Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume XX (x) (XXXX), 1 – 19
DOI : 10.15672/hujms.xx

Research Article

Ćirić contraction with graphical structure of bipolar metric spaces and related fixed point theorems

Mudasir Younis[1], Ali Mutlu*[2], Haroon Ahmad[3]

[1] Department of Mathematics, Faculty of Sciences, Sakarya University, Sakarya, 54050, Türkiye
[2] Department of Mathematics Faculty of Science Manisa Celal Bayar University Martyr Prof. Dr. İlhan Varank Campus 45140-Yunusemre Manisa, Türkiye
[3] Abdus Salam School of Mathematical Sciences, Government College University, Lahore , Pakistan, 54600, Pakistan

Abstract

We present the novel concept of graphical bipolar metric-type space in this article, which combines the notions of graph theory with fixed point theory. We prove that every bipolar metric space is a graphical bipolar metric space but the converse is not true in general. Various concepts like covariant mapping, contravariant mapping, and Cauchy bisequence are also discussed within the context of graphical bipolar metric-type spaces. Furthermore, in this study, we show that fixed point results exist in graphical structures of bipolar metric spaces and a series of examples are provided to support the main results within the realm of the graph structure.

Mathematics Subject Classification (2020). 47H10, 54H25

Keywords. bipolar metric space, Ćirić contraction, fixed point, graph

1. Introduction

In the literature of fixed point theory, the Banach contraction principle is very significant. Many authors have generalized this concept by utilizing various kinds of contraction mappings in several metric spaces (see, e.g., [1, 3, 5, 12, 13, 21, 25]). In 2016, Mutlu and Gürdal [14] introduced the notion of bipolar metric space, a sort of partial distance, as well as the relationship between metric spaces and bipolar metric spaces. Mutlu et al. [15] developed a generalization of coupled fixed point theorems in bipolar metric spaces in 2017. Fixed point methods for multivalued mappings in such spaces were recently proposed by Mutlu et al. [16]. For more synthesis on bipolar metric spaces one can refer to [11, 27].

In recent years, graph theory has become increasingly relevant in the study of fixed point theory. Many scientists are working on graphical fixed point theory, and its applications, e.g., [2, 8, 9, 17, 18, 20, 22–24]. Recently, Chuensupantharat et al. [2] generalized the

*Corresponding Author.
Email addresses: mudasiryouniscuk@gmail.com, myounis@iitk.ac.in (M. Younis), ali.mutlu@cbu.edu.tr, abgamutlu@gmail.com (A. Mutlu), haroonrao3@gmail.com (H. Ahmad)
Received: 25.05.2023; Accepted: 25.11.2023

notion of graphical metric spaces [19] and introduced graphical b-metric spaces along with suitable graphs. Most recently in 2019, Younis et al. [22] introduced graphical rectangular b-metric spaces, which extend the concepts given in ([2,6]). For a detailed summary of the graphical structures, we refer the reader to the informative article [7]. In 2019, Younis et al. [22, 23] introduced the notion of graphical rectangular b-metric spaces with an application related to engineering field. Recently, in 2021, Younis et al [26] graphical structure of extended b-metric space with an application of homogeneous bar. Mutlu and Gürdal [14], on the other hand, pioneered the idea of bipolar metric space as a sort of partial distance. The authors analyzed the relationship between metric spaces and bipolar metric spaces, and in particular, they proved for completeness and the unfolding of several of the well-known fixed-point theorems.

Keeping in view the applicability of the above mentioned theory, and motivated by the innovative work done in [4,14,15,22,23,26], in this paper, we aim to introduce the notion of graphical structure of bipolar metric spaces, which is the generalization of bipolar metric spaces and graphical metric spaces. Appropriate example equipped with suitable graphs are provided to endorse the validity of our findings.

2. Preliminaries

In this section, we present some basic facts and primary definitions which are essential for the further analysis of the article.
Jachymski [10] presents the following statement, let X be a nonempty set and Δ be the diagonal of $X \times X$. Also let $G = (V(G), E(G))$ be a directed graph without parallel edges, where $V(G)$ is vertex set of G such that it coincides with the set X and $E(G)$ is the edge set of G such that it contains all the loops of G, that is $\Delta \subseteq E(G)$. Also, we denote by G^{-1} the graph obtained by reversing the direction of $E(G)$. If the graph G contains symmetric edges, then it is denoted by the symbol $\breve{\mathcal{G}}$, that is,

$$E(\breve{\mathcal{G}}) = E(G^{-1}) \cup E(G).$$

Let v and w be the vertices of the directed graph G. A path in G is defined to be a sequence $\{v_j\}_{j=0}^{m}$ containing $(m+1)$ vertices such that $v_0 = v$, $v_m = w$ with $(v_{j-1}, v_j) \in E(G)$, where $j = 1, 2, \ldots, m$. A graph $\mathcal{G}$ is said to be a connected graph if there exists a path between every pair of its vertices. If the graph G is undirected and there is a path between its every two vertices, then we say G is weakly connected. A graph $H = (V(H), E(H))$ is called a subgraph of $G = (V(G), E(G))$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
Shukla [19] presents the notations $[v]_{\mathcal{G}}^{l} = \{w \in X :$ there exists a directed path from v to w in the graph G possessing length $l\}$ and a relation P on X is such that $(vPw)_G$ if there exists a path directing from v to w in G and $w \in (vPw)_{\mathcal{G}}$ if w lies in the path $(vPw)_{\mathcal{G}}$.
Further, a sequence $\{v_m\} \subset X$ is said to be $\mathcal{G}$-termwise connected (G-TWC) if $(v_m P v_{m+1})_G$ for all $m \in \mathbb{N}$. Henceforth, we consider all the graphs under consideration as directed unless otherwise stated.
Recently in [14], the authors introduced the bipolar metric space as follow:

