is reflexive if and only if J is surjective, that is, $\bigcup_{x \in E} J(x) = E^*$.

(T3) Let E be a Banach space and let J be the duality mapping on E. Then E is smooth if and only if J is single-valued.

(T4) Let E be a Banach space and let J be the duality mapping on E. If J is single-valued, then J is norm-to-weak continuous.*

(T5) Let E be a Banach space and let J be the duality mapping on E. Then E is strictly convex if and only if

$$1 - \langle x, y^* \rangle > 0$$

for any $x, y \in E$ with $x \neq y$ and $\|x\| = \|y\| = 1$ and for any $y^* \in J(y)$.

(T6) Let E be a Banach space and let E^* be the topological dual space of E. Then E is reflexive if and only if E^* is reflexive.

(T7) Let E be a Banach space and let E^* be the topological dual space of E. If E^* is strictly convex, then E is smooth. Conversely, E is reflexive and smooth, then E^* is strictly convex.

(T8) Let E be a Banach space and let E^* be the topological dual space of E. If E^* is smooth, then E is strictly convex. Conversely, E is reflexive and strictly convex, then E^* is smooth.

(T9) If a Banach space E is unformly convex, then E is reflexive.

(T10) Let E be a Banach space and let J be the duality mapping on E. If E has the Fréche differentiable norm, then J is norm-to-norm continuous.

(T11) Let E be a Banach space and let E^* be the topological dual space of E. E has uniformly Frécht differentiable norm if and only if E^* is uniformly convex.

(T12) Let E be a Banach space and let E^* be the topological dual space of E. E is strictly convex and reflexive and has Kadec-Klee property if and only if E^* has Fréchet differentiable norm.

Let E be a smooth Banach space, let J be the duality mapping on E and let ϕ be the mapping from $E \times E$ into $[0, \infty)$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for any $x, y \in E$. Since by (T3) J is single-valued, ϕ is well-defined. It is obvious that $x = y$ implies $\phi(x, y) = 0$. Conversely, by (T5)

Condition 2.2. *000000*

(T13) If E is also strictly convex, then $\phi(x, y) = 0$ implies $x = y$.

Let E be a strictly convex and smooth Banach space. By (T1) an (T3) J is a bijective mapping from E onto $J(E)$. In particular, if E is also reflective, then by (T2) J is a bijective mapping from E onto E^*. Suppose that E is strictly convex, reflective and smooth. Let ϕ_* be the mapping from $E^* \times E^*$ into $[0, \infty)$ defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2$$

for any $x^*, y^* \in E^*$. Then

$$\phi_*(x^*, y^*) = \phi(J^{-1}y^*, J^{-1}x^*) \tag{2.1}$$

holds. Therefore

Condition 2.3. *000000*

(T13)* $\phi_*(x^*, y^*) = 0$ if and only if $x^* = y^*$.

We use the following lemmas in this paper.

The following showed in [6].

Lemma 2.4. *Let E be a strictly convex and smooth Banach space and let C be a nonempty closed subset of E. Suppose that there exists a sunny generalized nonexpansive retraction of E onto C. Then the sunny generalized nonexpansive retraction is uniquely determined.*

Lemma 2.5. *Let E be a strictly convex and smooth Banach space, let C be a nonempty closed subset of E and let $(x, z) \in E \times C$. Suppose that there exists a sunny generalized nonexpansive retraction R_C of E onto C. Then the following hold.*

Condition 2.6. *0000*

(i) $z = R_C x$ if and only if $\langle x - z, Jz - Jy\rangle \geq 0$ for any $y \in C$;

(ii) $\phi(R_C x, y) + \phi(x, R_C x) \leq \phi(x, y)$ for any $y \in C$.

The following showed in [23].

Lemma 2.7. *Let E be a strictly convex, reflexive and smooth Banach space and let C be a nonempty closed subset of E. Then the following are equivalent:*

Condition 2.8. *0000*

(i) There exists a sunny generalized nonexpansive retraction of E onto C;

(ii) There exists a generalized nonexpansive retraction of E onto C;

(iii) $J(C)$ is closed and convex.

Lemma 2.9. *Let E be a strictly convex, reflexive and smooth Banach space, let C be a nonempty closed subset of E and let $(x, z) \in E \times C$. Suppose that there exists a sunny generalized nonexpansive retraction R_C of E onto C. Then the following are equivalent:*

Condition 2.10. *0000*

(i) $z = R_C x$;

(ii) $\phi(x, z) = \min_{y \in C} \phi(x, y)$.

The following showed in [28].

Lemma 2.11. *Let E be a uniformly convex and smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into itself with $B(T) \neq \emptyset$ and let R be the sunny generalized nonexpansive retraction of E onto $B(T)$. Then for any $x \in C$, $\{RT^n x\}$ is strongly convergent to an element in $B(T)$.*

The following lemmas are shown in [14–16].

Lemma 2.12. *Let E be a smooth Banach space, let C be a nonempty subset of E, let D be a nonempty convex subset of E, let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into D and let $\lambda \in [0, 1]$. Then T is a $((1-\lambda)\alpha_1 + \lambda\alpha_2), \lambda\alpha_1 + (1-\lambda)\alpha_2, (1-\lambda)\beta_1 + \lambda\gamma_2, \lambda\gamma_1 + (1-\lambda)\beta_2, (1-\lambda)\gamma_1 + \lambda\beta_2, \lambda\beta_1 + (1-\lambda)\gamma_2, (1-\lambda)\delta_1 + \lambda\delta_2, \lambda\delta_1 + (1-\lambda)\delta_2, (1-\lambda)\varepsilon_1 + \lambda\zeta_2, \lambda\zeta_1 + (1-\lambda)\varepsilon_2, (1-\lambda)\zeta_1 + \lambda\varepsilon_2, \lambda\varepsilon_1 + (1-\lambda)\zeta_2)$-generalized pseudocontraction from C into D.*

Lemma 2.13. *Let E^* be the topological dual space of a strictly convex, reflexive and smooth Banach space E, let C^* be a nonempty subset of E^*, let D^* be a nonempty convex subset of E^*, let T^* be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-*-generalized pseudocontraction from C^* into D^* and let $\lambda \in [0, 1]$. Then T^* is a $((1-\lambda)\alpha_1 + \lambda\alpha_2), \lambda\alpha_1 + (1-\lambda)\alpha_2, (1-\lambda)\beta_1 + \lambda\gamma_2, \lambda\gamma_1 + (1-\lambda)\beta_2, (1-\lambda)\gamma_1 + \lambda\beta_2, \lambda\beta_1 + (1-\lambda)\gamma_2, (1-\lambda)\delta_1 + \lambda\delta_2, \lambda\delta_1 + (1-\lambda)\delta_2, (1-\lambda)\varepsilon_1 + \lambda\zeta_2, \lambda\zeta_1 + (1-\lambda)\varepsilon_2, (1-\lambda)\zeta_1 + \lambda\varepsilon_2, \lambda\varepsilon_1 + (1-\lambda)\zeta_2)$-*-generalized pseudocontraction from C^* into D^*.*

Lemma 2.14. *Let E be a strictly convex, reflexive and smooth Banach space, let E^* be the topological dual space of E, let C and D be nonempty subsets of E and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into D. Put $T^* = JTJ^{-1}$, where J is the duality mapping on E. Then T^* is an $(\alpha_2, \alpha_1, \beta_2, \beta_1, \gamma_2, \gamma_1, \delta_2, \delta_1, \varepsilon_2, \varepsilon_1, \zeta_2, \zeta_1)$-*-generalized pseudocontraction from $J(C)$ into $J(D)$.*

3. Generalized acute and skew-acute point

Most of this section are included in [17], however, the following are described for completeness.

