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SOME RESULTS ON THE SMALLEST CARTESIAN GROUP PLANE

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ABSTRACT

Let π be the projective plane of order 25 coordinatised by elements of the smallest cartesian group. In this work, in some cases depending on the choice of the regular quadrangle it is shown that there is no any projective subplane of order 3 of π.

Keywords: Cartesian group, Algebraic structure, Projective plane

1. INTRODUCTION

Projective planes have applications in various branches of mathematics, including combinatorics, geometry, and coding theory. They are also studied for their interesting algebraic and combinatorial structures. it is well known that every projective plane has an algebraic structure obtained by coordinatization. Conversely, certain algebraic structures can be used to construct projective planes. For instance, a general method of generating Cartesian groups has been given by Panella in [6].

The algorithm for the classification of the k – arcs, some examples of the k –arcs, Fano planes, Baer subplanes in the projective planes of order 9 and 25 and embedding of the Projective Planes to the Projective Spaces are given in [2-5] are given.

Definition A projective plane $(P, L, \circ)$ consists of a set P of points, and a set L of subsets of P, called lines, such that every pair of points is contained in exactly one line, every two different lines intersect in exactly one point, and there exist four points, no three of which are collinear.

Definition A subplane of a projective plane $(P, L, \circ)$ is a *B* of points and lines which is itself a projective plane, relative to the incidence relation given in $(P, L, \circ)$. Let $(P, L, \circ)$ be a projective plane of order *n*. If $(P', L', \circ')$ is a subplane of order *m*, then either $n = m^2$ or $n \geq m^2 + m$. $B = (P', L', \circ')$ is called Baer subplane of $(P, L, \circ)$ if it satisfies the following conditions:

1) Every point of $(P, L, \circ)$ is incident with a line of *B*
2) Every line of $(P, L, \circ)$ is incident with a point of *B*.

It is clear that for the Baer subplane B of order $n = m^2$.

Cartesian Group: A system $(S, \oplus, \otimes)$ is a Cartesian group if and only if the following conditions are satisfied:

1) $(S, \oplus)$ a group

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2) Each of equations $a \otimes x = b$ and $x \otimes a = b$ has a unique solution for all Where 0 denotes the additive identity.
3) There exists an element $e \in S$ such that $e \otimes x = x \otimes e = x$ for all $x \in S$.
4) $0 \otimes x = x \otimes 0 = x$ for all $x \in S$.
5) Given $a, b, c, d \in S$ such that $a \neq c$, there exists a unique $x \in S$ such that

$$a \otimes x \oplus b = c \otimes x \oplus d$$

6) Given $a, b, c, d \in S$ such that $a \neq c$, there exists a unique pair $(x, y) \in S^2$ such that

$$x \otimes a \oplus y = b$$

and

$$x \otimes c \oplus y = d.$$

The construction of the cartesian group plane of order 25 in [1] is given. We shall be interested in the projective subplanes of order 3 of the smallest Cartesian Group Plane of order 25.

2. THE SMALLEST CARTESIAN GROUP PLANE

The algebraic structure of finite projective planes is a fascinating topic that combines algebraic and geometric concepts. It provides a mathematical framework for studying the properties and relationships of points and lines in a finite projective plane.

We consider the geometrical structure of the projective plane which is constructed on the known the smallest cartesian group.

Definition (See 1) Let $(F_5,+,.)$ be the field of integers modulo 5. Let $S = \{(a,b): a, b \in F_5\}$ and consider the addition and multiplication on S given by

$$(a,b) \oplus (c,d) = (a+b, c+d)$$

and

$$(a,b) \otimes (c,d) = \begin{cases} (a.c, a.d) & , if \;\; b = 0 \\ (a.c - (a^2-2).d.b^{-1}, b.c - a.d) & , if \;\; b \neq 0 \end{cases}.$$

The system $(S, \oplus, \otimes)$ is a proper Cartesian group.

We consider the geometrical structure of the projective plane which is constructed on the known the smallest cartesian group.

A finite projective plane of order n has n^2+n+1 points and n^2+n+1 lines. We shall be interested in the projective plane $\boldsymbol{\pi}$ over the smallest Cartesian Group of order 25. The 651 oints of $\boldsymbol{\pi}$ are the elements of the set $\{(x,y): x, y \in S\} \cup \{(m): m \in S\} \cup \{(\infty)\}$.
The points of the form (x, y), are called affine points and the points of the form (m) and the unique point (∞) are called ideal points. The 651 lines of $\boldsymbol{\pi}$ are defined to be set of points satisfying one of the three conditions:

$$[m,k]=\{(x,y)\in S^2 : y=m*x\oplus k\}\cup\{(m)\}$$
$$[\lambda]=\{(x,y)\in S^2 : x=\lambda\}\cup\{(\infty)\}$$
$$[\infty]=\{(m)\in S\ \}\cup\{(\infty)\}.$$

The 625 lines of **π** having form $y=m*x\oplus k$ and 25 lines of **π** having of the form $x=\lambda$ are called the affine lines and the unique line [∞] of **π** is called the ideal line. The system of points, lines and incidence relation given above defines a projective plane of order 25, which is the smallest Cartesian group plane.

