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INVESTIGATING PRE-SERVICE MATHEMATICS TEACHERS' CONCEPTIONS OF THE PROPERTY OF COMPLETENESS OF REAL NUMBERS[1]

MATEMATİK ÖĞRETMEN ADAYLARININ GERÇEL SAYILARIN TAMLIK ÖZELLİĞİNE İLİŞKİN KAVRAYIŞLARININ İNCELENMESİ

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Abstract

In recent years, the emphasis on concept teaching and conceptual understanding has played a central role in mathematics education curricula. The conceptualization of number systems is important to understand all areas of mathematics. Considering the difficulties experienced in the conceptualization of real numbers, it is seen that students have difficulty in understanding the relationship between real numbers and other number systems. This study aims to investigate pre-service mathematics teachers' conceptions of the property of completeness, which is the most basic element that distinguishes a real number set from other number systems. To this end, the APOS theory, which reveals the mental structures and mechanisms of the individual regarding a concept, was used. Data were collected via semi-structured interviews from three pre-service teachers. The data were analyzed using the descriptive analysis method, and the findings were presented under two themes, which are the epistemology of the concept of the property of completeness of real numbers and the mental structures in the schemas related to the property of completeness of real numbers. The findings revealed that the pre-service teachers' conceptions of the property of completeness of real numbers was mostly at the action level. The study found some mental structures such as the representation of rational numbers on a line, which should be included in the genetic decomposition to be created in the context of the APOS theory regarding the completeness of real numbers.

Keywords: APOS theory, completeness axiom, conceptual understanding, conception

Öz

Son yıllarda kavram öğretimi ve kavramsal anlama vurgusu matematik eğitimi müfredatlarında merkezi rol oynamaktadır. Sayı sistemlerinin kavramsallaştırılması matematiğin tüm alanlarının anlaşılması bakımından önemlidir. Gerçel sayıların kavramsallaştırılmasında yaşanan güçlüklere bakıldığında, öğrencilerin gerçel sayıların diğer sayı sistemleriyle olan ilişkilerini anlamakta zorlandıkları görülmektedir. Bu çalışmanın amacı matematik öğretmen adaylarının gerçel sayı kümesini diğer sayı sistemlerinden ayıran en temel unsur olan tamlık özelliğine ilişkin kavrayışlarını incelemektir. Bu amaç doğrultusunda bireyin bir kavrama ilişkin zihinsel yapılarını ve mekanizmalarını ortaya koyan APOS teorisi kullanılmıştır. Çalışmanın verileri üç öğretmen adayıyla yarı-yapılandırılmış görüşmeler yapılarak elde edilmiştir. Veriler betimsel analiz yöntemiyle incelenmiş ve analiz sonucunda bulgular belirlenen iki ana temaya göre, gerçel sayıların tamlık özelliği kavramının epistemolojisi ve gerçel sayıların tamlık özelliğine ilişkin şemalarda yer alan zihinsel yapılar olarak sunulmuştur. Bulgulara bakıldığında öğretmen adaylarının gerçel sayıların tamlık özelliğine ilişkin kavrayışlarının çoğunlukla eylem düzeyinde olduğu görülmüştür. Buna bağlı olarak gerçel sayıların tamlık özelliğine ilişkin APOS teorisi bağlamında oluşturulacak genetik ayrışımda yer alması gereken rasyonel sayıların bir doğru üzerinde temsil edilmesi gibi zihinsel yapılar belirlenmiştir.

Anahtar Kelimeler: APOS teori, kavramsal anlama, kavrayış, tamlık özelliği

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1. Introduction

A mathematical concept is the definition of an abstract object that has been agreed upon by mathematicians (McDonald, Mathews & Strobel, 2000). The connotations that the concept evokes in the individual and the set of all internal representations are defined as the individual's conception (Sfard, 1991). Conceptual understanding occurs when the individual's conception and the mathematical concept coincide.

The construction of concepts takes place through concrete situations, procedures, and processes and progresses towards the abstraction of mathematical concepts and understanding of symbols and mental concepts (Dubinsky, 1991; Gray & Tall, 1994; Pantziara & Philippou, 2012). Teaching mathematics improves the ability to understand mathematical concepts and helps students be aware of the relationships between concepts, to reach logical conclusions, and to use mathematical concepts in solving problems (National Council of Teachers of Mathematics, 2000).

In Türkiye, curricula have emphasized conceptual understanding and aimed to enable students to form mathematical concepts based on their concrete experiences and intuitions and to perform abstractions since 2009 (MoNE, 2009). The curriculum of 2013 emphasized the importance of concept teaching for conceptual understanding and mathematical concepts, the relationships between these concepts, and the mathematical meanings of basic mathematical operations (MoNE, 2013). The latest high school curriculum developed in 2018 defined mathematical competencies and highlighted the importance of effective use of mathematical concepts within this definition (MoNE, 2018). In this context, the way to teach concepts and the way to ensure conceptual understanding play a central role in research in the field of mathematics education.