Let E be a smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k, \ell, s \in \mathbb{R}$. $x \in E$ is called a (k, ℓ, s)-generalized acute point of T if

$$s(\phi(x, Ty) - \phi(x, y)) \leq k\phi(y, Ty) + \ell\phi(Ty, y) \tag{3.1}$$

for any $y \in C$. $x \in E$ is called a (k, ℓ, s)-generalized skew-acute point of T if

$$s(\phi(Ty, x) - \phi(y, x)) \leq k\phi(y, Ty) + \ell\phi(Ty, y) \tag{3.2}$$

for any $y \in C$. Let

$$\begin{aligned}
&\mathscr{A}_{k,\ell,s}(T) \\
&= \{x \in E \mid s(\phi(x, Ty) - \phi(x, y)) \leq k\phi(y, Ty) + \ell\phi(Ty, y) \text{ for any } y \in C\}; \\
&\mathscr{B}_{k,\ell,s}(T) \\
&= \{x \in E \mid s(\phi(Ty, x) - \phi(y, x)) \leq k\phi(y, Ty) + \ell\phi(Ty, y) \text{ for any } y \in C\}.
\end{aligned}$$

It is obvious that

$$\mathscr{A}_{k_1,\ell_1,s_1}(T) \subset \mathscr{A}_{k_2,\ell_2,s_2}(T),\ \mathscr{B}_{k_1,\ell_1,s_2}(T) \subset \mathscr{B}_{k_2,\ell_2,s_2}(T)$$

for any $k_1, k_2, \ell_1, \ell_2 \in \mathbb{R}$ and for any $s_1, s_2 \in (0, \infty)$ with $\frac{k_1}{s_1} \leq \frac{k_2}{s_2}$ and $\frac{\ell_1}{s_1} \leq \frac{\ell_2}{s_2}$;

$$\mathscr{A}_{k_1,\ell_1,s_1}(T) \supset \mathscr{A}_{k_2,\ell_2,s_2}(T),\ \mathscr{B}_{k_1,\ell_1,s_2}(T) \supset \mathscr{B}_{k_2,\ell_2,s_2}(T)$$

for any $k_1, k_2, \ell_1, \ell_2 \in \mathbb{R}$ and for any $s_1, s_2 \in (-\infty, 0)$ with $\frac{k_1}{s_1} \leq \frac{k_2}{s_2}$ and $\frac{\ell_1}{s_1} \leq \frac{\ell_2}{s_2}$. Furthermore

$$\mathscr{A}_{k,\ell,0}(T) = \mathscr{B}_{k,\ell,0}(T) = E$$

for any $(k, \ell) \in [0, \infty) \times [0, \infty)$;

$$\mathscr{A}_{k,\ell,0}(T) = \mathscr{B}_{k,\ell,0}(T) = \emptyset$$

for any $(k, \ell) \in (-\infty, 0] \times (-\infty, 0] \setminus \{(0, 0)\}$; otherwise,

$$\mathscr{A}_{k,\ell,0}(T) = E \text{ or } \emptyset,\ \mathscr{B}_{k,\ell,0}(T) = E \text{ or } \emptyset;$$

however, it is generally unknown which case holds. In this way, $\mathscr{A}_{k,\ell,0}(T)$ and $\mathscr{B}_{k,\ell,0}(T)$ may be empty. However, in later discussions, under some assumptions, such cases will be properly ruled out.

The following lemmas are important property characterizing them.

Lemma 3.1. *Let E be a smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k, \ell, s \in \mathbb{R}$. Then $\mathscr{A}_{k,\ell,s}(T)$ is closed and convex.*

Proof. Since

$$\phi(u, v) = \phi(u, w) + \phi(w, v) + 2\langle u - w, Jw - Jv\rangle \tag{3.3}$$

for any $u, v, w \in E$, (3.1) is equivalent to

$$2s\langle x, Jy - JTy\rangle \leq (k - s)\phi(y, Ty) + \ell\phi(Ty, y) + 2s\langle y, Jy - JTy\rangle.$$

Therefore $\mathscr{A}_{k,\ell,s}(T)$ is closed and convex. □

Lemma 3.2. *Let E be a smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k, \ell, s \in \mathbb{R}$. Then $\mathscr{B}_{k,\ell,s}(T)$ is closed.*

Proof. (3.2) is equivalent to

$$2s\langle y - Ty, Jx\rangle \leq k\phi(y, Ty) + (\ell - s)\phi(Ty, y) + 2s\langle y - Ty, Jy\rangle$$

from (3.3). Furthermore by (T4) J is norm-to-weak* continuous. Therefore $\mathscr{B}_{k,\ell,s}(T)$ is closed. □

Let E^* be the topological dual space of a strictly convex, reflexive and smooth Banach space E, let C^* be a nonempty subset of E^*, let T^* be a mapping from C^* into E^* and let $k, \ell, s \in \mathbb{R}$. $x^* \in E^*$ is called a (k, ℓ, s)-generalized-*-acute point of T^* if

$$s(\phi_*(x^*, T^*y^*) - \phi_*(x^*, y^*)) \leq k\phi_*(y^*, T^*y^*) + \ell\phi_*(T^*y^*, y^*) \tag{3.4}$$

for any $y^* \in C^*$. $x^* \in E^*$ is called a (k, ℓ, s)-generalized-*-skew-acute point of T^* if

$$s(\phi_*(T^*y^*, x^*) - \phi_*(y^*, x^*)) \leq k\phi_*(y^*, T^*y^*) + \ell\phi_*(T^*y^*, y^*) \tag{3.5}$$

for any $y^* \in C^*$. Let

$$\begin{aligned}
&\mathscr{A}^*_{k,\ell,s}(T^*) \\
&= \left\{ x^* \in E^* \;\middle|\; \begin{array}{l} s(\phi_*(x^*, T^*y^*) - \phi_*(x^*, y^*)) \leq k\phi_*(y^*, T^*y^*) + \ell\phi_*(T^*y^*, y^*) \\ \text{for any } y^* \in C^* \end{array} \right\}; \\
&\mathscr{B}^*_{k,\ell,s}(T^*) \\
&= \left\{ x^* \in E^* \;\middle|\; \begin{array}{l} s(\phi_*(T^*y^*, x^*) - \phi_*(y^*, x^*)) \leq k\phi_*(y^*, T^*y^*) + \ell\phi_*(T^*y^*, y^*) \\ \text{for any } y^* \in C^* \end{array} \right\}.
\end{aligned}$$

Lemma 3.3. *Let E^* be the topological dual space of a strictly convex, reflective and smooth Banach space E, let C^* be a nonempty subset of E^*, let T^* be a mapping from C^* into E^* and let $k, \ell, s \in \mathbb{R}$. Then $\mathscr{A}^*_{k,\ell,s}(T^*)$ is closed and convex.*