Definition 2.1 ([14]). A bipolar metric space is triple (X, Y, d) such that X, Y are non empty sets and $d : X \times Y \to \mathbb{R}^+$, is a function satisfying the properties

(1) If $d(x, y) = 0$, then $x = y$, for all $(x, y) \in X \times Y$.
(2) If $x = y$, then $d(x, y) = 0$, for all $(x, y) \in X \times Y$.
(3) $d(x, y) = d(y, x)$, for all $x, y \in X \cap Y$.
(4) $d(x_1, y_2) \leq d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)$,

for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $\mathbb{R}^+$ is set of all nonnegative real numbers. Then d is called a bipolar metric on the pair (X, Y).

3. Graphical bipolar metric spaces

We introduce the notion of graphical bipolar metric space and related fixed point theorems in this section.

Definition 3.1. Let (X, Y, d_G) be a graphical bipolar metric space if X, Y are non empty sets and equipped with a graph G. Let $d_G : X \times Y \to \mathbb{R}^+$, ($\mathbb{R}^+$ is set of all nonnegative real numbers) is said to be graphical bipolar metric space for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $(x, y) \in X \times Y$ if satisfying:

(1) If $d_G(x, y) = 0$, then $x = y$.
(2) If $x = y$, then $d_G(x, y) = 0$.
(3) $d_G(x, y) = d_G(y, x)$, for all $x, y \in X \cap Y$.
(4) $(x_1 P y_2)_G$, $x_2, y_1 \in (x_1 P y_2)_G \Rightarrow d_G(x_1, y_2) \leq d_G(x_1, y_1) + d_G(x_2, y_1) + d_G(x_2, y_2)$.

Remark 3.2. It is to notify that a graphical bipolar metric space (X, Y, d_G) is a proper generalization of bipolar metric space (X, Y, d).

Example 3.3. Every bipolar metric space is graphical bipolar metric space (X, Y, d_G).
Let $X = \{0, 1, 2, 3, 4\}$, $Y = \{3, 4, 5, 6, 7\}$ and the function $d_G : X \times Y \to \mathbb{R}^+$ is defined in the following table.

d_G	0	1	2	3	4	5	6	7
0	0	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$	$\frac{8}{9}$
1	$\frac{2}{3}$	0	$\frac{6}{7}$	$\frac{7}{8}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{10}{11}$	$\frac{11}{12}$
2	$\frac{3}{4}$	$\frac{6}{7}$	0	$\frac{12}{13}$	$\frac{13}{14}$	$\frac{14}{15}$	$\frac{15}{16}$	$\frac{16}{17}$
3	$\frac{4}{5}$	$\frac{7}{8}$	$\frac{12}{13}$	0	$\frac{8}{9}$	$\frac{7}{8}$	$\frac{6}{7}$	$\frac{5}{6}$
4	$\frac{5}{6}$	$\frac{4}{5}$	$\frac{13}{14}$	$\frac{8}{9}$	0	$\frac{3}{4}$	$\frac{5}{6}$	$\frac{4}{5}$
5	$\frac{6}{7}$	$\frac{5}{6}$	$\frac{14}{15}$	$\frac{7}{8}$	$\frac{3}{4}$	0	$\frac{12}{13}$	$\frac{13}{14}$
6	$\frac{7}{8}$	$\frac{10}{11}$	$\frac{15}{16}$	$\frac{6}{7}$	$\frac{5}{6}$	$\frac{12}{13}$	0	$\frac{17}{18}$
7	$\frac{8}{9}$	$\frac{11}{12}$	$\frac{16}{17}$	$\frac{5}{6}$	$\frac{4}{5}$	$\frac{13}{14}$	$\frac{17}{18}$	0

It is easy to show that (X, Y, d) is bipolar metric space. Now we have to show that bipolar metric space (X, Y, d) is graphical bipolar metric space (X, Y, d_G). For this, we inspect the graph G with vertex set $V(G) = X \cup Y = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and edge set obtained from $V(G)$ as follows:

$$E(G) = \triangle \cup \left\{ \begin{array}{c} \{(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(0,7), \\ (1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(2,3), \\ (2,4),(2,5),(2,6),(2,7),(3,4),(3,5),(3,6), \\ (3,7),(4,5),(4,6),(4,7),(5,6),(5,7),(6,7) \end{array} \right\},$$

then (X, Y, d_G) is graphical bipolar metric space as shown in Figure (1).