As stated in curricula, the conceptualization of number systems is important to understand all areas of mathematics (NCTM, 2006). On the other hand, many studies on students' understanding of number systems emphasize that rational (Vamvakoussi & Vosniadou, 2007), irrational (Uzun Erdem & Dost, 2023) and real number concepts are difficult to understand (Fischbein, Jehiam & Cohen, 1995).

Studies on high school mathematics courses revealed that the incommensurability of irrational numbers and the non-countability of real numbers regarding the conceptualization of real numbers are important obstacles to students' understanding (Fischbein et al., 1995). In addition, the results of some studies have shown that the discreteness of natural numbers is an obstacle for students and prospective teachers to understand the dense structure of rational and real numbers (Malara, 2001; Merenluoto & Lehtinen, 2002; Neumann, 1998; Vamvakoussi & Vosniadou, 2004). In this context, when related studies are examined, it is seen that in high school mathematics lessons, operations with singular points for real numbers are an obstacle to understanding the density of real and rational numbers. In the Analysis courses taught in mathematics and mathematics education departments at universities, real numbers are conceptualized not with singular features that deal with what happens at an isolated point, but with local features such as the neighborhood of a point (Maschietto, 2002). The property of completeness, which plays a fundamental role in the construction of the real number system, is also a kind of local property, since it is a property of sets of numbers, not of numbers.

Based on the literature, it can be said that one of the main reasons for the difficulties in the conceptualization of real numbers from the elementary grades to the university level is the difficulty in conceptualizing the completeness property of real numbers. In this context, this study aims to investigate the conceptions of the students studying in the department of mathematics education at a university to conceptualize the property of completeness seen in many theorems, starting with the construction of real numbers in Analysis courses.

2. Background

The most basic feature that distinguishes the set of real numbers from other sets of numbers is the property of completeness. Some equivalent axioms characterizing the property of completeness of real numbers are as follows.

- Every non-empty and upper bound subset of the set of real numbers has a least upper bound (the least upper bound property).
- Every finite and infinite subset of the ℝ set of real numbers has a concentration point in ℝ (Bolzano Continuity).
- Every Cauchy sequence in the set of real numbers ℝ is convergent (Cauchy continuity).
- For every A and B cut of the ℝ set of real numbers, there is at least one $c \in \mathbb{R}$ with $a \in A$ and $b \in B$, where $a \leq c \leq b$ (Dedekind Continuity).

In the construction of real numbers, the property of completeness is first determined by an axiom, in which real numbers are characterized by Dedekind cuts called "Dedekind's characterization of line completeness" (Awodey & Reck, 2002). Dedekind continuity reveals the necessity of real numbers to have this property, based on the completeness of the line. Equivalent to this axiom, the axioms known as the "Bolzano continuity" and the "Cauchy continuity", which characterize the completeness of real numbers with the sequence approach, and the axioms known as the “least upper bound property" were put forward by mathematicians.

When the curricula of analysis courses are examined (Thomas et al., 2003), it is seen that the property of completeness in the construction of real numbers is given with the least upper bound property, that is, the axiom that "Every upper bound and non-empty subset of the set of real numbers has a least upper bound". In high school mathematics courses and Calculus courses, the concept of line and the continuity of the line are considered as the premise, and real numbers are accepted as "whole numbers" by matching them with a number line (Berge, 2008). In this context, the mental construction of the explicit formulation of the property of completeness is important in many ways, such as being the basis for understanding many theorems, conceptualizing real numbers, and understanding why a line is continuous (Berge, 2008). In addition, it is important to determine the mental structures of the property of completeness to understand the concepts of the bound and the least upper bound of a set and to establish the relations between these concepts with real numbers.

Berge (2010) investigated university students' perceptions of the property of completeness of the set of real numbers and observed that the students viewed the property of completeness as a tool to define new concepts. It was revealed that in the later stages of the Analysis courses, the students did not know in which problems the property of completeness was used. The results revealed that this is due to the use of more advanced theorems in the later stages of the courses and the fact that the students do not encounter the property of completeness at these stages.

On the other hand, Durrand-Guerrier et al. 2019, investigated the difficulties experienced by university students in understanding the relationship between the concepts of density and completeness. The results revealed that the majority of the students understood the concept of density correctly, but they had difficulties in understanding the relationship and differences between this concept and the concept of completeness.

Considering the difficulties experienced by students from all grade levels regarding the conceptualization of real numbers in the literature, the latest secondary school mathematics curriculum in Türkiye includes the following learning outcomes regarding the concept of number sets (MoNE, 2018):

i) The symbols of natural number, integer, rational number, irrational number and real number sets are introduced, and the relationship among these number sets is emphasized.

ii) The place of numbers such as $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ on the number line is determined.

iii) The properties of addition and multiplication operations in the set of real numbers are emphasized.

iv) It is emphasized that the geometric representation of the set of real numbers $\mathbb{R}$ is the number line, and the geometric representation of $\mathbb{R} \times \mathbb{R}$ is the Cartesian coordinate system.