Proof. (3.4) is equivalent to

$$\begin{aligned}&2s\langle J^{-1}y^* - J^{-1}T^*y^*, x^*\rangle\\&\quad \le (k-s)\phi_*(y^*, T^*y^*) + \ell\phi_*(T^*y^*, y^*) + 2s\langle J^{-1}y^* - J^{-1}T^*y^*, y^*\rangle\end{aligned}$$

from (3.3) and (2.1), $\mathscr{A}^*_{k,\ell,s}(T^*)$ is closed and convex. □

Lemma 3.4. *Let E^* be the topological dual space of a strictly convex, reflexive and smooth Banach space E, let C^* be a nonempty subset of E^*, let T^* be a mapping from C^* into E^* and let $k, \ell, s \in \mathbb{R}$. Then $\mathscr{B}^*_{k,\ell,s}(T^*)$ is closed.*

Proof. (3.5) is equivalent to

$$\begin{aligned}&2s\langle J^{-1}x^*, y^* - T^*y^*\rangle\\&\quad \le k\phi_*(y^*, T^*y^*) + (\ell-s)\phi_*(T^*y^*, y^*) + 2s\langle J^{-1}y^*, y^* - T^*y^*\rangle\end{aligned}$$

from (3.3) and (2.1). Furthermore by (T4) J^{-1} is norm-to-weak* continuous. Therefore $\mathscr{B}^*_{k,\ell,s}(T^*)$ is closed. □

Lemma 3.5. *Let E be a strictly convex, reflective and smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E, let $T^* = JTJ^{-1}$ and let $k, \ell, s \in \mathbb{R}$. Then*

$$\mathscr{A}^*_{k,\ell,s}(T^*) = J(\mathscr{B}_{\ell,k,s}(T)),\ \mathscr{B}^*_{k,\ell,s}(T^*) = J(\mathscr{A}_{\ell,k,s}(T)).$$

In particular, $J(\mathscr{B}_{k,\ell,s}(T))$ is closed and convex and $J(\mathscr{A}_{k,\ell,s}(T))$ is closed.

Proof. Let $x^* \in \mathscr{A}^*_{k,\ell,s}(T^*)$. Then

$$s(\phi_*(x^*, T^*y^*) - \phi_*(x^*, y^*)) \le k\phi_*(y^*, T^*y^*) + \ell\phi_*(T^*y^*, y^*)$$

for any $y^* \in J(C)$. From (2.1)

$$\begin{aligned}&s(\phi(J^{-1}T^*y^*, J^{-1}x^*) - \phi(J^{-1}y^*, J^{-1}x^*))\\&\quad \le k\phi(J^{-1}T^*y^*, J^{-1}y^*) + \ell\phi(J^{-1}y^*, J^{-1}T^*y^*)\end{aligned}$$

for any $y^* \in J(C)$. Since $J^{-1}T^* = TJ^{-1}$, putting $y = J^{-1}y^*$, we obtain

$$s(\phi(Ty, J^{-1}x^*) - \phi(y, J^{-1}x^*)) \le \ell\phi(y, Ty) + k\phi(Ty, y).$$

Therefore $J^{-1}x^* \in \mathscr{B}_{\ell,k,s}(T)$ and hence $\mathscr{A}^*_{k,\ell,s}(T^*) = J(\mathscr{B}_{\ell,k,s}(T))$.

$\mathscr{B}^*_{k,\ell,s}(T^*) = J(\mathscr{A}_{\ell,k,s}(T))$ can be shown similarly.

Furthermore, by Lemma 3.3 $J(\mathscr{B}_{k,\ell,s}(T))$ is closed and convex and by Lemma 3.4 $J(\mathscr{A}_{k,\ell,s}(T))$ is closed. □

Lemma 3.6. *Let E be a strictly convex and smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k, \ell, s \in \mathbb{R}$. Then the following hold.*

Condition 3.7. *0000*

(1) If $(k, \ell) \in (-\infty, s] \times (-\infty, 0] \setminus \{(s, 0)\}$, then $C \cap \mathscr{A}_{k,\ell,s}(T)$ is a subset of the set of all fixed points of T;

(2) If $(k, \ell) \in (-\infty, 0] \times (-\infty, s] \setminus \{(0, s)\}$, then $C \cap \mathscr{B}_{k,\ell,s}(T)$ is a subset of the set of all fixed points of T.

Proof. Let $x \in C \cap \mathscr{A}_{k,\ell,s}(T)$. Then (3.1) holds for any $y \in C$. Putting $y = x$, we obtain $(s-k)\phi(x, Tx) - \ell\phi(Tx, x) \le 0$. If $(k, \ell) \in (-\infty, s] \times (-\infty, 0] \setminus \{(s, 0)\}$, then by (T13) we obtain $x = Tx$.

Let $x \in C \cap \mathscr{B}_{k,\ell,s}(T)$. Then (3.2) holds for any $y \in C$. Putting $y = x$, we obtain $-k\phi(x, Tx) + (s - \ell)\phi(Tx, x) \le 0$. If $(k, \ell) \in (-\infty, 0] \times (-\infty, s] \setminus \{(0, s)\}$, then by (T13) we obtain $x = Tx$. □

Lemma 3.8. *Let E^* be a strictly convex and smooth topological dual space of a Banach space, let C^* be a nonempty subset of E^*, let T^* be a mapping from C^* into E^* and let $k, \ell \in \mathbb{R}$. Then the following hold.*

Condition 3.9. *0000*

(1) If $(k, \ell) \in (-\infty, s] \times (-\infty, 0] \setminus \{(s, 0)\}$, then $C \cap \mathscr{A}^*_{k,\ell,s}(T^*)$ is a subset of the set of all fixed points of T^*;

(2) If $(k, \ell) \in (-\infty, 0] \times (-\infty, s] \setminus \{(0, 1)\}$, then $C \cap \mathscr{B}^*_{k,\ell,s}(T^*)$ is a subset of the set of all fixed points of T^*.

Proof. Let $x^* \in C^* \cap \mathscr{A}^*_{k,\ell,s}(T^*)$. Then (3.4) holds for any $y^* \in C^*$. Putting $y^* = x^*$, by we obtain $(s-k)\phi_*(x^*, T^*x^*) - \ell\phi_*(T^*x^*, x^*) \leq 0$. If $(k, \ell) \in (-\infty, s] \times (-\infty, 0] \setminus \{(s, 0)\}$, then by (T13)* we obtain $x^* = T^*x^*$.

Let $x^* \in C^* \cap \mathscr{B}^*_{k,\ell,s}(T^*)$. Then (3.5) holds for any $y^* \in C^*$. Putting $y^* = x^*$, by we obtain $-k\phi_*(x^*, T^*x^*) + (s-\ell)\phi_*(T^*x^*, x^*) \leq 0$. If $(k, \ell) \in (-\infty, 0] \times (-\infty, s] \setminus \{(0, s)\}$, then by (T13)* we obtain $x^* = T^*x^*$. □

4. Mean convergence theorems

Theorem 4.1. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm, let C be a nonempty subset of E and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_1 + \beta_1 + \gamma_1 + \delta_1) + \lambda(\alpha_2 + \beta_2 + \gamma_2 + \delta_2) \geq 0;\\
&\lambda(\alpha_1 + \gamma_1) + (1-\lambda)(\alpha_2 + \beta_2) \geq 0;\\
&\lambda(\beta_1 + \delta_1) + (1-\lambda)(\gamma_2 + \delta_2) \geq 0;\\
&(1-\lambda)\varepsilon_1 + \lambda\zeta_2 \geq 0;\\
&\lambda\zeta_1 + (1-\lambda)\varepsilon_2 \geq 0,
\end{aligned}$$

and suppose that

$$\mathscr{A}_{-((1-\lambda)\zeta_1+\lambda\varepsilon_2),-(\lambda\varepsilon_1+(1-\lambda)\zeta_2),(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}(T) \subset B(T) \neq \emptyset.$$

Let R be the sunny generalized nonexpansive retraction of E onto $B(T)$. Then for any $x \in C$,

$$S_n x \overset{def}{=} \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

is weakly convergent to an element

$$q \in \mathscr{A}_{-((1-\lambda)\zeta_1+\lambda\varepsilon_2),-(\lambda\varepsilon_1+(1-\lambda)\zeta_2),(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}(T),$$

where $q = \lim_{n\to\infty} RT^n x$.