Example 3.4. Graphical bipolar metric space (X, Y, d_G) **is not necessarily bipolar metric space.**
Let $X = \{1, 2, 3, 4, 5\}$, $Y = \{2, 3, 4, 5, 6\}$ and the function $d_G : X \times Y \to \mathbb{R}^+$ is defined as

$$d_G(x, y) = \begin{cases} 0, \text{ when } x = y, \\ |x - y|^2, \text{ otherwise,} \end{cases}$$

for all $x \in X$, $y \in Y$ and equipped with graph G. The graph $G = (V(G), E(G))$, having $V(G) = X \cup Y = \{1, 2, 3, 4, 5, 6\}$ be the set of vertices and $E(G)$ be the set of edges as shown in Figure (2). It is easy to verify that (X, Y, d_G) is graphical bipolar metric space. Now, observe that

$$d_G(1, 6) = 25 > d_G(1, 3) + d_G(3, 5) + d_G(5, 6) = 9$$

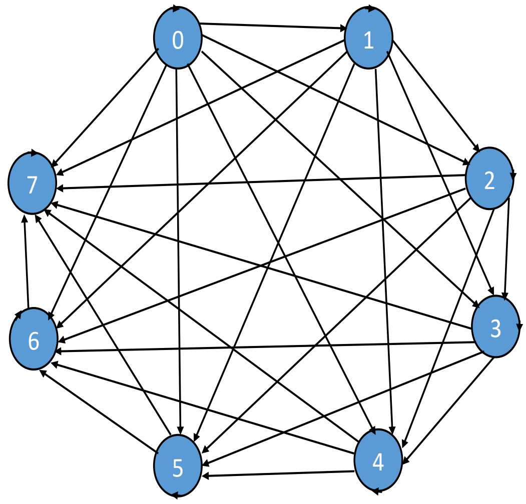

Figure 1. Graph associated with bipolar metric space

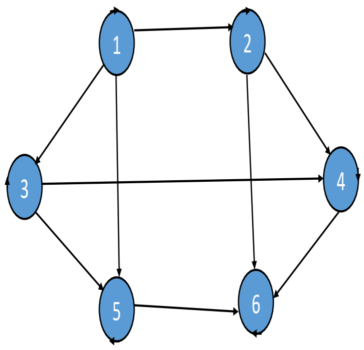

Figure 2. Graphical bipolar metric space

Hence, we conclude that graphical bipolar metric space is not necessarily bipolar metric space.

Now, we discuss Cauchy, complete, covariant and contravariant mappings as follows:

Definition 3.5. Let $T : X_1 \cup Y_1 \to X_2 \cup Y_2$ be a function on pairs of sets (X_1, Y_1) and (X_2, Y_2). If $T(X_1) \subseteq X_2$ and $T(Y_1) \subseteq Y_2$, is known as covariant mapping from (X_1, Y_1) to (X_2, Y_2) and denoted as

$$T : (X_1, Y_1) \rightrightarrows (X_2, Y_2).$$

If $T(X_1) \subseteq Y_2$ and $T(Y_1) \subseteq X_2$, is known as contravariant mapping from (X_1, Y_1) to (X_2, Y_2) and denoted as

$$T : (X_1, Y_1) \rightleftarrows (X_2, Y_2).$$

If d_{G_1} and d_{G_1} are graphical bipolar metric spaces on (X_1, Y_1) and (X_2, Y_2) respectively. Then we use the notions as

$$T : (X_1, Y_1, d_{G_1}) \rightrightarrows (X_2, Y_2, d_{G_2}), \text{ (covariant mapping)}$$

and

$$T : (X_1, Y_1, d_{G_1}) \rightleftarrows (X_2, Y_2, d_{G_2}), \text{ (contravariant mapping).}$$

Definition 3.6. Let (X, Y, d_G) be a graphical bipolar metric space:

(i): A point $z \in X \cup Y$, is left point if $z \in X$, right point if $z \in Y$ and a central point if $z \in X \cap Y$.

(ii): A sequence (x_n) on X is left sequence and (y_n) on Y is right sequence. A left or right sequence is simply a sequence in (X, Y, d_G). A sequence (z_n) is said to be convergent to a point z, iff (z_n) is left sequence and z as right point then

$$\lim_{n\to\infty} d_G(z_n, z) = 0.$$

Similarly, (z_n) is right sequence and z as left point then

$$\lim_{n\to\infty} d_G(z, z_n) = 0.$$

(iii): A sequence $(x_n, y_n) \in X \times Y$, is said to be bisequence on (X, Y, d_G). If both sequences (x_n) and (y_n) converges then the bisequence is said to be convergent and if both sequence converges to same point $z \in X \cap Y$, then bisequence is said to be biconvergent.

(iv): Every biconvergent bisequence in (X, Y, d_G) is a Cauchy bisequence and every convergent bisequence is biconvergent.

(v): A graphical bipolar metric space is called complete, if every Cauchy bisequence in (X, Y, d_G) is convergent.

Definition 3.7. Let (X_1, Y_1, d_{G_1}) and (X_2, Y_2, d_{G_2}) be graphical bipolar metric spaces:

(i): A mapping $T : (X_1, Y_1, d_{G_1}) \rightrightarrows (X_2, Y_2, d_{G_2})$ is said to be left continuous at a point $x_0 \in X_1$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_{G_1}(x_0, y) < \delta \implies d_{G_2}(T(x_0), T(y)) < \epsilon \text{ for all } y \in Y_1.$$

Similarly, right continuous at a point $y_0 \in Y_1$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_{G_1}(x, y_0) < \delta \implies d_{G_2}(T(x), T(y_0)) < \epsilon \text{ for all } x \in X_1,$$

and a mapping T is called continuous if it is left continuous at each point $x \in X_1$ and right continuous at each point $y \in Y_1$.