When these learning outcomes are examined, it is seen that there are learning outcomes centered on the property of completeness of real numbers, such as finding a rational number between two rational numbers, finding the place of the irrational number given with a square root expression on the number line, and finding the smallest/largest numbers with a certain property. In order to transfer these outcomes to students, it is thought that pre-service teachers should conceptualize local properties such as the property of completeness of real number sets to avoid the difficulties experienced as a result of the inability to conceptualize the density of real and rational number sets mentioned in the literature.

In this context, the present study investigates pre-service mathematics teachers' conceptions of the property of completeness of real numbers and the concepts related to this property. Therefore, the research problem of the study was constructed as, "What is the level of conception of the pre-service mathematics teachers about the property of completeness of real numbers in terms of the APOS theory?"

3. Theoretical Framework

In this study, the APOS theory (Dubinsky, 1991), which reveals the mental structures and mechanisms of the individual regarding a concept, was used to determine the pre-service teachers' conceptions of the property of completeness. The APOS theory consists of two main components: mental structures and mental mechanisms. Although mental structures are open to development, they are relatively stable (Stenger, et al., 2008). The individual uses mental structures to create a perception in mathematical situations. Mental mechanisms are the processes through which the individual can develop the structure formed in his mind. Mental structures are static, while mental mechanisms are dynamic. The APOS theory consists of the mental structures of Action, Process, Object, and Schema and the mental mechanisms of interiorization, encapsulation, coordination, reversal, de-encapsulation, and thematization.

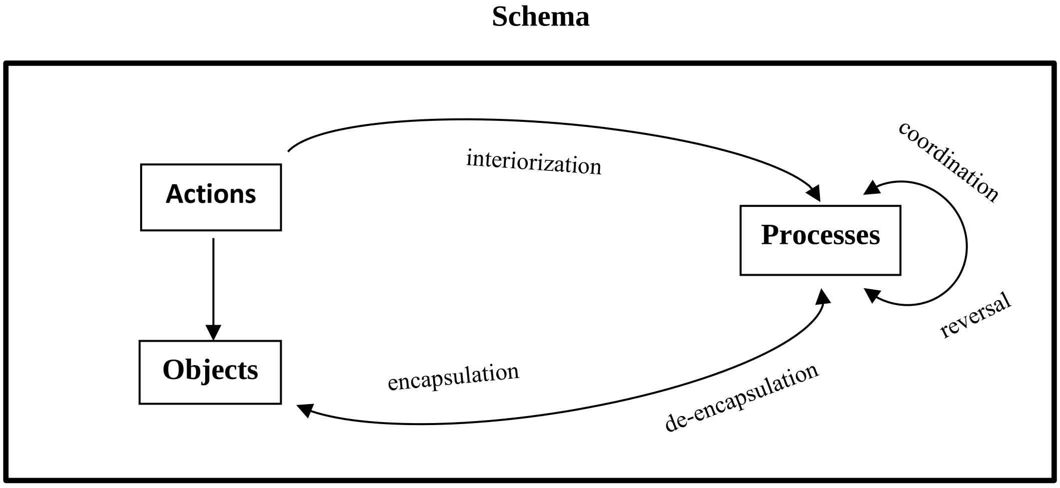

Figure 1. APOS Theory (based on Asiala vd., 1996)

Action structure is any transformation of mental objects according to a clear algorithm determined by external stimuli (Font et al., 2016). The individual reflects the action(s) on mental objects by repeating them and internalizes them into the Process structure (Dubinsky, 1991). Process is the repetition of actions, moving away from relying on external stimuli and providing internal control over actions (Arnon et al., 2014). New processes can be created with the mechanisms of reversal and coordination in the Process structure (Asiala et al., 1996). When the individual becomes aware of the Process as a whole and can apply a new action on the process, the Process is transformed into the Object structure (Dubinsky et al., 2005). Object structure means being aware of the processes related to a mathematical concept and understanding that transformations can be structured (Arnon et al., 2014). While making transformations on a mental object, it is possible to go back to the process that created the object with the rewind mechanism. Schema is defined as the structures containing the definitions, organization and examples of the mental structures that the individual builds on mathematical concepts (Arnon et al., 2014). It can be thematized and turned into an object so that actions can be applied on a schema (Font et al., 2016).

In APOS theory, a hypothetical model called genetic decomposition is used to determine mental structures and mechanisms related to mathematical concepts. A genetic decomposition consists of a detailed description of how individuals can construct actions, processes, objects or cognitive schemas related to a certain mathematical content (Bosch et al., 2017). The beginning of genetic decomposition is based on the experience and mathematical knowledge of the researcher in teaching and learning the concept as a hypothesis, previous research on the concept, the historical development of the concept, and previous research on students' thinking about the concept (Arnon et al., 2014).

According to APOS theory, the genetic decomposition of the conceptual understanding of a concept cannot be considered independent of the individual's conceptions (Arnon et al., 2014). In the context of APOS theory, it is necessary to reveal individuals' conceptions of the concept, which is one of the steps of the genetic decomposition study for the conceptualization of the property of completeness. In this context, in this study, it was aimed to determine student conceptions that would contribute to the emergence of the genetic decomposition of the concept with the interview questions prepared in line with the expert opinion on the property of completeness.