Additionally, if C is closed and convex and one of the following holds:

Condition 4.2. *0000*

(1) $(1-\lambda)(\alpha_1 + \beta_1 + \zeta_1) + \lambda(\alpha_2 + \gamma_2 + \varepsilon_2) > 0$ and $\lambda\varepsilon_1 + (1-\lambda)\zeta_2 \geq 0$;

(2) $(1-\lambda)(\alpha_1 + \beta_1 + \zeta_1) + \lambda(\alpha_2 + \gamma_2 + \varepsilon_2) \geq 0$ and $\lambda\varepsilon_1 + (1-\lambda)\zeta_2 > 0$,

then q is a fixed point of T.

Proof. By the assumption E is strictly convex and smooth, and by (T9) E is reflexive. By Lemma 3.2 $B(T)$ is closed and by Lemma 3.5 $J(B(T))$ is closed and convex. Therefore by Lemmas 2.7 and 2.4 there exists a unique sunny nonexpansive retraction R of E onto $B(T)$.

By Lemma 2.12 T is a $((1-\lambda)\alpha_1+\lambda\alpha_2), \lambda\alpha_1+(1-\lambda)\alpha_2, (1-\lambda)\beta_1+\lambda\gamma_2, \lambda\gamma_1+(1-\lambda)\beta_2, (1-\lambda)\gamma_1+\lambda\beta_2, \lambda\beta_1+(1-\lambda)\gamma_2, (1-\lambda)\delta_1+\lambda\delta_2, \lambda\delta_1+(1-\lambda)\delta_2, (1-\lambda)\varepsilon_1+\lambda\zeta_2, \lambda\zeta_1+(1-\lambda)\varepsilon_2, (1-\lambda)\zeta_1+\lambda\varepsilon_2, \lambda\varepsilon_1+(1-\lambda)\zeta_2)$-generalized pseudocontraction. From (3.3) we obtain

$$
\begin{aligned}
&((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(Tx,Ty)+(\lambda\alpha_1+(1-\lambda)\alpha_2)\phi(Ty,Tx)\\
&\quad+((1-\lambda)\beta_1+\lambda\gamma_2)\phi(x,Ty)+(\lambda\gamma_1+(1-\lambda)\beta_2)\phi(Ty,x)\\
&\quad+((1-\lambda)\gamma_1+\lambda\beta_2)\phi(Tx,y)+(\lambda\beta_1+(1-\lambda)\gamma_2)\phi(y,Tx)\\
&\quad+((1-\lambda)\delta_1+\lambda\delta_2)\phi(x,y)+(\lambda\delta_1+(1-\lambda)\delta_2)\phi(y,x)\\
&\quad+((1-\lambda)\varepsilon_1+\lambda\zeta_2)\phi(Tx,x)+(\lambda\zeta_1+(1-\lambda)\varepsilon_2)\phi(x,Tx)\\
&\quad+((1-\lambda)\zeta_1+\lambda\varepsilon_2)\phi(y,Ty)+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&=((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(Tx,Ty)+(\lambda\alpha_1+(1-\lambda)\alpha_2)\phi(Ty,Tx)\\
&\quad-((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(x,Ty)\\
&\quad+((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))(\phi(x,y)+\phi(y,Ty)+2\langle x-y,Jy-JTy\rangle)\\
&\quad+(\lambda\gamma_1+(1-\lambda)\beta_2)\phi(Ty,x)\\
&\quad+((1-\lambda)\gamma_1+\lambda\beta_2)\phi(Tx,y)+(\lambda\beta_1+(1-\lambda)\gamma_2)\phi(y,Tx)\\
&\quad+((1-\lambda)\delta_1+\lambda\delta_2)\phi(x,y)+(\lambda\delta_1+(1-\lambda)\delta_2)\phi(y,x)\\
&\quad+((1-\lambda)\varepsilon_1+\lambda\zeta_2)\phi(Tx,x)+(\lambda\zeta_1+(1-\lambda)\varepsilon_2)\phi(x,Tx)\\
&\quad+((1-\lambda)\zeta_1+\lambda\varepsilon_2)\phi(y,Ty)+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&=((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(Tx,Ty)+(\lambda\alpha_1+(1-\lambda)\alpha_2)\phi(Ty,Tx)\\
&\quad-((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(x,Ty)+(\lambda\gamma_1+(1-\lambda)\beta_2)\phi(Ty,x)\\
&\quad+((1-\lambda)\gamma_1+\lambda\beta_2)\phi(Tx,y)+(\lambda\beta_1+(1-\lambda)\gamma_2)\phi(y,Tx)\\
&\quad+((1-\lambda)(\alpha_1+\beta_1+\delta_1)+\lambda(\alpha_2+\gamma_2+\delta_2))\phi(x,y)\\
&\quad+(\lambda\delta_1+(1-\lambda)\delta_2)\phi(y,x)\\
&\quad+((1-\lambda)\varepsilon_1+\lambda\zeta_2)\phi(Tx,x)+(\lambda\zeta_1+(1-\lambda)\varepsilon_2)\phi(x,Tx)\\
&\quad+((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&\quad+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&\quad+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle x-y,Jy-JTy\rangle.
\end{aligned}
$$

Since

$$
\begin{aligned}
&(1-\lambda)(\alpha_1+\beta_1+\delta_1)+\lambda(\alpha_2+\gamma_2+\delta_2)\geq-((1-\lambda)\gamma_1+\lambda\beta_2);\\
&\lambda\gamma_1+(1-\lambda)\beta_2\geq-(\lambda\alpha_1+(1-\lambda)\alpha_2);\\
&\lambda\delta_1+(1-\lambda)\delta_2\geq-(\lambda\beta_1+(1-\lambda)\gamma_2);\\
&(1-\lambda)\varepsilon_1+\lambda\zeta_2\geq 0;\\
&\lambda\zeta_1+(1-\lambda)\varepsilon_2\geq 0,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(Tx,Ty)+(\lambda\alpha_1+(1-\lambda)\alpha_2)\phi(Ty,Tx)\\
&\quad-((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(x,Ty)+(\lambda\gamma_1+(1-\lambda)\beta_2)\phi(Ty,x)\\
&\quad+((1-\lambda)\gamma_1+\lambda\beta_2)\phi(Tx,y)+(\lambda\beta_1+(1-\lambda)\gamma_2)\phi(y,Tx)\\
&\quad+((1-\lambda)(\alpha_1+\beta_1+\delta_1)+\lambda(\alpha_2+\gamma_2+\delta_2))\phi(x,y)\\
&\quad+(\lambda\delta_1+(1-\lambda)\delta_2)\phi(y,x)\\
&\quad+((1-\lambda)\varepsilon_1+\lambda\zeta_2)\phi(Tx,x)+(\lambda\zeta_1+(1-\lambda)\varepsilon_2)\phi(x,Tx)\\
&\quad+((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)
\end{aligned}
$$