(ii): A mapping $T : (X_1, Y_1, d_{G_1}) \rightleftarrows (X_2, Y_2, d_{G_2})$ is continuous iff it is continuous as a covariant mapping $T : (X_1, Y_1, d_{G_1}) \rightrightarrows (X_2, Y_2, d_{G_2})$.
Now, we can say that a covariant and contravariant mapping is continuous from (X_1, Y_1, d_{G_1}) to (X_2, Y_2, d_{G_2}) iff:

$$(z_n) \to v \text{ on } (X_1, Y_1, d_{G_1}) \text{ implies that } (T(z_n)) \to T(v) \text{ on } (X_2, Y_2, d_{G_2}).$$

4. Main results

In this section, we will discuss some fixed point related to graphical bipolar metric space (X, Y, d_G), where the graph G is a weighted graph. Let $x_0 \in X$ and $y_0 \in Y$ be the initial values of the bisequence (x_m, y_m), we say (x_m, y_m) to be $T-$Picards bisequence $(T - P_bS)$ if $x_m = Tx_{m-1}$ and $y_m = Ty_{m-1}$ for all $m \in \mathbb{N}$.

Definition 4.1. A relation P on $X \times Y$ is such that $(x_1 P y_2)_G$ if there exists a path directing from x_1 to y_2 in G and if $x_2, y_1 \in (x_1 P y_2)_G$, then a bisequence $(x_m, y_m) \in X \times Y$ a G^*-termwise connected $(G^* - TWC)$ if $(x_m P y_{m+1})$ for all $x \in X$, $y \in Y$ and $m \in \mathbb{N}$. Furthermore, we say a graph $G = (V(G), E(G))$ satisfies the property (P^*), if a $G^* - TWC$ $T-$Picards bisequence (x_m, y_m) biconvergent in $X \cap Y$, which guarantees that there is a limit v such that $((x_m, y_m), v) \in E(G)$ or $(v, (x_m, y_m)) \in E(G)$ for all $m > m_0$.

Definition 4.2. Let G be a graph containing all the loops associated with graphical bipolar metric space (X, Y, d_G). A covariant mapping $T : (X, Y, d_G) \rightrightarrows (X, Y, d_G)$ is said to be G_b-contraction (on $X \cup Y$) on graphical bipolar metric space (X, Y, d_G) such that:

(i): $(x, y) \in E(G)$, then we have

$$(Tx, Ty) \in E(G), \text{ for all } x \in X, \; y \in Y. \tag{4.1}$$

(ii): there exists $\gamma \in (0, 1)$ for all $x \in X$, $y \in Y$ with $(x, y) \in E(G)$ implies

$$d_G(Tx, Ty) \leq \gamma d(x, y). \tag{4.2}$$

Theorem 4.3. *Let $T : (X, Y, d_G) \rightrightarrows (X, Y, d_G)$ be a graphical G_b-contraction on a $G-$complete graphical bipolar metric space (X, Y, d_G). If the following conditions hold:*

(i): G satisfies the property (P^*)
(ii): there exist $x_0 \in X$, $y_0 \in Y$ with $(Tx_0, Ty_0) \in [(x_0, y_0)]^l_G$ for some $l \in \mathbb{N}$.

Then, there exists $v \in X \cap Y$ such that $T - P_bS$ (x_m, y_m) with initial value $x_0 \in X$, $y_0 \in Y$ is $G^* - TWC$ and biconverges to both v and Tv.

Proof. Let $x_0 \in X$ and $y_0 \in Y$ such that $(Tx_0, Ty_0) \in [(x_0, y_0)]^l_G$ for some $l \in \mathbb{N}$. By taking $x_0 \in X$ and $y_0 \in Y$ be initial values of $T - P_bS$ (x_m, y_m), $\exists$ a path $\{(x_j, y_j)\}^l_{j=0}$, such that $x_{n+1} = Tx_n$ and $y_{n+1} = Ty_n$ where $(x_j, y_j) \in E(G)$ for $j = 0, 1, 2 \ldots, l$. By using (4.1), we have $(Tx_{j-1}, Ty_j) \in E(G)$ for $j = 1, , 2 \ldots, l$. This implies that $\{(Tx_j, Ty_j)\}^l_{j=0}$ is a path from $x_2 = Ty_1 = T^2x_1$ to $y_2 = Tx_2 = T^2y_1$ having length l such that $(x_2, y_2) \in [(x_1, y_1)]^l_G$. Continuing this procedure we conclude that $\{(T^mx_j, T^my_j)\}^l_{j=0}$ is a path from $(T^{m-1}x_0, T^{m-1}y_0) = (x_m, y_m)$ to $(T^mx_l, T^my_l) = (x_{m+1}, y_{m+1})$ of length l and hence $(x_{m+1}, y_{m+1}) \in [(x_m, y_m)]^l_G$ for all $m \in \mathbb{N}$. This confirms that (x_m, y_m) is a $G^* - TWC$ bisequence, which shows that

$$(T^mx_j, T^my_j) \in E(G) \text{ for } j = 1, , 2 \ldots, l \text{ and } m \in \mathbb{N}.$$

