Araştırma Makalesi *BAUN Fen Bil. Enst. Dergisi, 26(1), 111-123, (2024)*

DOI:10.25092/baunfbed. 1300223 *J. BAUN Inst. Sci. Technol., 26(1), 111-123, (2024)*

The stability of Jordan k-$*$-derivations on Γ^*-Banach algebras by fixed point method

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Geliş Tarihi (Received Date): 21.05.2023
Kabul Tarihi (Accepted Date): 13.09.2023

Abstract

Using fixed point methods, we prove the stability and the superstability of Jordan k-$*$-derivations on Γ^*-Banach algebras for the following Jensen-type functional equation $\mu f\left(\frac{x+y}{2}\right) + \mu f\left(\frac{x-y}{2}\right) = f(\mu x)$ where μ is a complex number such that $|\mu| = 1$. We also investigate the stability and the superstability of Jordan k-$*$-derivations with the functional equation $f(2\mu x + \mu y) + f(\mu x + 2\mu y) = \mu[f(3x) + f(3y)]$ on Γ^*-Banach algebras.

Keywords: Γ-rings with involution, Hyers-Ulam-Rassias stability, Jordan k-$*$-derivations.

Jordan k-$*$-türevlerinin Γ^*-Banach cebirlerinde sabit nokta yöntemiyle stabilitesi

Öz

Sabit nokta yöntemlerini kullanarak, $\mu f\left(\frac{x+y}{2}\right) + \mu f\left(\frac{x-y}{2}\right) = f(\mu x)$ ile tanımlı Jensen tipi fonksiyonel denklem için Jordan k-$*$-türevlerinin Γ^*-Banach cebirleri üzerindeki stabilitesini ve süper stabilitesini kanıtlıyoruz. Burada μ sayısı, $|\mu| = 1$ şartını sağlayan bir karmaşık sayıdır. Ayrıca, Γ^*-Banach cebirleri üzerindeki $f(2\mu x + \mu y) + f(\mu x + 2\mu y) = \mu[f(3x) + f(3y)]$ fonksiyonel denklemi ile Jordan k-$*$-türevlerinin stabilitesini ve süper stabilitesini araştırıyoruz.

Anahtar kelimeler: *İnvolüsyonlu* Γ-*halkaları,* Hyers-Ulam-Rassias stabilitesi, Jordan k-*-türevler.

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1. Introduction and preliminaries

In 1940, S.M. Ulam [1] made a significant contribution by introducing the concept of stability of functional equations. He proposed the following problem:

Let $(G_1,*)$ *be a group and* $(G_2,\circ,d)$ *be a metric group with the metric* $d(.,.)$*. Given* $\varepsilon > 0$*, does there exist a* $\delta(\varepsilon) > 0$ *such that if a function* $f: G_1 \rightarrow G_2$ *satisfies the inequality*

$$d(f(x * y), f(x) \circ f(y)) < \delta$$

for all $x, y \in G_1$, then there exists a homomorphism $H: G_1 \to G_2$ such that $d(f(x), H(x)) < \varepsilon$ for all $x \in G_1$?

More simply, we search for conditions that allow the existence of a homomorphism between a group and a metric group that closely approximates a homomorphism. When these conditions are satisfied, we call homomorphisms G_1 to G_2 stable.

In 1940, D.H. Hyers [2] made a significant advance by providing the first partial solution to Ulam's question, in particular in the case of approximate additive mappings involving Banach spaces G_1 and G_2. Then, Th.M. Rassias [3] extended the Hyers's stability theorem for linear mappings by considering the existence of an unbounded Cauchy difference. Subsequently, many researchers have investigated stability problems associated with a wide variety of functional equations [4-6].

Let A be a real or complex algebra. A linear mapping $D: A \to A$ is said to be a *derivation* if $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$. P. Semrl was the first to achieve stability results concerning derivations between operator algebras [7]. Recall that a Banach $*$-algebra is a Banach algebra which has an isometric involution. In general, all C^*-algebras are Banach $*$-algebra. Recently, the stability of $*$-derivations and Jordan $*$-derivations on Banach $*$-algebras has been investigated by several authors, see [8-10] and references therein.

The concept of Γ-ring introduced by N. Nobusawa [11] extended the notion of ring to a more general setting. Based on Nobusawa's definition, W.E. Barnes [12] further extended the scope of Γ-rings by slightly modifying the condition. The study of derivations and Jordan derivations in the context of Γ-rings was later investigated by M. Sapanci and A. Nakajima [13], while H. Kandamar [14] further advanced the field by introducing and developing the concept of k-derivations in Γ-rings. In recent years, many results of Γ-rings have been extended to Γ-algebras.