$$
\begin{aligned}
&+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle x-y,Jy-JTy\rangle\\
\geq\ &((1-\lambda)\alpha_1+\lambda\alpha_2)(\phi(Tx,Ty)-\phi(x,Ty))\\
&+(\lambda\alpha_1+(1-\lambda)\alpha_2)(\phi(Ty,Tx)-\phi(Ty,x))\\
&+((1-\lambda)\gamma_1+\lambda\beta_2)(\phi(Tx,y)-\phi(x,y))\\
&+(\lambda\beta_1+(1-\lambda)\gamma_2)(\phi(y,Tx)-\phi(y,x))\\
&+((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle x-y,Jy-JTy\rangle.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&((1-\lambda)\alpha_1+\lambda\alpha_2)(\phi(Tx,Ty)-\phi(x,Ty))\\
&\quad+(\lambda\alpha_1+(1-\lambda)\alpha_2)(\phi(Ty,Tx)-\phi(Ty,x))\\
&\quad+((1-\lambda)\gamma_1+\lambda\beta_2)(\phi(Tx,y)-\phi(x,y))\\
&\quad+(\lambda\beta_1+(1-\lambda)\gamma_2)(\phi(y,Tx)-\phi(y,x))\\
&\quad+((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&\quad+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&\quad+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle x-y,Jy-JTy\rangle\\
&\leq 0.
\end{aligned}
$$

Replacing x by T^kx, we obtain

$$
\begin{aligned}
&((1-\lambda)\alpha_1+\lambda\alpha_2)(\phi(T^{k+1}x,Ty)-\phi(T^kx,Ty))\\
&\quad+(\lambda\alpha_1+(1-\lambda)\alpha_2)(\phi(Ty,T^{k+1}x)-\phi(Ty,T^kx))\\
&\quad+((1-\lambda)\gamma_1+\lambda\beta_2)(\phi(T^{k+1}x,y)-\phi(T^kx,y))\\
&\quad+(\lambda\beta_1+(1-\lambda)\gamma_2)(\phi(y,T^{k+1}x)-\phi(y,T^kx))\\
&\quad+((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&\quad+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&\quad+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle T^kx-y,Jy-JTy\rangle\\
&\leq 0.
\end{aligned}
$$

Summing up these inequalities for $k=0,\dots,n-1$ and dividing by n, we obtain

$$
\begin{aligned}
&\frac{(1-\lambda)\alpha_1+\lambda\alpha_2}{n}(\phi(T^nx,Ty)-\phi(x,Ty))\\
&\quad+\frac{\lambda\alpha_1+(1-\lambda)\alpha_2}{n}(\phi(Ty,T^nx)-\phi(Ty,x))\\
&\quad+\frac{(1-\lambda)\gamma_1+\lambda\beta_2}{n}(\phi(T^nx,y)-\phi(x,y))\\
&\quad+\frac{\lambda\beta_1+(1-\lambda)\gamma_2}{n}(\phi(y,T^nx)-\phi(y,x))\\
&\quad+((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&\quad+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&\quad+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle S_nx-y,Jy-JTy\rangle\\
&\leq 0.
\end{aligned}
$$

Since $B(T) \neq \emptyset$, we obtain

$$\phi(T^n x, y) \leq \phi(T^{n-1}x, y)$$

for any $x \in C$, for any $y \in B(T)$ and for any $n \in \mathbb{N}$. Therefore $\{T^n x\}$ is bounded and hence $\{S_n x\}$ is also bounded. Therefore there exists a subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$ such that $\{S_{n_i} x\}$ is weakly convergent to an element $p \in E$. Replacing n by n_i, we obtain

$$\begin{aligned}
&\frac{(1-\lambda)\alpha_1 + \lambda\alpha_2}{n_i}(\phi(T^{n_i}x, Ty) - \phi(x, Ty)) \\
&\quad + \frac{\lambda\alpha_1 + (1-\lambda)\alpha_2}{n_i}(\phi(Ty, T^{n_i}x) - \phi(Ty, x)) \\
&\quad + \frac{(1-\lambda)\gamma_1 + \lambda\beta_2}{n_i}(\phi(T^{n_i}x, y) - \phi(x, y)) \\
&\quad + \frac{\lambda\beta_1 + (1-\lambda)\gamma_2}{n_i}(\phi(y, T^{n_i}x) - \phi(y, x)) \\
&\quad + ((1-\lambda)(\alpha_1 + \beta_1 + \zeta_1) + \lambda(\alpha_2 + \gamma_2 + \varepsilon_2))\phi(y, Ty) \\
&\quad + (\lambda\varepsilon_1 + (1-\lambda)\zeta_2)\phi(Ty, y) \\
&\quad + 2((1-\lambda)(\alpha_1 + \beta_1) + \lambda(\alpha_2 + \gamma_2))\langle S_{n_i}x - y, Jy - JTy\rangle \\
&\leq 0.
\end{aligned}$$

Putting $i \to \infty$, we obtain

$$\begin{aligned}
&((1-\lambda)(\alpha_1 + \beta_1 + \zeta_1) + \lambda(\alpha_2 + \gamma_2 + \varepsilon_2))\phi(y, Ty) + (\lambda\varepsilon_1 + (1-\lambda)\zeta_2)\phi(Ty, y) \\
&\quad + 2((1-\lambda)(\alpha_1 + \beta_1) + \lambda(\alpha_2 + \gamma_2))\langle p - y, Jy - JTy\rangle \\
&\leq 0.
\end{aligned}$$

From (3.3) we obtain

$$\begin{aligned}
&((1-\lambda)\zeta_1 + \lambda\varepsilon_2)\phi(y, Ty) + (\lambda\varepsilon_1 + (1-\lambda)\zeta_2)\phi(Ty, y) \\
&\quad + ((1-\lambda)(\alpha_1 + \beta_1) + \lambda(\alpha_2 + \gamma_2))(\phi(p, Ty) - \phi(p, y)) \\
&\leq 0.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&((1-\lambda)(\alpha_1 + \beta_1) + \lambda(\alpha_2 + \gamma_2))(\phi(p, Ty) - \phi(p, y)) \\
&\quad \leq -((1-\lambda)\zeta_1 + \lambda\varepsilon_2)\phi(y, Ty) - (\lambda\varepsilon_1 + (1-\lambda)\zeta_2)\phi(Ty, y)
\end{aligned}$$

and hence

$$p \in \mathscr{A}_{-((1-\lambda)\zeta_1+\lambda\varepsilon_2), -(\lambda\varepsilon_1+(1-\lambda)\zeta_2), (1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}(T).$$

Next by Lemma 2.5 we obtain

$$\langle T^k x - RT^k x, Jy - JRT^k x\rangle \leq 0$$

for any $y \in B(T)$. By Lemma 2.11 for any $x \in C$, $\{RT^n x\}$ is strongly convergent to an element in $B(T)$. Let $q = \lim_{n\to\infty} RT^n x$. Since $\{T^n x\}$ is bounded, by Lemma 2.5 $\{RT^n x\}$ is also bounded. Putting $K = \max_{n\in\mathbb{N}\cup\{0\}, x\in C} \|T^n x - RT^n x\|$, we obtain

$$\begin{aligned}
\langle T^k x - RT^k x, Jy - Jq\rangle &\leq \langle T^k x - RT^k x, JRT^k x - Jq\rangle \\
&\leq \|T^k x - RT^k x\| \cdot \|JRT^k x - Jq\| \\
&\leq K\|JRT^k x - Jq\|.
\end{aligned}$$