In this study, pre-service mathematics teachers' conceptions of this concept were investigated in order to provide data for the problem of generating a genetic decomposition regarding the property of completeness of real numbers in the context of the APOS theory within the scope of the first author's doctoral thesis.

4. Methodology

In this study, a case study, which is one of the qualitative research designs (Yin, 2003) and an approach in which one or more cases are investigated in depth, was used to examine the level of conception of pre-service mathematics teachers regarding the property of completeness of real numbers.

In the study, APOS methodology was used to answer the research questions that emerged from the literature review. Accordingly, the situation of creating a road map for the conceptual understanding of the completeness property of real numbers was taken into consideration. In this context, semi-structured interview questions were created in order to examine the individuals' conceptions of the concept of completeness property, which is one of the steps of the genetic decomposition of the concept of completeness property, and descriptive analysis was conducted within the scope of the APOS theoretical framework based on the answers given to these questions.

4.1. Participants

The study was conducted with three senior pre-service teachers in the mathematics teaching program. The purposeful sampling method was used to obtain rich data on the property of completeness of real numbers and the mathematical connections in the minds of prospective teachers regarding other concepts related to this property. Participants were selected using the maximum variation sampling method, one of the purposive sampling methods. The aim here was to create as much diversity as possible by creating a relatively small sample (Patton, 1987; Yıldırım & Şimşek, 2013). In this context, the participants were selected according to the following criteria.

i. Pre-service teachers who took the Analysis 1-2-3-4 courses, which included the concepts such as completeness, the bound of a set, and the supremum-infimum of a set and who were successful in these courses were selected.
ii. In line with the views of the lecturers who teach Analysis 1-2-3-4, pre-service teachers who can provide rich data and express their thoughts easily were determined.
iii. In order to ensure maximum diversity, one pre-service teacher with relatively low, medium and high general academic point averages among the pre-service teachers who met the first two criteria was included in the study.

4.2. Data Collection

A semi-structured interview was conducted in the study to reveal the conceptions of pre-service mathematics teachers regarding the property of completeness and other concepts related to this property. The interview form was developed by an expert (also one of the authors of the study) who has been teaching Analysis for about 20 years. The other author of the study reviewed the comprehensiveness of the interview form based on the components of the APOS theory of the concept of completeness property and revised it. Then, the interview questions were tested as a pilot study with another group showing the above participant characteristics. At the end of this pilot study, some changes were made to the interview form. An interview form was created and the interviews were conducted with three pre-service teachers on one-on-

one basis. Each interview lasted an average of 15 minutes. They were recorded and then transcribed. The questions in the interview form are as follows:

1- What do you understand when I say “complete”?
2- What does it mean for the set of real numbers ℝ to be complete?
3- Is a line complete?
4- Is the completeness of the line the same as the completeness of the set of real numbers ℝ?
5- What does the bound and supremum-infimum of a set mean?
6- What is the relationship between "a non-empty upper bound set having a least upper bound in the set of real numbers ℝ " and "having at least one real number between any two points in the set of real numbers ℝ "?
7- What would happen if the set of real numbers ℝ was not complete?

Depending on the answers given by the pre-service teachers in the interviews, additional questions such as “Do you mean this?”, “But is there a relationship between the theorem you just mentioned and the completeness of the line?” were asked to help the participants clarify their answers.

4.3. Data Analysis

The data were analyzed using the descriptive analysis method. According to this method, data are summarized and interpreted according to previously determined themes (Yıldırım & Şimşek, 2013). The data obtained from the semi-structured interviews were analyzed and interpreted depending on the cause-effect relationship according to the mental structures in the APOS theory.

Our data analysis revealed two main themes as “the epistemology of the concept” and “the schemas related to the concept”. The sub-themes of “etymological differences” and “relationships between concepts” were created under the theme of epistemology of the concept. The sub-themes of “related concept definitions” and “real number object” were created under the theme of the schemas related to the concept. Under these themes, mental structures related to the property of completeness of real numbers were put forward depending on the APOS theory. Below is a table showing the main themes and sub-themes, semi-structured interview questions related to the main themes, mental structures, and sample quotes.