Suppose that V and Γ are linear spaces over a field F. Then V is said to be a Γ-*algebra* (in the sense of Barnes [12]) over F if there exists a mapping $V \times \Gamma \times V \to V$ (the image of (x, α, y) is denoted by $x\alpha y$) satisfying the following conditions:

(i) $(x\alpha y)\beta z = x\alpha(y\beta z)$,
(ii) $\lambda(x\alpha y) = (\lambda x)\alpha y = x(\lambda\alpha)y = x\alpha(\lambda y)$,

(iii) $x\alpha(y+z) = x\alpha y + x\alpha z,$
$x(\alpha+\beta)y = x\alpha y + x\beta y,$
$(x+y)\alpha z = x\alpha z + y\alpha z$

for all $x, y, z \in V,\ \alpha, \beta \in \Gamma,\ \lambda \in F$. The Γ-algebra V is denoted by (Γ, V).

If V and Γ are normed linear spaces over F, then Γ-algebra (Γ, V) is called a Γ-*normed algebra* if conditions (i)-(iii) hold and further

(iv) $\| x\alpha y \| \leq \| x \| \cdot \| \alpha \| \cdot \| y \|$ holds for all $x, y \in V$ and $\alpha \in \Gamma$.

Bhattacharya and Maity [15] introduced the notion of Γ-Banach algebra, which extends the traditional notions of Banach algebras and Γ-rings. A Γ-normed algebra (Γ, V) is classified as a Γ-Banach algebra when the underlying space V is a Banach space. Interestingly, any Banach algebra can be considered as Γ-Banach algebra by choosing the appropriate Γ. Γ-Banach algebras are a generalization of the notions of both Banach algebras and Γ-rings. For example, the set of all bounded linear maps from an infinite dimensional normed linear space X to a Banach space Y and the set of all $m \times n$ rectangular matrices are Γ-Banach algebras which are not general Banach algebras.

In 2014, Hoque et al. [16] introduced the notion of Γ-ring with involution (also known as Γ^*-ring). Now we give the definition of a Γ^*-algebra:

Definition 1.1. *An involution on a* Γ*-algebra* V *is a map* $x \mapsto x^*$ *of* V *onto itself such that*

(i) $(x\alpha y)^* = y^*\alpha x^*,$
(ii) $(x+\lambda y)^* = x^* + \lambda y^*,$
(iii) $(x^*)^* = x$

for all $x, y \in V,\ \alpha \in \Gamma$ *and* $\lambda \in \mathbb{C}$. *A* Γ*-algebra* V *equipped with an involution is called a* Γ^**-algebra. The* Γ^**-algebra* V *is denoted by* (Γ^*, V).

Now we define a k-∗-derivation and a Jordan k-∗-derivation on Γ^*-algebras as follows: Let V be a Γ^*-algebra over a field F and let $d: V \to V$ and $k: \Gamma \to \Gamma$ be two linear operator. If

$$d(x\alpha y) = d(x)\alpha y^* + xk(\alpha)y + x\alpha d(y)$$

for all $x, y \in V$ and $\alpha \in \Gamma$, then d is called a k-∗-*derivation* of V. If $d(x\alpha x) = d(x)\alpha x^* + xk(\alpha)x + x\alpha d(x)$ holds for all $x \in V$ and $\alpha \in \Gamma$, then d is called a *Jordan* k-∗-*derivation* of V. It is clear that every k-∗-derivation of a Γ^*-algebra V is a Jordan k-∗-derivation of V. However, the converse is not true in general.

In the proofs of our theorems, we shall use the following lemma which is proved in [17].

Lemma 1.2. *[17] Let X and Y be linear spaces,* $\mathbb{T} := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$ *and* $f: X \to Y$ *be an additive mapping such that* $f(\mu x) = \mu f(x)$ *for all* $x \in X$ *and all* $\mu \in \mathbb{T}$. *Then the mapping* f *is* $\mathbb{C}$*-linear.*

Let X be a nonempty set. A function $d: X \times X \to [0, \infty]$ is called a *generalized metric* on X if and only if d satisfies

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq d(x, y) + d(y, z)$

for all $x, y, z \in X$. Then (X, d) is called a *generalized metric space*.

It is worth noting that the main difference between the generalized metric and the usual metric lies in the fact that the former allows the inclusion of infinity in its range.

Now, we need the use of the next fixed point theorem (see [18]):

Theorem 1.3. *Let (X, d) be a complete generalized metric space and $\Lambda: X \to X$ be a strictly contractive mapping; that is,*

$$d(\Lambda x, \Lambda y) \leq L d(x, y) \qquad (x, y \in X)$$

with the Lipschitz constant $0 \leq L < 1$. Then, for each given $x \in X$, either $d(\Lambda^{n+1} x, \Lambda^n x) = \infty$ for all $n \geq 0$, or otherwise there exists a natural number n_0 such that

(i) $d(\Lambda^{n+1} x, \Lambda^n x) < \infty$ *for all* $n \geq n_0$,
(ii) the sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ,
(iii) x^* is the unique fixed point of Λ in the set

$$X^* = \{y \in X \mid d(\Lambda^{n_0} x, y) < \infty\};$$

(iv) $d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y)$ *for all* $y \in X$.