Summing up these inequalities for $k = 0, \dots, n-1$ and dividing by n, we obtain

$$\left\langle S_n x - \frac{1}{n}\sum_{k=0}^{n-1} RT^k x, Jy - Jq \right\rangle \leq \frac{K}{n}\sum_{k=0}^{n-1} \|JRT^k x - Jq\|.$$

Since $\{S_{n_i} x\}$ is weakly convergent to p and by (T11) J is norm-to-norm continuous, we obtain

$$\langle p - q, Jy - Jq\rangle \leq 0.$$

Since $\mathscr{A}_{-((1-\lambda)\zeta_1+\lambda\varepsilon_2),-(\lambda\varepsilon_1+(1-\lambda)\zeta_2),(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}(T) \subset B(T)$, putting $y = p$, from (3.3) we obtain

$$\begin{aligned} 0 &\leq 2\langle p - q, Jq - Jp\rangle \\ &= -\phi(p, q) - \phi(q, p) \end{aligned}$$

and by (T13) we obtain $p = q$. Therefore $\{S_n x\}$ is weakly convergent to q.

Additionally, if C is closed and convex and (1) or (2) holds, then $\{S_n x\} \subset C$ and hence $q \in C$. By Lemma 3.6 q is a fixed point of T. □

Theorem 4.3. *Let E^* be a uniformly convex topological dual space with a Fréchet differentiable norm, let C^* be a nonempty subset of E^* and let T^* be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-$*$-generalized pseudocontraction from C^* into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

$$\begin{aligned} &(1-\lambda)(\alpha_1 + \beta_1 + \gamma_1 + \delta_1) + \lambda(\alpha_2 + \beta_2 + \gamma_2 + \delta_2) \geq 0; \\ &\lambda(\alpha_1 + \gamma_1) + (1-\lambda)(\alpha_2 + \beta_2) \geq 0; \\ &\lambda(\beta_1 + \delta_1) + (1-\lambda)(\gamma_2 + \delta_2) \geq 0; \\ &(1-\lambda)\varepsilon_1 + \lambda\zeta_2 \geq 0; \\ &\lambda\zeta_1 + (1-\lambda)\varepsilon_2 \geq 0, \end{aligned}$$

and suppose that

$$\mathscr{A}^*_{-((1-\lambda)\zeta_1+\lambda\varepsilon_2),-(\lambda\varepsilon_1+(1-\lambda)\zeta_2),(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}(T^*) \subset \mathscr{B}^*_{0,0}(T^*) \neq \emptyset.$$

Let R^* be the sunny generalized nonexpansive retraction of E^* onto $\mathscr{B}^*_{0,0}(T^*)$. Then for any $x^* \in C^*$,

$$S_n^* x^* \stackrel{def}{=} \frac{1}{n}\sum_{k=0}^{n-1} (T^*)^k x^*$$

is weakly convergent to an element

$$q^* \in \mathscr{A}^*_{-((1-\lambda)\zeta_1+\lambda\varepsilon_2),-(\lambda\varepsilon_1+(1-\lambda)\zeta_2),(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}(T^*),$$

where $q^* = \lim_{n\to\infty} R^*(T^*)^n x^*$.

Additionally, if C^* is closed and convex and one of the following holds:

Condition 4.4. *0000*

(1) $(1-\lambda)(\alpha_1 + \beta_1 + \zeta_1) + \lambda(\alpha_2 + \gamma_2 + \varepsilon_2) > 0$ *and* $\lambda\varepsilon_1 + (1-\lambda)\zeta_2 \geq 0$*;*

(2) $(1-\lambda)(\alpha_1 + \beta_1 + \zeta_1) + \lambda(\alpha_2 + \gamma_2 + \varepsilon_2) \geq 0$ *and* $\lambda\varepsilon_1 + (1-\lambda)\zeta_2 > 0$*,*

then q^* is a fixed point of T^*.

Proof. By (T11) and (T12) E is a strictly convex, reflexive and smooth Banach space. Therefore ϕ_* is well-defined. From (2.1) and (3.3) we obtain

$$\phi_*(u^*, v^*) = \phi_*(u^*, w^*) + \phi_*(w^*, v^*) + 2\langle J^{-1}w^* - J^{-1}v^*, u^* - w^*\rangle. \tag{4.1}$$

Therefore we obtain similarly to the proof of Theorem 4.1

$$\begin{aligned}
&((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi_*(y^*, T^*y^*) \\
&\quad +(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi_*(T^*y^*, y^*) \\
&\quad +2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle J^{-1}y^* - J^{-1}T^*y^*, p^* - y^*\rangle \\
&\le 0
\end{aligned}$$

for any $x^*, y^* \in C^*$, where $p^* \in E^*$ is a weak limit of a subsequence $\{S^*_{n_i}x^*\}$ of $\{S^*_n x^*\}$. From (4.1) we obtain

$$\begin{aligned}
&((1-\lambda)\zeta_1+\lambda\varepsilon_2)\phi_*(y^*, T^*y^*) + (\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi_*(T^*y^*, y^*) \\
&\quad +((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))(\phi_*(p^*, T^*y^*) - \phi_*(p^*, y^*)) \\
&\le 0.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))(\phi_*(p^*, T^*y^*) - \phi_*(p^*, y^*)) \\
&\le -((1-\lambda)\zeta_1+\lambda\varepsilon_2)\phi_*(y^*, T^*y^*) - (\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi_*(T^*y^*, y^*)
\end{aligned}$$

and hence

$$p^* \in \mathscr{A}^*_{-((1-\lambda)\zeta_1+\lambda\varepsilon_2), -(\lambda\varepsilon_1+(1-\lambda)\zeta_2), (1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}(T^*).$$

Next by Lemma 2.5 we obtain

$$\langle J^{-1}y^* - J^{-1}R^*(T^*)^k x^*, (T^*)^k x^* - R^*(T^*)^k x^*\rangle \le 0$$

for any $y^* \in \mathscr{B}^*_{0,0}(T^*)$. By Lemma 2.11 for any $x^* \in C^*$, $\{R^*(T^*)^n x^*\}$ is strongly convergent to an element in $\mathscr{B}^*_{0,0}(T^*)$. Putting $q^* = \lim_{n\to\infty} R^*(T^*)^n x^*$ and $K = \max_{n\in\mathbb{N}\cup\{0\}, x^*\in C^*} \|(T^*)^n x^* - R^*(T^*)^n x^*\|$, we obtain

$$\begin{aligned}
&\langle J^{-1}y^* - J^{-1}q^*, (T^*)^k x^* - R^*(T^*)^k x^*\rangle \\
&\quad \le \langle J^{-1}R^*(T^*)^k x^* - J^{-1}q^*, (T^*)^k x^* - R^*(T^*)^k x^*\rangle \\
&\quad \le \|J^{-1}R^*(T^*)^k x^* - J^{-1}q^*\| \cdot \|(T^*)^k x^* - R^*(T^*)^k x^*\| \\
&\quad \le K\|J^{-1}R^*(T^*)^k x^* - J^{-1}q^*\|.
\end{aligned}$$

Summing up these inequalities for $k = 0, \dots, n-1$ and dividing by n, we obtain

$$\left\langle J^{-1}y^* - J^{-1}q^*, S^*_n x^* - \frac{1}{n}\sum_{k=0}^{n-1} R^*(T^*)^k x^* \right\rangle \le \frac{K}{n}\sum_{k=0}^{n-1} \|J^{-1}R^*(T^*)^k x^* - J^{-1}q^*\|.$$

Since $\{S^*_{n_i}x^*\}$ is weakly convergent to p^* and by (T9) J^{-1} is norm-to-norm continuous, we obtain

$$\langle J^{-1}y^* - J^{-1}q^*, p^* - q^*\rangle \le 0.$$

Since $\mathscr{A}^*_{-((1-\lambda)\zeta_1+\lambda\varepsilon_2), -(\lambda\varepsilon_1+(1-\lambda)\zeta_2), (1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}(T^*) \subset \mathscr{B}^*_{0,0}(T^*)$, putting $y^* = p^*$, from (3.3) we obtain

$$\begin{aligned}
0 &\le 2\langle J^{-1}y^* - J^{-1}q^*, p^* - q^*\rangle \\
&= -\phi_*(p^*, q^*) - \phi_*(q^*, p^*)
\end{aligned}$$

and by (T13)* we obtain $p^* = q^*$. Therefore $\{S^*_n x^*\}$ is weakly convergent to q^*.