Then by using (4.2), we obtain

$$\begin{aligned} d_G(T^mx_j, T^my_j) &\leq \gamma d_G(T^{m-1}x_j, T^{m-1}y_j) \\ &\leq \gamma^2 d_G(T^{m-2}x_j, T^{m-2}y_j) \end{aligned}$$

continuing the same procedure we conclude that

$$d_G(T^mx_j, T^my_j) \leq \gamma^m d_G(x_j, y_j).$$

Now, we have to show that (x_n, y_n) is Cauchy bisequence in graphical bipolar metric space (X, Y, d_G) for this we suppose that $P = d_G(x_0, y_1) + d_G(x_0, y_0)$ and $Q_n = \frac{\gamma^n P}{1-\gamma}$. Then for each positive integer n and p we have

$$\begin{aligned} d_G(x_n, y_n) &= d_G(T(x_{n-1}), T(y_{n-1})) \\ &\leq \gamma d_G(x_{n-1}, y_{n-1}) \\ &\leq \gamma^2 d_G(x_{n-2}, y_{n-2}) \\ &\vdots \\ &\leq \gamma^n d_G(x_0, y_0), \end{aligned}$$

and also we have

$$\begin{aligned} d_G(x_n, y_{n+1}) &= d_G(T(x_{n-1}), T(y_n)) \\ &\leq \gamma d_G(x_{n-1}, y_n) \\ &\leq \gamma^2 d_G(x_{n-2}, y_{n-1}) \\ &\vdots \\ &\leq \gamma^n d_G(x_0, y_1). \end{aligned}$$

As we know that

$$\begin{aligned} d_G(x_{n+p}, y_n) &\leq d_G(x_{n+p}, y_{n+1}) + d_G(x_n, y_{n+1}) + d_G(x_n, y_n) \\ &\leq d_G(x_{n+p}, y_{n+1}) + \gamma^n P \\ &\leq d_G(x_{n+p}, y_{n+2}) + d_G(x_{n+1}, y_{n+2}) + d_G(x_{n+1}, y_{n+1}) + \gamma^n P \\ &\leq d_G(x_{n+p}, y_{n+2}) + (\gamma^{n+1} + \gamma^n)P \\ &\vdots \\ &\leq d_G(x_{n+p}, y_{n+p}) + (\gamma^{n+p-1} + \cdots + \gamma^{n+1} + \gamma^n)P \\ &\leq (\gamma^{n+p} + \cdots + \gamma^{n+1} + \gamma^n)P \\ &\leq \gamma^n(1 + \gamma + \gamma^2 + \cdots + \gamma^{p-1} + \gamma^p)P \\ &\leq \gamma^n P \sum_{k=0}^{\infty} \gamma^k = Q_n. \end{aligned}$$

Similarly, we can prove that

$$\begin{aligned} d_G(x_n, y_{n+p}) &\leq d_G(x_n, y_n) + d_G(x_{n+1}, y_n) + d_G(x_{n+1}, y_{n+p}) \\ &\leq \gamma^n P + d_G(x_{n+1}, y_{n+p}) \\ &\leq \gamma^n P + d_G(x_{n+1}, y_{n+1}) + d_G(x_{n+2}, y_{n+1}) + d_G(x_{n+2}, y_{n+p}) \\ &\leq (\gamma^{n+1} + \gamma^n)P + d_G(x_{n+2}, y_{n+p}) \\ &\vdots \\ &\leq (\gamma^{n+p-1} + \cdots + \gamma^{n+1} + \gamma^n)P + d_G(x_{n+p}, y_{n+p}) \\ &\leq (\gamma^{n+p} + \cdots + \gamma^{n+1} + \gamma^n)P \\ &\leq \gamma^n(1 + \gamma + \gamma^2 + \cdots + \gamma^{p-1} + \gamma^p)P \\ &\leq \gamma^n P \sum_{k=0}^{\infty} \gamma^k = Q_n. \end{aligned}$$

Let $\epsilon > 0$ and $\gamma \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $Q_{n_0} = \frac{\gamma^{n_0} P}{1-\gamma} < \frac{\epsilon}{3}$. Then

$$\begin{aligned} d_G(x_n, y_m) &\leq d_G(x_n, y_{n_0}) + d_G(x_{n_0}, y_{n_0}) + d_G(x_{n_0}, y_m) \\ &\leq 3Q_n < \epsilon, \end{aligned}$$

which shows that (x_n, y_n) is a Cauchy bisequence. Since (X, Y, d_G) is a $G-$complete graphical bipolar metric space, then bisequence (x_n, y_n) is biconvergent in $X \cap Y$, and from the given condition (i), there exists some $v \in X \cap Y$, $n_0 \in \mathbb{N}$ such that $((x_n, y_n), v) \in E(G)$ and $(v, (x_n, y_n)) \in E(G)$ for all $n > n_0$ and

$$\lim_{n\to\infty} d_G\left((x_n, y_n), v\right) = 0,$$

which confirms that bisequence (x_n, y_n) is biconvergent to v. If $((x_n, y_n), v) \in E(G)$ then by using (4.2)

$$\begin{aligned} d_G\left((x_{n+1}, y_{n+1}), Tv\right) &= d_G\left((Tx_n, Ty_n), Tv\right) \\ &\leq \gamma d_G\left((x_n, y_n), v\right), \end{aligned}$$

for all $n > n_0$, which shows that

$$\lim_{n\to\infty} d_G\left((x_{n+1}, y_{n+1}), Tv\right) = 0.$$

If $(v, (x_n, y_n)) \in E(G)$, similar argument that we used above,

$$\lim_{n\to\infty} d_G\left(Tv, (x_{n+1}, y_{n+1})\right) = 0,$$

hence bisequence (x_n, y_n) is biconvergent to v and Tv. □

Now we have to show that mapping T has a fixed point. To prove this, we first define a property (M^*) as follows:

Definition 4.4. Let (X, Y, d_G) be a graphical bipolar metric space and $T : X \cup Y \to X \cup Y$ be a mapping. We say that quadruple (X, Y, d_G, T) satisfies the property (M^*) if corresponding two limits $u \in X \cap Y$ and $v \in T(X \cap Y)$ of a $G^* - TWC\ T - P_bS\ (x_n, y_n)$, then we have $u = v$.