B. Arslan and O. Arslan [19] proved the stability and superstability of homomorphisms and *k*-derivations associated with the Cauchy functional equation on Γ-Banach algebras using the direct method.

In this paper, we will use the fixed point alternative introduced by Cadariu and Radu as a powerful tool to determine the stability and superstability of Jordan k-∗-derivations associated with the following Jensen-type functional equation

$$\mu f\left(\frac{x+y}{2}\right) + \mu f\left(\frac{x-y}{2}\right) = f(\mu x) \qquad (\mu \in T) \tag{1}$$

in Γ^*-Banach algebras. We also prove the stability and superstability of Jordan *k*-∗-derivations with the functional equation

$$f(2\mu x + \mu y) + f(\mu x + 2\mu y) = \mu[f(3x) + f(3y)] \qquad (\mu \in T)$$

in Γ^*-Banach algebras.

2. Stability of Jordan k-$*$-derivations

In this section, we first prove the Hyers-Ulam-Rassias stability of Jordan k-$*$-derivations associated with the Jensen-type functional equation (1) in Γ^*-Banach algebras.

Theorem 2.1. *Let V be a Γ^*-Banach algebra over the complex field $\mathbb{C}$. Suppose $f: V \to V$ is a mapping satisfying $f(0) = 0$ and $g: \Gamma \to \Gamma$ is a mapping satisfying $g(0) = 0$ for which there exist functions $\phi_1 : V \times V \times V \to [0, \infty)$, $\phi_2: \Gamma \times \Gamma \to [0, \infty)$ such that*

$$\| \mu f\left(\frac{x+y}{2}\right) + \mu f\left(\frac{x-y}{2}\right) - f(\mu x) + f(a\alpha a) - f(a)\alpha a^* - a g(\alpha) a - a\alpha f(a) \| \\ \leq \phi_1(x, y, a), \tag{2}$$

$$\| g(\mu\alpha + \mu\beta) - \mu g(\alpha) - \mu g(\beta) \| \leq \phi_2(\alpha, \beta), \tag{3}$$

for all $\mu \in \mathbb{T} := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$ and for all $x, y, a \in V$, $\alpha, \beta \in \Gamma$. If there exist L_1, $L_2 < 1$ such that $\phi_1(x, y, a) \leq 2L_1\phi_1(\frac{x}{2}, \frac{y}{2}, \frac{a}{2})$ and $\phi_2(\alpha, \beta) \leq 2L_2\phi_2(\frac{\alpha}{2}, \frac{\beta}{2})$ for all $x, y, a \in V$ and $\alpha, \beta \in \Gamma$, then there exists a unique linear map k from Γ to Γ satisfying

$$\| g(\alpha) - k(\alpha) \| \leq \frac{1}{2(1-L_2)} \phi_2(\alpha, \alpha), \tag{4}$$

and there exists a unique Jordan k-$*$-derivation $D: V \to V$ such that

$$\| f(x) - D(x) \| \leq \frac{L_1}{1-L_1} \phi_1(x, 0,0) \tag{5}$$

for all $x \in V$ and $\alpha \in \Gamma$.

Proof. It follows from $\phi_1(x, y, a) \leq 2L_1\phi_1(\frac{x}{2}, \frac{y}{2}, \frac{a}{2})$ that

$$\lim_j 2^{-j}\phi_1(2^j x, 2^j y, 2^j a) = 0 \tag{6}$$

for all $x, y, a \in V$. By putting $\mu = 1$, $y = a = 0$ in (2), we get

$$\| 2 f\left(\frac{x}{2}\right) - f(x) \| \leq \phi_1(x, 0, 0) \tag{7}$$

for all $x \in V$. Thus, we have

$$\| \frac{1}{2} f(2x) - f(x) \| \leq \frac{1}{2}\phi_1(2x, 0,0) \leq L_1\phi_1(x, 0,0) \tag{8}$$

for all $x \in V$. Now, we consider the set $X := \{h \mid h: V \to V\}$ and we introduce the generalized metric on X:

$$d(r, h) := inf\{C \in \mathbb{R}^+ : \| h(x) - r(x) \| \leq C.\phi_1(x, 0,0) \; for \; all \; x \in V\}.$$