Table 1. Descriptive Analysis of Data According to APOS Theoretical Framework

Themes	Sub-themes	Mental structures	Sample quotes
Epistemology of the Concept	Etymological Differences		*“To be complete is the completeness of something.”* *“To cover a space completely, to cover it completely”*
	Relationships Between Concepts	Coordinating processes	*“But how do we get a sequence on it right… I can't make a connection between the two right now.”*
		Actions for the definition of the concept	*“There are some theorems I remember about ℝ being complete, if we think mathematically.”*
		Number line object	*“If I think of it as a number line, all the numbers in the number line are already real numbers, so I can actually say “yes” because of the similarities at one point.”*
Schemas related to the Concept	Related Concept Definitions	Actions regarding the concept of upper bound and their internalization into the process	*“For example, if we consider the open interval (0,1), that is, when we think about it, it is logically the upper bound of 1, but it does not include it. That is, it is possible if we do not include a number larger than ten in the neighborhood of an epsilon.”*
		Internalizing the actions regarding the concept of supremum-infimum into the process	*“Yes, -1 is actually the lower bound. I just thought of the maximum lower bound.”*
	Real Number Object	Coordinating processes related to the boundary and supremum-infimum of a set	*“Every set has it because if the set has an upper bound, the set of upper bounds can be created because it definitely has an upper bound and all those greater than it are included in that set. When I choose the smallest element of the set, I find the supremun."*
		Coordinating the processes of real and rational number sets	*“Here is the case of taking all the values in R, that is, there is no gap because it includes all real numbers, but all rational and irrational numbers in between. That is because it is complete.”*

5. Findings

This section presents the findings of data analysis. The findings were presented under the two main themes determined in data analysis: the epistemology of the concept of the property of completeness of real numbers and the mental structures in the schemas related to the property of completeness of real numbers. The data were presented with direct quotations from the three pre-service teachers in the study. The pre-service teachers were coded as T1, T2, and T3, and the researcher was coded as R.

5.1. Epistemology of the Concept of the Property of Completeness of Real Numbers

In order to reveal the mental structures in the pre-service teachers' schemas related to the property of completeness of real numbers, questions about the epistemological structure of the property of completeness were asked. In the interviews, first, the etymological definition of the concept of completeness was discussed.

R: What do you understand when I say “being complete”?
T1: If we consider the word meaning, to be complete means nothing is missing.
T2: Covering a space completely, covering something completely.
T3: When I leave mathematics aside, I think being complete means having no deficiencies.

In accordance with the etymology of the concept, the pre-service teachers defined being complete as "having nothing missing" (T1, T3) and "to cover a space" (T3). These two different perspectives show parallelism with the answers given to the question of whether a line is complete and what it means for real numbers to be complete.

R: Is a line complete?
T2: The line is something that goes on forever, so no matter which space we look at, there is a point in it for a certain value. I think it is complete. But now, when I think about R^2, for example, there are points that are not in it, so I don't know.
T3: Yes, it is complete, because when the points come together without a gap, a line is formed and it is formed completely.

R: Well, do you think real numbers are complete? Or what does it mean for real numbers to be complete?
T1: Well, to be complete means… real numbers… I mean, if we ignore complex numbers, I'd say complete… is complete.

The answer of T2 to the question of whether the line is complete reveals that according to this teacher, the line is complete in the sense that its points are infinite and "covers" the space it is in. T1 stated that the points are spaces, that is, the line is complete on the grounds that it has "no deficiencies". On the other hand, as for the question of whether the real numbers are complete, T1 stated that they are complete because they have "a complete structure". Despite two different etymological approaches to the concept of completeness, the idea that the line does not "cover" the R^2 space and that the real numbers are "missing" in the set of complex numbers isomorphic to the R^2 space are similar.

T3 explained the property of completeness of real numbers with Cauchy sequences. On the other hand, when asked whether there is a relationship between the concept of completeness and the property of completeness, T3 established a relationship between the etymology of the concept and the definition of the property of completeness of real numbers.

R: What does it mean for R to be complete?
T3: If we think mathematically, there are some theorems I remember about R being complete. It must be ensured that every sequence selected from R must be a Cauchy sequence and each Cauchy sequence must be convergent.

R: Is there a relationship between being complete and the theorem you mentioned?
T3: Actually, there is a relationship. A connection can be made since every convergent sequence I get will be a Cauchy sequence, and this will be a complete one.

T3 could not establish a connection between the completeness of the line and the completeness of real numbers.

R: So, is a line complete?
T3: Yes, it is complete, because when the points come together without a gap, a line is formed and it is formed completely.
R: Well, is there a relationship between the theorem you mentioned and the completeness of the line?
T3: If I remember the theorem correctly, there should be. A sequence on it must be convergent.
R: A sequence on the line?
T3: But how can we get a sequence on the line? I cannot establish a connection between the two.

The fact that T3 could not establish a relationship between the theorem in the definition of the property of completeness of real numbers and the completeness of the line may be due to the inability to coordinate these two processes. On the other hand, the explanation of the property of completeness by the convergence of the Cauchy sequences is a sequential approach, and the concepts of set and sequence must have become objects in the mental scheme of the property of completeness. The fact that the pre-service teacher remembers the theorem regarding the property of completeness but cannot coordinate it with the processes related to the concept of line shows that T3's understanding of the definition of this concept is at the action level.

T1 and T2 were asked about the relationship between the completeness of line and real numbers. Their answers are given below.

R: Do you think the completeness of the line and the completeness of *R* are the same thing?
T1: Actually, could be. Yes. If I think of it as a number line, all real numbers are already included in the number line, so actually I can say yes because of the similarities at one point, for R.
T2: Yes, we already see real numbers and a line as the same; we see them as equal; there was such a thing.