Theorem 4.5. *If all of the hypotheses in Theorem 4.3 are true, and we further assume that the quadruple (X, Y, d_G, T) meets the condition (M^*), then T admits a fixed point.*

Proof. Theorem 4.3 proves that $T - P_bS\ (x_n, y_n)$ with initial values $x_0 \in X$, $y_0 \in Y$ biconverges to both v and Tv. Since $v \in X \cap Y$ and $Tv \in T(X \cap Y)$, by hypothesis, we obtain $v = Tv$ and hence T concedes a fixed point. □

Now, for contravariant mapping, we shall establish a similar conclusion.

Definition 4.6. Let G be a graph containing all the loops associated with graphical bipolar metric space (X, Y, d_G). A contravariant mapping $T : (X, Y, d_G) \rightleftarrows (X, Y, d_G)$ is said to be G_b^*-contraction (on $X \cup Y$) on graphical bipolar metric space (X, Y, d_G) such that:

(i): $(x, y) \in E(G)$, then we have

$$(Tx, Ty) \in E(G), \text{ for all } x \in X,\ y \in Y. \tag{4.3}$$

(ii): there exists $\gamma \in (0, 1)$ for all $x \in X$, $y \in Y$ with $(x, y) \in E(G)$ implies

$$d_G(Tx, Ty) \leq \gamma d(y, x). \tag{4.4}$$

Theorem 4.7. *Let $T : (X, Y, d_G) \rightleftarrows (X, Y, d_G)$ be a graphical G_b^*-contraction on a $G-$complete graphical bipolar metric space (X, Y, d_G). If the following conditions hold:*

(i): G satisfies the property (P^*)

(ii): there exist $x_0 \in X$, $y_0 \in Y$ with $(Tx_0, Ty_0) \in [(x_0, y_0)]_G^l$ for some $l \in \mathbb{N}$.

Then, there exists $v \in X \cap Y$ such that $T - P_bS\ (x_m, y_m)$ with initial values $x_0 \in X$, $y_0 \in Y$ is $G^* - TWC$ and biconverges to both v and Tv.

Proof. Let $x_0 \in X$ such that $(Tx_0, Ty_0) \in [(x_0, y_0)]_G^l$ for some $l \in \mathbb{N}$. By taking $x_0 \in X$ be initial value of $T - P_bS\ (x_m, y_m)$, $\exists$ a path $\{(x_j, y_j)\}_{j=0}^l$, such that $y_n = Tx_n$ and $x_{n+1} = Ty_n$ where $(x_j, y_j) \in E(G)$ for $j = 0, 1, 2 \ldots, l$. By using (4.3), we have $(Tx_{j-1}, Ty_j) \in E(G)$ for $j = 1, 2 \ldots, l$. This implies that $\{(Tx_j, Ty_j)\}_{j=0}^l$ is a path from $x_2 = T^2x_1 = T^3x_0$ to $y_2 = T^2y_1 = T^3y_0$ having length l such that $(x_2, y_2) \in [(x_1, y_1)]_G^l$. Continuing this procedure we conclude that $\{(T^mx_j, T^my_j)\}_{j=0}^l$ is a path from $(T^{m+1}x_0, T^{m+1}y_0) = (x_m, y_m)$ to $(T^{m+2}x_l, T^{m+2}y_l) = (x_{m+1}, y_{m+1})$ of length l and hence

$(x_{m+1}, y_{m+1}) \in [(x_m, y_m)]_G^l$ for all $m \in \mathbb{N}$. This confirms that (x_m, y_m) is a $G^* - TWC$ bisequence, which shows that

$$(T^m x_j, T^m y_j) \in E(G) \text{ for } j = 1, , 2 \ldots, l \text{ and } m \in \mathbb{N}.$$

Then by using (4.4), we obtain

$$\begin{aligned} d_G(T^m x_j, T^m y_j) &\leq \gamma d_G(T^{m-1} y_j, T^m x_j) \\ &\leq \gamma^2 d_G(T^{m-1} y_j, T^{m-1} x_j) \\ &= \gamma^2 d_G(T^{m-1} x_j, T^{m-1} y_j). \end{aligned}$$

Continuing the same procedure we conclude that

$$d_G(T^m x_j, T^m y_j) \leq \gamma^{2m} d_G(x_j, y_j).$$

Now, we have to show that (x_n, y_n) is Cauchy bisequence in graphical bipolar metric space (X, Y, d_G). For this we suppose that $Q_n = \frac{\gamma^{2n}}{1-\gamma} d(x_0, y_0)$, then for each positive integer n and p we have

$$\begin{aligned} d_G(x_n, y_n) &= d_G(T(y_{n-1}), T(x_n)) \\ &\leq \gamma d_G(x_n, y_{n-1}) \\ &\leq \gamma^2 d_G(x_{n-1}, y_{n-1}) \\ &\vdots \\ &\leq \gamma^{2n} d_G(x_0, y_0). \end{aligned}$$

Also, we have

$$\begin{aligned} d_G(x_n, y_{n+1}) &= d_G(T(x_{n-1}), T(y_n)) \\ &\leq \gamma d_G(x_{n-1}, y_n) \\ &\vdots \\ &\leq \gamma^{2n+1} d_G(x_0, y_0). \end{aligned}$$