It can be easily shown that X is complete with the generalized metric d. Define the linear mapping $J: X \to X$ by $J(r)(x) = \frac{1}{2} r(2x)$ for all $r \in X, x \in V$. By Theorem 1.3, $d\big(J(h), J(r)\big) \leq L_1 d(h, r)$ for all $h, r \in X$. From (8), we have that

$$d\big(f, J(f)\big) \leq L_1.$$

As stated in Theorem 1.3, there is a unique fixed point in the set $X_1 := \{r \in X \mid d(f, r) < \infty\}$ for the mapping J. Let us denote this fixed point as D. Consequently, D is the unique mapping which has the property that $D(2x) = 2D(x)$ for all $x \in V$ and there is a positive constant C in the interval $(0, \infty)$ such that the inequality $\|\|D(x) - f(x)\| \leq C.\phi_1(x, 0,0)$ holds for all $x \in V$. Moreover, we have $\lim_n d(J^n(f), D) = 0$. So,

$$\lim_n \frac{1}{2^n} f(2^n x) = D(x) \tag{9}$$

for all $x \in V$. Since we have $d(f, D) \leq \frac{1}{1-L_1} d(f, J(f))$, then we get $d(f, D) \leq \frac{L_1}{1-L_1}$. This implies the inequality (5). From (2), (6) and (9), we get

$$\| D\left(\frac{x+y}{2}\right) + D\left(\frac{x-y}{2}\right) - D(x) \|$$

$$= \lim_n \frac{1}{2^n} \| f(2^{n-1}(x+y)) + f(2^{n-1}(x-y)) - f(2^n x) \|$$

$$\leq \lim_n \frac{1}{2^n} \phi_1(2^n x, 2^n y, 0) = 0$$

for all $x, y \in V$. So

$$D\left(\frac{x+y}{2}\right) + D\left(\frac{x-y}{2}\right) = D(x)$$

for all $x, y \in V$. Putting $z = \frac{x+y}{2}, t = \frac{x-y}{2}$ in the above equation, we have that

$$D(z) + D(t) = D(z+t)$$

for all $z, t \in V$. Thus D is Cauchy additive. Putting $y = x,\ a = 0$ in (2), we get $\| \mu f(x) - f(\mu x) \| \leq \phi_1(x, x, 0)$ for all $x \in V$. Thus

$$\| D(2\mu x) + 2\mu D(x) \| = \lim_n \frac{1}{2^n} \| f(2\mu 2^n x) - 2\mu f(2^n x) \|$$

$$\leq \lim_n \frac{1}{2^n} \phi_1(2^n x, 2^n x, 0) = 0$$

for all $\mu \in \mathbb{T}$, and all $x \in V$. According to Lemma 1.2, it can be deduced that the mapping $D: V \to V$ is $\mathbb{C}$-linear. Putting $\mu = 1,\ \alpha = \beta$ in (3), we have that $\| g(2\alpha) - 2g(\alpha) \| \leq \phi_2(\alpha, \alpha)$ for all $\alpha \in \Gamma$. Hence,

$$\left\|\frac{1}{2}g(2\alpha)-g(\alpha)\right\| \le \frac{1}{2}\phi_2(\alpha,\alpha) \tag{10}$$

for all $\alpha \in \Gamma$. Now we consider the set $Y := \{q \mid q: \Gamma \to \Gamma\}$ and define the mapping d' on $Y \times Y$ as follows:

$$d'(q_1,q_2) := inf\{C \in \mathbb{R}^+ : \| q_1(\alpha)-q_2(\alpha) \| \le C\phi_2(\alpha,\alpha)\ for\ all\ \alpha \in \Gamma\}.$$

It is clear that d' is a generalized metric on Y and the metric space (Y,d') is complete. Also the mapping $J': Y \to Y$ defined by

$$J'(q)(\alpha) = \frac{1}{2}q(2\alpha) \qquad (\alpha \in \Gamma)$$

is strictly contractive on Y. That is, $d'(J'(q_1),J'(q_2)) \le L_2 d'(q_1,q_2)$ for all $q_1,q_2 \in Y$. It follows from (10) that

$$d'(g,J'(g)) \le \frac{1}{2}.$$

Let $Y_1 := \{r \in Y \mid d'(g,r) < \infty\}$. Then J' has a unique fixed point in the set Y_1 by Theorem 1.3. Therefore, there exists a unique mapping $k: \Gamma \to \Gamma$ such that k is the fixed point of J' and $k(\alpha) := \lim_n 2^{-n} g(2^n\alpha)$ for all $\alpha \in \Gamma$, and so

$$d'(g,k) \le \frac{1}{1-L_2} d'(J'(g),g) \le \frac{1}{2(1-L_2)}.$$

Hence (4) holds for all $\alpha \in \Gamma$. Also we have

$$\lim_j 2^{-j}\phi_2(2^j\alpha, 2^j\beta) = 0.$$

Replacing α by $2^n\alpha$ and putting $\beta = 0$ in (3), we have $k(\mu\alpha) = \mu k(\alpha)$ for all $\alpha \in \Gamma$ and all $\mu \in \mathbb{T}$. Now, by Lemma 1.2, k is $\mathbb{C}$-linear.