Looking at the answers of both pre-service teachers, it is seen that they put forward a mental structure related to the concept of completeness by matching the number line with the real numbers. It is noted that they do not have the number line object structure obtained by the one-to-one matching method between the set of real numbers and the set of points forming a line. It is seen that the actions that make up the number line object cannot be internalized by the pre-service teachers. Nevertheless, in their schemas, they have a mental structure for accepting real numbers as equal regarding the property of completeness of real numbers. Since this structure was asked by the researcher, it is an action structure created by an external stimulus effect. The findings obtained according to their answers regarding the epistemology of the property of completeness are presented in the figure below.

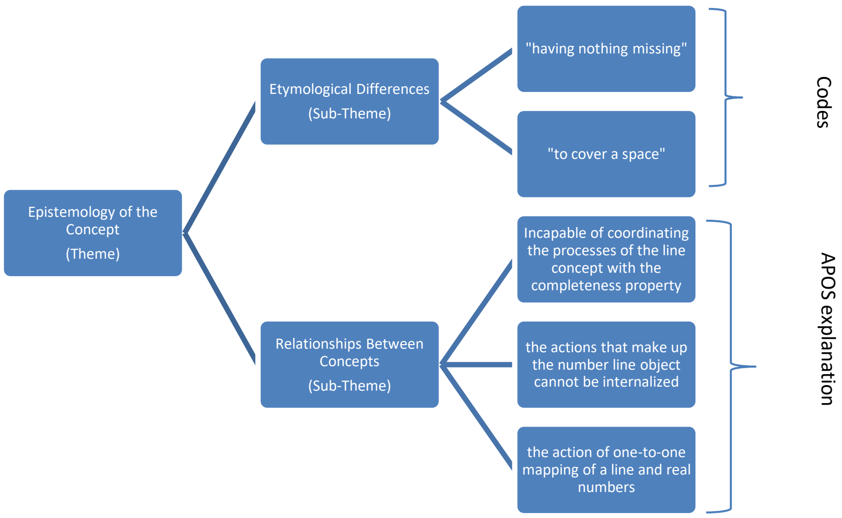

Figure 2. Findings according to the theme of epistemology of the concept of completeness property

5.2. Mental Structures in Schemas Regarding the Property of Completeness of Real Numbers

In order to determine the schemas for the property of completeness of real numbers, the pre-service teachers were asked questions that would reveal the mental structures of the bound of a set and supremum-infimum objects. First, the answers given by T1 are presented below.

R: What is the upper bound of a set?
T1: The upper bound of a set simply means the largest number in that set. It means that there is no number greater than that; it is the largest of all numbers in that set.

T1 explained the upper bound of a set as the largest number in the set. Therefore, the researcher made an intervention and asked whether the upper bound of a set should be included in the set.

R: Should it be within that set?
T1: Hmm… It may not be in that set.
R: I see.
T1: For example, if we consider the open interval (0,1), 1 is the upper bound, but it does not include it. This is the case if we do not include a number larger than that in the neighborhood of an epsilon. So, it does not have to be closed. It does not need to be included.

T1 gave the open interval (0,1) as an example, and took the action of determining the upper bounds of a given set of real numbers for the boundary of the set. By reflecting on this action, T1 internalized the process that the upper bound of a set does not have to be included in the set. It should be noted that T1 used the concept of neighborhood in the boundary concept process. From this point of view, the researcher asked T1 what she meant when she said epsilon neighborhood.

R: What do you mean by saying "if we do not include a number larger than that in the neighborhood of an epsilon"?
T1: Well, for example, let's say we have the open interval (0,1). I take the number 1, that is, 1 is not included. But, for example, if I said (0,1) merging 2, I wouldn't be able to set an upper bound as 1 here. I mean this, because 2 is bigger.

Here, it is seen that T1 tried to determine the least upper bound in the open interval (0,1). A similar situation was observed in the interview with T3.

R: What is the boundary of a set? You can consider the R.
T3: The lower bound of a set means that all elements are not smaller than it. Let a be a bound. All elements of the set a must be within any epsilon neighborhood.
R: Should all elements of the set stay within neighborhood? For example, let's consider the open interval (1,2). Is 0 a lower bound?
T3: Yes.
R: In any neighborhood of 0, will the elements of the set remain in the set?
T3: If I take the 3 neighborhood of 0, they will.
R: But you said "any neighborhood".
T3: Then, the definition I thought is not correct.

As can be seen, T3 tried to give the definition of the maximum lower bound while giving the definition of the lower bound of a set. The concept of the maximum lower bound, that is, infimum, was defined as the fact that all the elements of a set must remain in the neighborhood in every epsilon neighborhood of the infimum. With the intervention of the researcher, T3 was able to reflect that all the elements of a set should not be in the neighborhood.

Similarly, it was observed that T2 explained the concept of the bound of a set with the concept of the least upper bound or the maximum lower bound.