As we know that

$$\begin{aligned} d_G(x_{n+p}, y_n) &\leq d_G(x_{n+p}, y_{n+1}) + d_G(x_{n+1}, y_{n+1}) + d_G(x_{n+1}, y_n) \\ &\leq d_G(x_{n+p}, y_{n+1}) + (\gamma^{2n+2} + \gamma^{2n+1}) d(x_0, y_0) \\ &\leq d_G(x_{n+p}, y_{n+2}) + d_G(x_{n+2}, y_{n+2}) + d_G(x_{n+2}, y_{n+1}) \\ &+ (\gamma^{2n+2} + \gamma^{2n+1}) d(x_0, y_0) \\ &\leq d_G(x_{n+p}, y_{n+2}) + (\gamma^{2n+4} + \gamma^{2n+3} + \gamma^{2n+2} + \gamma^{2n+1}) d(x_0, y_0) \\ &\vdots \\ &\leq d_G(x_{n+p}, y_{n+p-1}) + (\gamma^{2n+2p-2} + \cdots + \gamma^{2n+2} + \gamma^{2n+1}) d(x_0, y_0) \\ &\leq (\gamma^{2n+2p-1} + \gamma^{2n+2p-2} + \cdots + \gamma^{2n+2} + \gamma^{2n+1}) d(x_0, y_0) \\ &\leq \gamma^{2n+1} \sum_{k=0}^{\infty} \gamma^k = \gamma Q_n < Q_n. \end{aligned}$$

Similarly, we can prove that

$$\begin{aligned} d_G(x_n, y_{n+p}) &\le d_G(x_n, y_n) + d_G(x_{n+1}, y_n) + d_G(x_{n+1}, y_{n+p}) \\ &\le (\gamma^{2n} + \gamma^{2n+1}) d(x_0, y_0) + d_G(x_{n+1}, y_{n+p}) \\ &\le (\gamma^{2n} + \gamma^{2n+1}) d(x_0, y_0) + d_G(x_{n+1}, y_{n+1}) + d_G(x_{n+2}, y_{n+1}) \\ &+ d_G(x_{n+2}, y_{n+p}) \\ &\le (\gamma^{2n} + \gamma^{2n+1} + \gamma^{2n+2} + \gamma^{2n+3}) d(x_0, y_0) + d_G(x_{n+2}, y_{n+p}) \\ &\vdots \\ &\le (\gamma^{2n} + \gamma^{2n+1} + \cdots + \gamma^{2n+2p-1}) d(x_0, y_0) + d_G(x_{n+p}, y_{n+p}) \\ &\le (\gamma^{2n} + \gamma^{2n+1} + \cdots + \gamma^{2n+2p-1} + \gamma^{2n+2p}) d(x_0, y_0) \\ &\le \gamma^{2n} \sum_{k=0}^{\infty} \gamma^k d(x_0, y_0) = Q_n. \end{aligned}$$

Let $\epsilon > 0$ and $\gamma \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $Q_{n_0} = \frac{\gamma^{2n_0+1}}{1-\gamma} d(x_0, y_0) < \frac{\epsilon}{3}$. Then

$$\begin{aligned} d_G(x_n, y_m) &\le d_G(x_n, y_{n_0}) + d_G(x_{n_0}, y_{n_0}) + d_G(x_{n_0}, y_m) \\ &\le 3Q_{n_0} < \epsilon, \end{aligned}$$

which shows that (x_n, y_n) is a Cauchy bisequence. Since (X, Y, d_G) is a $G-$complete graphical bipolar metric space, then bisequence (x_n, y_n) biconvergent to in $X \cap Y$ and from the given condition (i) there exists some $v \in X \cap Y$, $n_0 \in \mathbb{N}$ such that $((x_n, y_n), v) \in E(G)$ and $(v, (x_n, y_n)) \in E(G)$ for all $n > n_0$ and

$$\lim_{n \to \infty} d_G\left((x_n, y_n), v\right) = 0,$$

which confirms that bisequence (x_n, y_n) biconvergent to v. If $((x_n, y_n), v) \in E(G)$ then by using (4.4)

$$\begin{aligned} d_G\left((x_{n+1}, y_{n+1}), Tv\right) &= d_G\left((Ty_n, Tx_{n+1}), Tv\right) \\ &\le \gamma d_G\left((x_{n+1}, y_n), v\right) \\ &\le \gamma^2 d_G\left((x_n, y_n), v\right) \end{aligned}$$

for all $n > n_0$, which shows that

$$\lim_{n \to \infty} d_G\left((x_{n+1}, y_{n+1}), Tv\right) = 0.$$

If $(v, (x_n, y_n)) \in E(G)$, with the similar argument as above, we obtain

$$\lim_{n \to \infty} d_G\left(Tv, (x_{n+1}, y_{n+1})\right) = 0.$$

Hence the bisequence (x_n, y_n) is biconvergent and converges to to v and Tv. □

Now we have to show that mapping T have a fixed point for this we use the property (M^*).