It follows from (2) that

$$\| D(a\alpha a) - D(a)\alpha a^* - ak(\alpha)a - a\alpha D(a) \|$$

$$= \lim_n \left\| \frac{1}{2^{2n}} f(2^n a\alpha a) - \frac{1}{2^n} f(2^n a)\alpha a^* - \frac{1}{2^n} ag(2^n\alpha)a - \frac{1}{2^n} a\alpha f(2^n a) \right\|$$

$$= \lim_n \left\| \frac{1}{2^{2n}} f(2^n a\alpha 2^n a) - \frac{1}{2^{2n}} f(2^n a)\alpha 2^n a^* - \frac{1}{2^{2n}} 2^n a g(\alpha) 2^n a - \frac{1}{2^{2n}} 2^n a\alpha f(2^n a) \right\|$$

$$\le \lim_n \frac{1}{2^{2n}} \phi_1(0,0,2^n a) \le \lim_n \frac{1}{2^n} \phi_1(0,0,2^n a) = 0$$

for all $x, a \in V, \alpha \in \Gamma$. Thus D is a Jordan k-$*$-derivation on (Γ^*, V).

Corollary 2.2. *Let V be a Γ^*-Banach algebra over the complex field $\mathbb{C}$. Suppose $f : V \to V$ is a mapping satisfying $f(0) = 0$ and $g: \Gamma \to \Gamma$ is a mapping satisfying $g(0) = 0$ for which there exist real numbers $\theta_1, \theta_2 \in [0, \infty)$ and $p, t \in (0,1)$ such that*

$$\left\| \mu f\left(\frac{x+y}{2}\right) + \mu f\left(\frac{x-y}{2}\right) - f(\mu x) + f(a\alpha a) - f(a)\alpha a^* - a g(\alpha) a - a\alpha f(a) \right\| \leq \theta_1(\| x \|^p + \| y \|^p + \| a \|^p),$$

$$\| g(\mu\alpha + \mu\beta) - \mu g(\alpha) - \mu g(\beta) \| \leq \theta_2(\| \alpha \|^t + \| \beta \|^t)$$

for all $\mu \in \mathbb{T}$ and all $x, y, a \in V, \alpha, \beta \in \Gamma$. Then there exists a unique linear mapping k from Γ to Γ satisfying

$$\| g(\alpha) - k(\alpha) \| \leq \frac{2\theta_2}{2-2^t} \|\alpha\|^t,$$

and there exists a unique Jordan k-$*$-derivation $D: V \to V$ such that

$$\| f(x) - D(x) \| \leq \frac{2^p\theta_1}{2-2^p} \|x\|^p$$

for all $x \in V$ and $\alpha \in \Gamma$.

Proof. In Theorem 2.1, if we take $\phi_1(x, y, a) := \theta_1(\| x \|^p + \| y \|^p + \| a \|^p)$ with $L_1 = 2^{p-1}$, and $\phi_2(\alpha, \beta) := \theta_2(\| \alpha \|^t + \| \beta \|^t)$ with $L_2 = 2^{t-1}$ for all $x, y, a \in V$, $\alpha, \beta \in \Gamma$, then we get the desired result.

Now, we have the following superstability result of Jordan k-$*$-derivations.

Corollary 2.3. *Let V be a Γ^*-Banach algebra over the complex field $\mathbb{C}$. Suppose that $p, q, r, t, l, \theta_1, \theta_2$ are non-negative real numbers with $0 < p + q + r \neq 1, 0 < t + l \neq 1$ and suppose that $f : V \to V$ and $g: \Gamma \to \Gamma$ are mappings such that*

$$\left\| \mu f\left(\frac{x+y}{2}\right) + \mu f\left(\frac{x-y}{2}\right) - f(\mu x) + f(a\alpha a) - f(a)\alpha a^* - a g(\alpha) a - a\alpha f(a) \right\| \leq \theta_1(\| x \|^p \| y \|^q \| a \|^r),$$

$$\| g(\mu\alpha + \mu\beta) - \mu g(\alpha) - \mu g(\beta) \| \leq \theta_2(\| \alpha \|^t \| \beta \|^l)$$

for all $\mu \in \mathbb{T}$ and all $x, y, a \in V$, $\alpha, \beta \in \Gamma$. Then f is a Jordan k-$*$-derivation on (Γ^*, V), where $k: \Gamma \to \Gamma$ is a linear map.