Additionally, if C^* is closed and convex and (1) or (2) holds, then $\{S^*_n x^*\} \subset C^*$ and hence $q^* \in C^*$. By Lemma 3.8 q^* is a fixed point of T^*. □

By Theorem 4.3 we obtain the following.

Theorem 4.5. *Let E be a strictly convex and reflexive Banach space with Kadec-Klee property and a uniformly Fréchet differentiable norm, let C be a nonempty subset of E and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_2+\beta_2+\gamma_2+\delta_2)+\lambda(\alpha_1+\beta_1+\gamma_1+\delta_1) \geq 0;\\
&\lambda(\alpha_2+\gamma_2)+(1-\lambda)(\alpha_1+\beta_1) \geq 0;\\
&\lambda(\beta_2+\delta_2)+(1-\lambda)(\gamma_1+\delta_1) \geq 0;\\
&(1-\lambda)\varepsilon_2+\lambda\zeta_1 \geq 0;\\
&\lambda\zeta_2+(1-\lambda)\varepsilon_1 \geq 0,
\end{aligned}$$

suppose that

$$\mathscr{B}_{-(\lambda\varepsilon_2+(1-\lambda)\zeta_1),-((1-\lambda)\zeta_2+\lambda\varepsilon_1),(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)}(T) \subset A(T) \neq \emptyset$$

and suppose that J^{-1} is weakly sequentially continuous. Let R^* be the sunny generalized nonexpansive retraction of E^* onto $J(A(T))$. Then for any $x \in C$,

$$S_n x \overset{def}{=} J^{-1}\left(\frac{1}{n}\sum_{k=0}^{n-1} JT^k x\right)$$

is weakly convergent to an element

$$q \in \mathscr{B}_{-(\lambda\varepsilon_2+(1-\lambda)\zeta_1),-((1-\lambda)\zeta_2+\lambda\varepsilon_1),(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)}(T),$$

where $q = \lim_{n\to\infty} J^{-1}R^*JT^n x$.

Additionally, if $J(C)$ is closed and convexx and one of the following holds:

Condition 4.6. *0000*

(1) $(1-\lambda)(\alpha_2+\beta_2+\zeta_2)+\lambda(\alpha_1+\gamma_1+\varepsilon_1) > 0$ *and* $\lambda\varepsilon_2+(1-\lambda)\zeta_1 \geq 0$*;*

(2) $(1-\lambda)(\alpha_2+\beta_2+\zeta_2)+\lambda(\alpha_1+\gamma_1+\varepsilon_1) \geq 0$ *and* $\lambda\varepsilon_2+(1-\lambda)\zeta_1 > 0$*,*

then q is a fixed point of T.

Proof. By (T11) and (T12) E^* is uniformly convex with a Fréchet differentiable norm. Let $T^* = JTJ^{-1}$. Then T^* is a mapping from $J(C)$ into itself. Putting $x^* = Jx$ and $y^* = Jy$, By Lemma 2.14 T^* is an $(\alpha_2, \alpha_1, \beta_2, \beta_1, \gamma_2, \gamma_1, \delta_2, \delta_1, \varepsilon_2, \varepsilon_1, \zeta_2, \zeta_1)$-*-generalized pseudocontraction from $J(C)$ into itself. Since $(T^*)^n x^* = JT^n x$, $\|(T^*)^n x^*\| = \|JT^n x\| = \|T^n x\|$ and hence $\{(T^*)^n x^* \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. By Lemma 3.5

$$\begin{aligned}
&\mathscr{A}^*_{-((1-\lambda)\zeta_2+\lambda\varepsilon_1),-(\lambda\varepsilon_2+(1-\lambda)\zeta_1,(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_2+\gamma_2)}(T^*)\\
&\qquad = J\left(\mathscr{B}_{-(\lambda\varepsilon_2+(1-\lambda)\zeta_1),-((1-\lambda)\zeta_2+\lambda\varepsilon_1),(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)}(T)\right),\\
&B^*_{0,0}(T^*) = J(A(T)).
\end{aligned}$$

By Theorem 4.3 for any $x \in C$,

$$S_n^* x^* = \frac{1}{n}\sum_{k=0}^{n-1}(T^*)^k x^*$$

is weakly convergent to an element

$$q^* \in J\left(\mathscr{B}_{-(\lambda\varepsilon_2+(1-\lambda)\zeta_1),-((1-\lambda)\zeta_2+\lambda\varepsilon_1),(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)}(T)\right),$$

where $q^* = \lim_{n\to\infty} R^* J T^n x$. Since J^{-1} is weakly sequentially continuous and by (T9) J^{-1} is norm-to-norm continuous,

$$S_n x = J^{-1} S_n^* J x = J^{-1} \left(\frac{1}{n} \sum_{k=0}^{n-1} J T^k x \right)$$

is weakly convergent to the element

$$q = J^{-1} q^* \in \mathscr{B}_{-(\lambda\varepsilon_2+(1-\lambda)\zeta_1),-((1-\lambda)\zeta_2+\lambda\varepsilon_1),(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)}(T),$$

where $q = \lim_{n\to\infty} J^{-1} R^* J T^n x$.

Additionally, if $J(C)$ is closed and convex and (1) or (2) holds, then q^* is a fixed point of T^* and hence $q = J^{-1} q^*$ is a fixed point of T. □

5. Remark and example

In the proof using the concept of acute or skew-acute point we needed the assumption $(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2) > 0$ or $(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1) > 0$ in addition to

$$\begin{aligned}
&(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1)+\lambda(\alpha_2+\beta_2+\gamma_2+\delta_2) \geq 0;\\
&\lambda(\alpha_1+\gamma_1)+(1-\lambda)(\alpha_2+\beta_2) \geq 0;\\
&\lambda(\beta_1+\delta_1)+(1-\lambda)(\gamma_2+\delta_2) \geq 0;\\
&(1-\lambda)\varepsilon_1+\lambda\zeta_2 \geq 0;\\
&\lambda\zeta_1+(1-\lambda)\varepsilon_2 \geq 0,
\end{aligned}$$

or

$$\begin{aligned}
&(1-\lambda)(\alpha_2+\beta_2+\gamma_2+\delta_2)+\lambda(\alpha_1+\beta_1+\gamma_1+\delta_1) \geq 0;\\
&\lambda(\alpha_2+\gamma_2)+(1-\lambda)(\alpha_1+\beta_1) \geq 0;\\
&\lambda(\beta_2+\delta_2)+(1-\lambda)(\gamma_1+\delta_1) \geq 0;\\
&(1-\lambda)\varepsilon_2+\lambda\zeta_1 \geq 0;\\
&\lambda\zeta_2+(1-\lambda)\varepsilon_1 \geq 0;
\end{aligned}$$

see [14].