R: Do you remember what the bound of a set is?
T2: Yes. If there is a number or a value that is smaller than all its elements, and that value is the bound.
R: Can you give an example?
T2: Natural numbers have a lower bound of 0, no upper bound.
R: Isn't 0 and -1 lower bound for example?
T2: Yes, -1 is actually the lower bound. I thought of the maximum lower bound.

All three pre-service teachers' explanation of the concept of the bound of a set with the concepts of infimum-supremum shows that the pre-service teachers' understanding of the bound of a set is at the action level. In this case, it is not possible to internalize the definition of the concept into a process. Accordingly, it is seen that the mental object item in the interview, i.e., "every upper bound and non-empty subset of a set has a supremum" is a high-level question for these pre-service teachers, and therefore, they could not give correct answers to the questions. Below are some excerpts from the relevant interviews.

R: Does every upper bound set have a supremum in R?
T3: Every set has one because if the set has an upper bound, the set of upper bounds can be created. It definitely has an upper bound and all values greater than that are included in that set. When I select the smallest element of the set, I find the supremum.
R: Let's consider Q, with the same logic. Consider the upper bound set of "numbers whose square are less than 2" of Q. Does it have an upper bound?
T3: Yes, it has an upper bound, but not a lower bound.
R: What is the supremum of this set?
T3: $\sqrt{2}$.
R: Is $\sqrt{2}$ rational?
T3: No, it is not.
R: Then, what you said does not work.
T3: Yes, it does not work.
R: So, what's the difference? What is the reason for this difference between R and Q?
T3: R is complete.
R: Why did you think so? Isn't Q complete?
T3: I don't know.
R: What is the real number? Why is there a real number? Why is real number needed when there is a rational number?
T3: Because there are irrational numbers, and we cannot express them with rational numbers. We combine these two sets of numbers to form R.

T3 stated that there is a supremum of every upper bound and non-empty set of real numbers. Nevertheless, T3 also said that the relevant proposition would also be realized for the set of rational numbers, since he did not have an object understanding of the concept of supremum. With the intervention of the researcher, T3 performed actions on the example of a subset of rational numbers given and reflected that an upper bound subset of rational numbers cannot have a supremum. Based on this, T3 stated that real numbers and rational numbers are different due to the property of completeness. However, he stated that he did not know whether the set of rational numbers was complete. He could not associate the concept of completeness with the fact that every non-empty and upper bound subset of a set has a supremum. It is thought that this result is due to the fact that the understanding of the concepts of bound, supremum and infimum of a set should be at the action level and that the processes of these concepts should be coordinated in order to establish this relationship.

The interview with T2 revealed that an association was formed between the objects in the schemas related to the property of completeness through decomposition.

R: Does every subset of R have such a supremum?
T2: So, for example, ıı real numbers do not. Sets that go to infinity like real numbers
R: When does a subset have a supremum?
T2: If it has an upper bound, it has an upper limit; if it has a lower bound, it has a lower limit.
R: You say that there is a supremum if it has an upper bound, right? So, every upper bound set of real numbers has a supremum you mean?
T2: Yes, because it has a bound; there is a place where it ends; it cannot continue.
R: Does this require a supremum?
T2: I think it does.
R: Well, let's consider the same thing for rational numbers. Let us also take an upper bound subset of rational numbers. Is there still a supremum?
T2: Yes, but it doesn't have to be in rational numbers.
R: You said R has it. Why? What is the difference between the two?
T2: That is the case of taking all the values in R, that is, all real numbers, but since all rational and irrational numbers are covered, there is no gap, because it is complete. Again, it is a matter of completeness.

As can be seen, T2 coordinated the processes of the concepts of the bound and supremum of a set with the processes of real and rational number sets, and created the object of having the supremum of the upper bound and non-empty subsets of a set, with the property of completeness. The findings obtained according to their answers regarding the schemas related to the property of completeness concept are presented in the figure below.