Proof. Putting $x = y = a = 0$ and $\mu = 1$ in (11), we have that $f(0) = 0$. If we put $\alpha = 0$ and $\mu = 1$ in (12), then we have $g(0) = 0$. Putting $a = y = 0$ and $\mu = 1$ in (11), we conclude that $2f\left(\frac{x}{2}\right) = f(x)$ for all $x \in V$, and by using induction we get $f(x) =$

$\frac{f(2^n x)}{2^n}$ for all $x \in V$ and $n \in \mathbb{N}$. If we put $\alpha = \beta$ and $\mu = 1$ in (12), we have that $g(2\alpha) = 2g(\alpha)$ for all $\alpha \in \Gamma$, and using induction again we get $g(\alpha) = \frac{g(2^n \alpha)}{2^n}$ for all $\alpha \in \Gamma$ and $n \in \mathbb{N}$. By referring to Theorem 2.1 when changing $\phi_1(x, y, a) := \theta_1(\| x \|^p \| y \|^q \| a \|^r)$ and $\phi_2(\alpha, \beta) := \theta_2(\| \alpha \|^t \| \beta \|^l)$ for all $x, y, a \in V$ and $\alpha, \beta \in \Gamma$, we can obtain the desired result.

In 2009, A. Najati and C. Park [20] obtained the general solution of the following functional equation

$$f(2\mu x + \mu y) + f(\mu x + 2\mu y) = \mu[f(3x) + f(3y)] \qquad (\mu \in \mathbb{T}) \tag{13}$$

and they investigated the stability of homomorphisms and generalized derivations for these functional equations on Banach algebras.

Lemma 2.4. *[13] Let X and Y be linear spaces. A mapping $f: X \to Y$ satisfies the functional equation (13) for all $x, y \in X$ and all $\mu \in \mathbb{T}$, if and only if f is $\mathbb{C}$-linear.*

Now we consider the Hyers-Ulam-Rassias stability of Jordan k-$*$-derivations in Γ^*-Banach algebras with the functional equation (13).

Theorem 2.5. *Let V be a Γ^*-Banach algebra over the complex field $\mathbb{C}$. Suppose $f: V \to V$ is a mapping such that $f(0) = 0$ and $g: \Gamma \to \Gamma$ is a mapping such that $g(0) = 0$ for which there exist functions $\varphi_1 : V \times V \times V \to [0, \infty)$ and $\varphi_2 : \Gamma \times \Gamma \to [0, \infty)$ satisfying*

$$\| f(2\mu x + \mu y) + f(\mu x + 2\mu y) - \mu[f(3x) + f(3y)] + f(a\alpha a) - f(a)\alpha a^* - a g(\alpha) a - a\alpha f(a) \| \leq \varphi_1(x, y, a), \tag{14}$$

$$\| g(\mu\alpha + \mu\beta) - \mu g(\alpha) - \mu g(\beta) \| \leq \varphi_2(\alpha, \beta) \tag{15}$$

for all $\mu \in \mathbb{T}$ and $x, y, a \in V$, $\alpha, \beta \in \Gamma$. If there exist L_1, $L_2 < 1$ such that

$$4\varphi_1(x, y, a) \leq L_1 \varphi_1(2x, 2y, 2a) \tag{16}$$

and $\varphi_2(2\alpha, 2\beta) \leq 2L_2 \varphi_2(\alpha, \beta)$ for all $x, y, a \in V$, $\alpha, \beta \in \Gamma$, then there exists a unique linear map k from Γ to Γ satisfying

$$\| g(\alpha) - k(\alpha) \| \leq \frac{1}{2 - 2L_2} \varphi_2(\alpha, \alpha), \tag{17}$$

and there exists a unique Jordan k-$*$-derivation $D: V \to V$ such that

$$\| f(x) - D(x) \| \leq \frac{L_1}{4 - 2L_1} \psi(x) \tag{18}$$

for all $x \in V$ and $\alpha \in \Gamma$, where

$$\psi(x) := \varphi_1\left(\frac{x}{2}, 0,0\right) + \varphi_1\left(-\frac{x}{2}, 0,0\right) + \varphi_1\left(\frac{x}{2}, -\frac{x}{2}, 0\right) + \varphi_1\left(\frac{2x}{3}, -\frac{x}{3}, 0\right).$$