However the assumption $(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2) > 0$ or $(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1) > 0$ is not needed [17]. Therefore we wondered if the condition was unnecessary. In this paper we generalize the concept of acute point and by using the concept of generalized acute and skew-acute point we do not need the assumptions $(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2) > 0$ and $(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1) > 0$.

We consider an example.

Example 5.1. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm, let C be a nonempty subset of E and let T be an $(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,\delta_1,\delta_2,\varepsilon_1,\varepsilon_2,\zeta_1,\zeta_2)$-generalized pseudocontraction from C into itself. Suppose that*

$$\begin{aligned}
&\alpha_1,\alpha_2 \in \mathbb{R};\\
&\beta_1=\gamma_1=-\alpha_1;\ \delta_1=\alpha_1;\\
&\beta_2=\gamma_2=-\alpha_2;\ \delta_2=\alpha_2;\\
&\varepsilon_1,\varepsilon_2 \in [0,\infty); \zeta_1=\varepsilon_2; \zeta_2=\varepsilon_1,
\end{aligned}$$

and suppose that

$$\mathscr{A}_{-((1-\lambda)\zeta_1+\lambda\varepsilon_2),-(\lambda\varepsilon_1+(1-\lambda)\zeta_2),(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}(T) \subset B(T) \neq \emptyset.$$

Then for any $\lambda \in [0,1]$ *we obtain*

$$\begin{aligned}
&(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1)+\lambda(\alpha_2+\beta_2+\gamma_2+\delta_2)=0;\\
&\lambda(\alpha_1+\gamma_1)+(1-\lambda)(\alpha_2+\beta_2)=0;\\
&\lambda(\beta_1+\delta_1)+(1-\lambda)(\gamma_2+\delta_2)=0;\\
&(1-\lambda)\varepsilon_1+\lambda\zeta_2\geq 0;\\
&\lambda\zeta_1+(1-\lambda)\varepsilon_2\geq 0.
\end{aligned}$$

Furthermore

$$(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)=(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)=0.$$

Unfortunately, by the previous theorem [14, Theorem 4.1] we cannot show the mean convergence theorem to acute point, and of course, the mean convergence theorem to fixed point. However, by using Theorem 4.1 we can show the mean convergence theorem to generalized acute point and the mean convergence theorem to fixed point.

References

[1] S. Atsushiba, S. Iemoto, R. Kubota, and Y. Takeuchi, *Convergence theorems for some classes of nonlinear mappings in Hilbert spaces*, Linear and Nonlinear Analysis **2** (2016), 125–153.

[2] J.-B. Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences. Séries A et B **280** (1975), 1511–1514.

[3] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, Journal of Mathematical Analysis and Applications **20** (1967), 197–228.

[4] D. Butnariu, A. N. Iusem, and E. Resmerita, *Total convexity for powers of the norm in uniformly convex Banach spaces*, Journal of Convex Analysis **7** (2000), 319–334.

[5] I. Ciorănescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Mathematics and its applications, vol. 62, Kluwer Academic Publishers, Dordrecht, 1990.

[6] T. Ibaraki and W. Takahashi, *A new projection and convergence theorems for the projections in Banach spaces*, J. Approx. Theory **149** (2007), 1–14.

[7] T. Kawasaki, *An extension of existence and mean approximation of fixed points of generalized hybrid non-self mappings in Hilbert spaces*, preprint.

[8] ______, *Fixed points theorems and mean convergence theorems for generalized hybrid self mappings and non-self mappings in Hilbert spaces*, Pacific Journal of Optimization **12** (2016), 133–150.

[9] ______, *Fixed point theorem for widely more generalized hybrid demicontinuous mappings in Hilbert spaces*, Proceedings of Nonlinear Analysis and Convex Analysis, Yokohama Publishers, Yokohama, to appear.

[10] ______, *Fixed point theorems for widely more generalized hybrid mappings in metric spaces, Banach spaces and Hilbert spaces*, Journal of Nonlinear and Convex Analysis **19** (2018), 1675–1683.

[11] ______, *On convergence of orbits to a fixed point for widely more generalized hybrid mappings*, Nihonkai Mathematical Journal **27** (2016), 89–97.

[12] ______, *Fixed point theorems for widely more generalized hybrid mappings in a metric space, a Banach space and a Hilbert space*, Journal of Nonlinear and Convex Analysis **19** (2018), 1675–1683.

[13] ______, *Fixed point and acute point theorems for new mappings in a Banach space*, Journal of Mathematics **2019** (2019), 12 pages.

[14] ______, *Mean convergence theorems for new mappings in a Banach space*, Journal of Nonlinear and Variational Analysis **3** (2019), 61–78.

[15] ______, *Fixed point and acute point theorems for generalized pseudocontractions in a Banach space*, Proceedings of Nonlinear Analysis and Convex Analysis, Yokohama Publishers, Yokohama, submitted.

[16] ______, *Weak convergence theorems for new mappings in a Banach space*, Linear and Nonlinear Analysis **5** (2019), 147–171.

[17] ______, *Generalized acute point theorems for generalized pseudocontractions in a Banach space*, Linear and Nonlinear Analysis **6** (2020), 73–90.

[18] T. Kawasaki and T. Kobayashi, *Existence and mean approximation of fixed points of generalized hybrid non-self mappings in Hilbert spaces*, Scientiae Mathematicae Japonicae **77 (Online Version: e-2014)** (2014), 13–26 (Online Version: 29–42).

[19] T. Kawasaki and W. Takahashi, *Existence and mean approximation of fixed points of generalized hybrid mappings in Hilbert spaces*, Journal of Nonlinear and Convex Analysis **14** (2013), 71–87.

[20] ______, *Fixed point and nonlinear ergodic theorems for widely more generalized hybrid mappings in Hilbert spaces and applications*, Proceedings of Nonlinear Analysis and Convex Analysis, Yokohama Publishers, Yokohama, to appear.

[21] ______, *Fixed point theorems for generalized hybrid demicontinuous mappings in Hilbert spaces*, Linear and Nonlinear Analysis **1** (2015), 125–138.

[22] P. Kocourek, W. Takahashi, and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese Journal of Mathematics **14** (2010), 2497–2511.

[23] F. Kohsaka and W. Takahashi, *Generalized nonexpansive retractions and a proximal-type algorithm in Banach spaces*, Journal of Nonlinear and Convex Analysis **8** (2007), 197–209.

[24] ______, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Archiv der Mathematik **91** (2008), 166–177.

[25] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[26] ______, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, Journal of Nonlinear and Convex Analysis **11** (2010), 79–88.

[27] W. Takahashi and Y. Takeuchi, *Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space*, Journal of Nonlinear and Convex Analysis **12** (2011), 399–406.

[28] W. Takahashi, N.-C. Wong, and J.-C. Yao, *Attractive point and mean convergence theorems for new generalized nonspreading mappings in Banach spaces*, Infinite Products of Operators and Their Applications (S. Reich and A. J. Zaslavski, eds.), Contemporary Mathematics, vol. 636, American Mathematical Society, Providence, 2015, pp. 225–248.