3. SUBPLANES OF ORDER 3 OF THE SMALLEST CARTESIAN GROUP PLANE

Let's assume the vertices of the regular quadrangle are O,I,X,P. If the diagonal points E,F,G of this regular quadrangle are not collinear, then this quadrangle does not determine a Fano plane [1].

Let $O=((0,0),(0,0)), I=((1,0),(1,0)), X=((0,0))$ and $P=((0,0),(a,b))$ for $a=b=0$ and $a=1,\ b=0$ be any four points that are known not to form a regular quadrangle.

The following lemmas are taken from [1].

Lemma If $P=((0,0),(a,b))$ with $b\neq 0$, then each a regular quadrangle O,I,X,P determines a Fano subplane of π.

Lemma If $P=((0,0),(a,b))$ with $a\neq 0,1$ and $b=0$, then non of the regular quadrangle O,I,X,P determines a Fano configuration of π.

Lemma If $P=(\infty)$, then the regular quadrangles O,I,X,P doesn't determine a Fano configuration of π.

Theorem Let $O=((0,0),(0,0)),\ I=((1,0),(1,0)),\ X=((0,0))$ and $P=((0,0),(a,b))$ be a regular quadrangle in π. The configurations obtained from completing the some regular quadrangles O,I,X,P of π do not form the respective subplanes of order 3 of π.

Proof To obtain configurations that form a projective plane of order 3 from completing the regular quadrangles O,I,X,P of π, it is necessary for these configurations to satisfy the conditions in Lemma 2. Indeed, the number of such the quadrangles with diagonal points are not collinear is three in Lemma 2. Because in the other conditions, the completions of these quadrangles, known as Fano planes, cannot be subplanes of projective planes of order 3 . Now, let's examine these three cases:

Case 1 If $a=2, b=0$, then P is $((0,0),(2,0))$. The coordinates of the opposite sides and diagonal points of this quadrangle are obtained as follows:

$$OP=[(0,0)],\ IX=[(0,0),(1,0)],\ OI=[(1,0),(0,0)],\ PX=[(0,0),(2,0)],$$
$$OX=[(0,0),(0,0)],\ PI=[(4,0),(2,0)]$$

and

$$E:=OP\wedge IX=((0,0),(1,0)),\ F:=OI\wedge PX=((2,0),(2,0)),\ G:=PI\wedge OX=((2,0),(0,0)).$$

The configuration that completes this the regular quadrangles O, I, X, P should have the property of being a plane of order 3. For this to happen, four lines must pass through each point, and there should be four points on each line.

By performing the necessary calculations, the remaining 6 points are found as follows:

$$N := EF \wedge OX = ((3,0),(0,0)),\ N' := EF \wedge PI = ((2,1),(2,3)),\ L := FG \wedge IX = ((2,0),(1,0)),$$
$$L' := FG \wedge OP = (\infty),\ M := EG \wedge OI = ((4,0),(4,0)),\ M' := PX \wedge EG = ((3,0),(2,0)).$$

Since there should be four points on each line, two missing points on the line $PM = [(3,0),(2,0)]$ should be determined.
However, the points L and N can be on this line PM.

$$L \circ PM \Leftrightarrow (1,0) = (3,0) \otimes (2,0) \oplus (2,0)$$

and

$$N \circ PM \Leftrightarrow (0,0) = (3,0) \otimes (3,0) \oplus (2,0)$$

Since the above equations are not satisfied and there are no two points on line PM among these 13 points, the resulting configuration cannot be a projective plane of order 3.

Case 2 If $a = 3,\ b = 0$, then P is $((0,0),(3,0))$. The coordinates of the opposite sides and diagonal points of this quadrangle are obtained as follows:

$$OP = [(0,0)],\ IX = [(0,0),(1,0)],\ OI = [(1,0),(0,0)],\ PX = [(0,0),(3,0)],$$
$$OX = [(0,0),(0,0)],\ PI = [(3,0),(0,0)]$$

and

$$E := OP \wedge IX = ((0,0),(1,0)),\ F := OI \wedge PX = ((3,0),(3,0)),\ G := PI \wedge OX = ((4,0),(0,0)).$$

By performing the necessary calculations, the remaining 6 points are found as follows:

$$N := ((1,0),(0,0)),\ N' := ((2,0),(4,0)),\ L := ((2,0),(1,0)),$$
$$L' := ((0,0),(2,0)),\ M := (1,0),\ M' := PX \wedge EG = ((2,0),(3,0)),$$

Since there should be four points on each line, two missing points on the line $PM = [(1,0),(3,0)]$ should be determined.

However, the points L and N can be on this line PM.

If calculations are done as in the first case, two points still cannot be found on line PM, the resulting configuration cannot be a projective plane of order 3.

Case 3 If $a = 4,\ b = 0,$ then P is $((0,0),(4,0))$.

If calculations are done as in the first and second cases, two points still cannot be found on line *PM*. Therefore, a projective plane of order 3 cannot be constructed.

4. CONCLUSION

In this paper, we have presented an approach for finding projective subplanes of order 3 within the cartesian group plane of order 25. By combining principles from algebraic geometry, linear algebra, and computational techniques, we have developed a systematic methodology that enables efficient identification, characterization of some subplanes.

Investigating projective subplanes of order 3 within the cartesian group plane of order 25 by considering different the regular quadrangles is an open problem.

CONFLICT OF INTEREST

The author stated that there are no conflicts of interest regarding the publication of this article.

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