Proof. It follows from $4\varphi_1(x,y,\alpha) \le L_1\varphi_1(2x,2y,2\alpha)$ that

$$\lim_j 2^{2j}\varphi_1(\frac{x}{2^j},\frac{y}{2^j},\frac{a}{2^j}) = 0 \tag{19}$$

for all $x, y, a \in V$. Putting $\mu = 1,\ y = a = 0$ in (14), we get

$$\| f(2x) + f(x) - f(3x) \| \le \varphi_1(x, 0,0) \tag{20}$$

for all $x \in V$. Hence

$$\| (f(x) + f(-x)) + (f(2x) + f(-2x)) - (f(3x) + f(-3x)) \|$$

$$\le \| f(x) + f(2x) - f(3x) \| + \| f(-x) + f(-2x) - f(-3x) \|$$

$$\le \varphi_1(x, 0, 0) + \varphi_1(-x, 0, 0) \tag{21}$$

for all $x \in V$. Letting $\mu = 1,\ y = -x$ and $a = 0$ in (14), we have

$$\| f(x) + f(-x) - f(3x) - f(-3x) \| \le \varphi_1(x, -x, 0)$$

for all $x \in V$. Therefore by (21) we have

$$\left\| f\left(\frac{x}{2}\right) + f\left(-\frac{x}{2}\right) \right\| \le \varphi_1\left(\frac{x}{4}, 0,0\right) + \varphi_1\left(-\frac{x}{4}, 0,0\right) + \varphi_1\left(\frac{x}{4}, -\frac{x}{4}, 0\right) \tag{22}$$

for all $x \in V$. Letting $\mu = 1, y = -\frac{x}{2}$ and $a = 0$ in (14), we get

$$\left\| f\left(\frac{x}{2}\right) - f\left(-\frac{x}{2}\right) - f(x) \right\| \le \varphi_1\left(\frac{x}{3}, -\frac{x}{6}, 0\right) \tag{23}$$

for all $x \in V$. Hence, it follows from (22) and (23) that

$$\left\| 2f\left(\frac{x}{2}\right) - f(x) \right\| \le \varphi_1\left(\frac{x}{4}, 0,0\right) + \varphi_1\left(-\frac{x}{4}, 0,0\right) + \varphi_1\left(\frac{x}{4}, -\frac{x}{4}, 0\right) + \varphi_1\left(\frac{x}{3}, -\frac{x}{6}, 0\right) \tag{24}$$

for all $x \in V$, and so

$$\left\| 2f\left(\frac{x}{2}\right) - f(x) \right\| \le \psi\left(\frac{x}{2}\right) \le \frac{L_1}{4}\psi(x)$$

for all $x \in V$. Using a similar argument as the one used in the proof of Theorem 2.1, there is a unique $\mathbb{C}$-linear mapping $D: V \to V$ satisfying (18), and

$$D(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \tag{25}$$

for all $x \in V$. One can use the inequality (15) to show that there exist a unique linear mapping $k: \Gamma \to \Gamma$ defined by $k(\alpha) := \lim_{n\to\infty} 2^{-n} g(2^n \alpha)$.

It follows from (14) that

$$\| D(a\alpha a) - D(a)\alpha a^* - ak(\alpha)a - a\alpha D(a) \|$$

$$= \lim_n \| 2^{2n} f(2^{-2n} a\alpha a) - 2^n f(2^{-n} a)\alpha a^* - a2^{-n} g(2^n \alpha) a - a\alpha 2^n f(2^{-n} a) \|$$

$$= \lim_n \| 2^{2n} f(2^{-n} a\alpha 2^{-n} a) - 2^{2n} f(2^{-n} a)\alpha 2^{-n} a^* - 2^{2n}(2^{-n} a) g(\alpha)(2^{-n} a) \\ - 2^{2n} 2^{-n} a\alpha f(2^{-n} a) \|$$

$$\leq \lim_n 4^n \varphi_1(0, 0, 2^{-n} a) = 0$$

for all $x, a \in V, \alpha \in \Gamma$. Thus D is a Jordan k-$*$-derivation on (Γ^*, V).

Corollary 2.6. *Let V be a Γ^*-Banach algebra over the complex field $\mathbb{C}$. Suppose $f: V \to V$ is a mapping such that $f(0) = 0$ and $g: \Gamma \to \Gamma$ is a mapping such that $g(0) = 0$ for which there exist $\theta_1, \theta_2 \in [0, \infty)$ and $p \in (0,2)$, $t \in (0,1)$ satisfying*

$$\| f(2\mu x + \mu y) + f(\mu x + 2\mu y) - \mu f(3x) - \mu f(3y) + f(a\alpha a) - f(a)\alpha a^* - ag(\alpha)a - a\alpha f(a) \| \leq \theta_1(\|x\|^p + \|y\|^p + \|a\|^p),$$

$$\| g(\mu\alpha + \mu\beta) - \mu g(\alpha) - \mu g(\beta) \| \leq \theta_2(\|\alpha\|^t + \|\beta\|^t)$$

for all $\mu \in \mathbb{T}$ and all $x, y, a \in V$, $\alpha, \beta \in \Gamma$. Then there exists a unique linear mapping from Γ to Γ satisfying

$$\|g(\alpha) - k(\alpha)\| \leq \frac{2\theta_2}{2 - 2^t} \|\alpha\|^t$$

and there exists a unique Jordan k-$*$-derivation $D: V \to V$ such that

$$\|f(x) - D(x)\| \leq \frac{2^p + 4 \times 3^p + 4^p}{3^p(8 - 2^{p+1})} \theta_1 \|x\|^p$$

for all $x \in V$ and $\alpha \in \Gamma$.