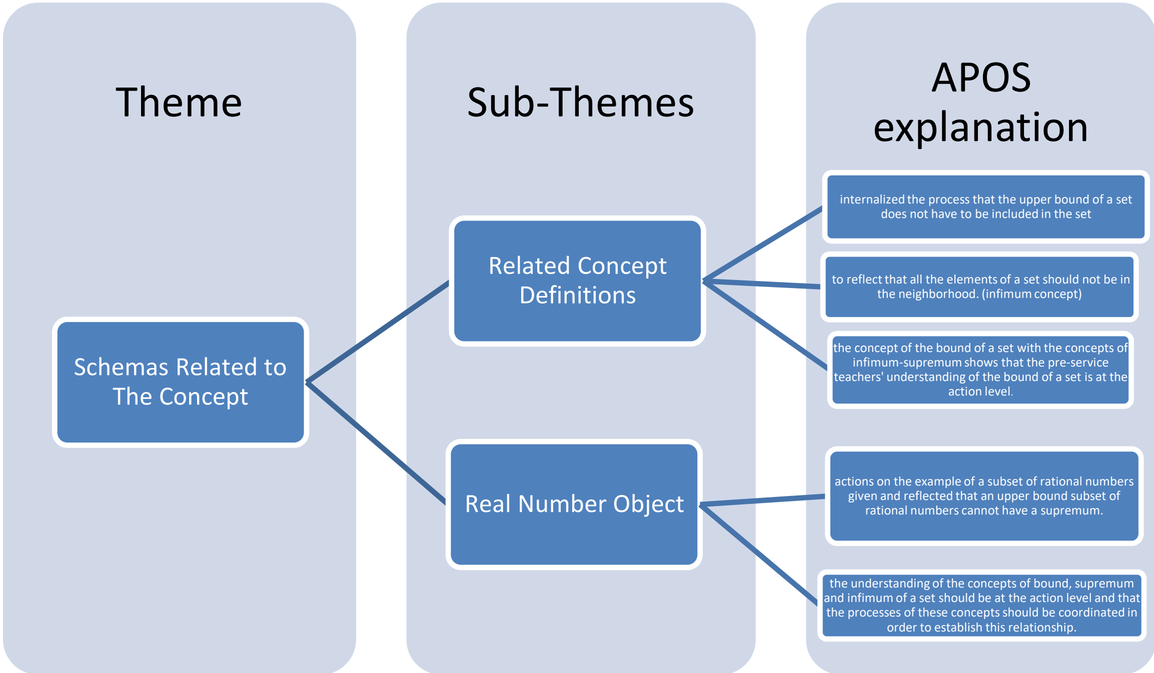

Figure 2. Findings according to the theme of Schemas Related to The Concept of completeness property

6. Discussion and Conclusion

This study investigated pre-service teachers' conceptions of the property of completeness of real numbers. Semi-structured interviews were conducted with pre-service teachers on the property of completeness of real numbers. Revealing insights about a concept in the context of the APOS theory is an important step for the genetic decomposition to be created for this concept (Arnon et al., 2014). In this respect, it is thought that the study may guide the genetic decomposition to be created to reveal the mental structures related to the concept of the property of completeness of real numbers.

The findings revealed that etymological differences such as "having nothing missing" and "covering something" were observed in the statements of pre-service teachers regarding the concept of completeness. This difference seems to affect their conception of the completeness of real numbers. For example, T1 defined completeness as "having nothing missing" and explained the completeness of real numbers by saying that they have a complete sequence. On the other hand, T2 defined completeness as "covering something completely" and stated that line is complete because the points on a line continue forever. In this context, it can be said that etymological differences regarding the concept affect the mental structures regarding the concept. For example, the expression "having nothing missing" is important in terms of revealing the difference between rational numbers and real numbers, while the expression "covering something" is important in terms of the construction of completeness of real numbers with a number line and the fact that rational numbers are concentrated on real numbers. In this context, the necessity of

- mental structures related to the representation of rational numbers on a line, and

- mental structures related to the concentration of rational numbers on real numbers

is emphasized in the genetic decomposition to be created for the related concept of the study. It is thought that this finding is significant in terms of eliminating the difficulties experienced by students in revealing the difference between intensity and completeness as reported in Durrand-Guerrier et al.'s (2019) study.

T3 explained the completeness of the real numbers by the convergence of the Cauchy sequences, but could not establish a relationship between the completeness of the line and the completeness of the real numbers. Thus, it is important to reveal the concept of sequences, Dedekind cut, the concepts of supremum-infimum, and the relationship between them in the context of the historical development of the property of completeness for the genetic decomposition to be created within this context. This result coincides with the conclusion in Berge's (2010) study that the students' inability to establish a relationship between the analysis concepts and the property of completeness stems from the fact that they do not encounter the property of completeness after they finish taking the analysis courses. In addition, it is thought that this study shows the necessity of emphasizing the historical development of the property of completeness in the mental construction of the property.

When the schemas of the pre-service teachers regarding the property of completeness are examined, it is seen that their mental structures of the concepts such as the bound of a set and supremum-infimum are at the action level according to the APOS theory. For example, all three pre-service teachers explained the bound of a set with the least upper bound, which showed that their conception of the upper bound concept was at the action level. Accordingly, they could not construct the object of the completeness of real numbers, that is, the supremum of the upper bound and non-empty subsets of real numbers. Nevertheless, it was observed that T2 was able to coordinate the process of the property of completeness of real numbers and the supremum of the non-empty and upper bound subsets of real numbers set. It can be said that this situation arises from the fact that the mental structures of the property of completeness schemas are not included in detail in the questions in the semi-structured interviews. Based on the APOS theory, it is important to create a conceptual understanding test based on this genetic decomposition together with the genetic decomposition to be created regarding the property of completeness in order to reveal the mental structures of the related concept.

Considering the findings of this study and the importance of the completeness property in the context of real numbers, it is thought that studies on the completeness property should be increased in terms of the mental construction of real numbers. In this context, it is suggested that studies should be conducted on how the completeness property should be conceptualized and classroom activities should be observed and reported. The findings of this study show that pre-service mathematics teachers' conceptions of the property of completeness are inadequate. Since real numbers and related concepts such as the property of completeness are fundamental for mathematics education, it is thought that it is important to focus on the understanding of these concepts.

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