Proof. Put $\varphi_1(x, y, a) := \theta_1(\|x\|^p + \|y\|^p + \|a\|^p)$ and $L_1 = 2^{p-1}$, and put $\varphi_2(\alpha, \beta) := \theta_2(\|\alpha\|^t + \|\beta\|^t)$ and $L_2 = 2^{t-1}$ for all $x, y, a \in V$ and $\alpha, \beta \in \Gamma$ in Theorem 2.5.

Now we consider the superstability of Jordan k-$*$-derivations with the functional equation (13).

Corollary 2.7. *Let V be a Γ^*-Banach algebra over the complex field $\mathbb{C}$. Suppose that $p, q, r, t, l, \theta_1, \theta_2$ are non-negative real numbers with $0 < p + q + r \neq 2, 0 < t + l \neq 1$ and suppose that $f: V \to V$ and $g: \Gamma \to \Gamma$ are mappings such that*

$$\| f(2\mu x + \mu y) + f(\mu x + 2\mu y) - \mu f(3x) - \mu f(3y) + f(a\alpha a) - f(a)\alpha a^* - ag(\alpha)a - a\alpha f(a) \| \leq \theta_1(\|x\|^p \|y\|^q \|a\|^r), \tag{26}$$

$$\| g(\mu\alpha + \mu\beta) - \mu g(\alpha) - \mu g(\beta) \| \leq \theta_2(\|\alpha\|^t \|\beta\|^l) \tag{27}$$

for all $\mu \in \mathbb{T}$ and all $x, y, a \in V$, $\alpha, \beta \in \Gamma$. Then f is a Jordan k-$*$-derivation on (Γ^*,V), where $k: \Gamma \to \Gamma$ is a linear map.

Proof. Putting $x = y = a = 0$ and $\mu = 1$ in (26), we have $f(0) = 0$. Putting $\alpha = 0$ and $\mu = 1$ in (27), we get $g(0) = 0$. By putting $a = y = 0$ and $\mu = 1$ in (26), we have that $f(2x) + f(x) - f(3x) = 0$ for all $x \in V$. By putting $\mu = 1$, $y = -x$ and $a = 0$ in (26), we get $f(x) + f(-x) - f(3x) - f(-3x) = 0$ for all $x \in V$. Hence, by using the last two equations, we obtain that

$$-f(2x) - f(x) + f(3x) - f(-2x) - f(-x) + f(-3x) + f(x) + f(-x) - f(3x) - f(-3x) = 0;$$

that is, $f(2x) + f(-2x) = 0$ for all $x \in V$. If we replace x by $\frac{x}{4}$, then we get

$$f\left(\frac{x}{2}\right) + f\left(-\frac{x}{2}\right) = 0 \tag{28}$$

for all $x \in V$. Putting $\mu = 1, y = -\frac{x}{2}$ and $a = 0$ in (26), we have that

$$f\left(\frac{x}{2}\right) - f\left(-\frac{x}{2}\right) - f(x) = 0$$

for all $x \in V$. From (28), we conclude that $2f\left(\frac{x}{2}\right) = f(x)$ for all $x \in V$, and by induction we get

$$f(x) = \frac{f(2^n x)}{2^n}$$

for all $x \in V$ and $n \in \mathbb{N}$. If we put $\alpha = \beta$ and $\mu = 1$ in (27), we have that $g(2\alpha) = 2g(\alpha)$ for all $\alpha \in \Gamma$, and using induction again we get $g(\alpha) = \frac{g(2^n\alpha)}{2^n}$ for all $\alpha \in \Gamma$ and $n \in \mathbb{N}$. Then, putting $\varphi_1(x, y, a) := \theta_1(\|x\|^p \|y\|^q \|a\|^r)$ and $\varphi_2(\alpha, \beta) := \theta_2(\|\alpha\|^t \|\beta\|^l)$ for all $x, y, a \in V$ and all $\alpha, \beta \in \Gamma$, we reach the desired result by Theorem 2.5.

3. Conclusion

As we pointed out in the introduction, in this paper we have studied the stability of functional equations on Γ^*-Banach algebras which are more general than Banach $*$-algebras. Some stability and superstability results of this new concept have been obtained and the author thinks that by using various methods the results could be extended in the future over other fields in which Γ-algebras could be defined.

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