Theorem 4.8. *If all the hypothesis embodied in Theorem 4.7 and further suppose that the quadruple (X, Y, d_G, T) satisfies the property (M^*), then T concedes a fixed point.*

Proof. From the Theorem 4.7 provide that $T - P_b S$ (x_n, y_n) with initial value $x_0 \in X$ biconverges to both v and Tv. Since $v \in X \cap Y$ and $Tv \in T(X \cap Y)$, by hypothesis we obtain $v = Tv$ and T concedes a fixed point. □

Example 4.9. Let $X = \{0,1,2\}$, $Y = \{1,2,3\}$ and $d_G : X \times Y \to \mathbb{R}^+$ be defined in the Table 4.9:

d_G	0	1	2	3
0	0	10	11	7
1	10	0	12	8
2	11	12	0	9
3	7	8	9	0

for all $x \in X$ and $y \in Y$. It is easy to see that (X, Y, d_G) is graphical bipolar metric space with vertex set $V(G) = X \cup Y$ and edges set $E(G) = \{(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)\}$ as shown in the Figure (3).

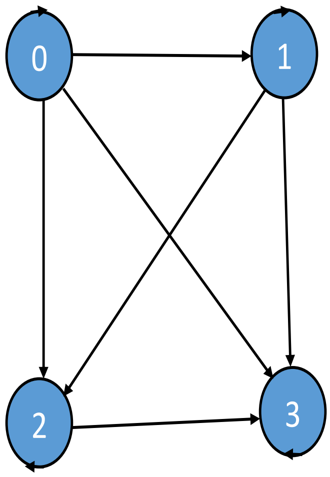

Figure 3. Graph associated with Example 4.9

Now, we have to define a covariant mapping

$$T : (X, Y, d_G) \rightrightarrows (X, Y, d_G),$$

having $T(X) \subseteq X$, $T(Y) \subseteq Y$ by the following

$$Tx = \begin{cases} 1, & \text{when } x = 0, \\ 2, & \text{when } x = \{1,2\}, \end{cases}$$

and

$$Ty = \{3, \text{ when } y = 2,$$

for all $x \in X$, $y \in Y$.

Case (i): When $x = 0$ and $y = 2$, then

$$\begin{aligned} d_G(T0, T2) &\leq \gamma d_G(0,2) \\ d_G(1,3) &\leq \gamma d_G(0,2) \\ 8 &\leq \gamma(11). \end{aligned}$$

Case (ii): For $x = 1$ and $y = 2$,

$$\begin{aligned} d_G(T1, T2) &\leq \gamma d_G(1,2) \\ d_G(2,3) &\leq \gamma d_G(1,2) \\ 9 &\leq \gamma(12), \end{aligned}$$

Hence G_b-contraction is satisfied if $\gamma = \frac{5}{6}$ for all $x \in X$, $y \in Y$. All the conditions of the Theorem (4.3) are satisfied and T concedes a fixed point, which is $2 \in X \cap Y$.

5. Ćirić type graphical contraction

In this section, we will discuss some fixed point theorems using Ćirić type contractive conditions in the setup of graphical bipolar metric spaces (X, Y, d_G).

Definition 5.1. Let G be a graph containing all the loops associated with graphical bipolar metric space (X, Y, d_G). A covariant mapping $T : (X, Y, d_G) \rightrightarrows (X, Y, d_G)$ is said to be graphical $G_\zeta-$contraction (on $X \cup Y$) on graphical bipolar metric space (X, Y, d_G) such that:

(i): $(x, y) \in E(G)$, then we have

$$(Tx, Ty) \in E(G), \text{ for all } x \in X, \; y \in Y. \tag{5.1}$$

(ii): there exists $\zeta \in (0, 1)$ for all $x \in X$, $y \in Y$ with $(x, y) \in E(G)$ implies

$$d_G(Tx, Ty) \leq \zeta \max\{d_G(x, y), d_G(x, Tx), d_G(y, Ty), d_G(x, Ty), d_G(y, Tx)\}. \tag{5.2}$$

Theorem 5.2. *Let $T : (X, Y, d_G) \rightrightarrows (X, Y, d_G)$ be a graphical $G_\zeta-$contraction on a $G-$complete graphical bipolar metric space (X, Y, d_G). If the following conditions hold:*

(i): G satisfies the property (P^*)
(ii): there exist $x_0 \in X$, $y_0 \in Y$ with $(Tx_0, Ty_0) \in [(x_0, y_0)]^l_G$ for some $l \in \mathbb{N}$.

Then, there exists $v \in X \cap Y$ such that $T - P_bS$ (x_m, y_m) with initial values $x_0 \in X$, $y_0 \in Y$ is $G^* - TWC$ and biconverges to both v and Tv.

Proof. Let $x_0 \in X$ and $y_0 \in Y$ such that $(Tx_0, Ty_0) \in [(x_0, y_0)]^l_G$ for some $l \in \mathbb{N}$. By taking $x_0 \in X$ and $y_0 \in Y$ be initial values of $T - P_bS$ (x_m, y_m), $\exists$ a path $\{(x_j, y_j)\}^l_{j=0}$, such that $x_{n+1} = Tx_n$ and $y_{n+1} = Ty_n$ where $(x_j, y_j) \in E(G)$ for $j = 0, 1, 2 \ldots, l$. By using (5.1), we have $(Tx_{j-1}, Ty_j) \in E(G)$ for $j = 1, , 2 \ldots, l$. This implies that $\{(Tx_j, Ty_j)\}^l_{j=0}$ is a path from $x_2 = Ty_1 = T^2x_1$ to $y_2 = Tx_2 = T^2y_1$ having length l such that $(x_2, y_2) \in [(x_1, y_1)]^l_G$. Continuing this procedure we conclude that $\{(T^m x_j, T^m y_j)\}^l_{j=0}$ is a path from $(T^{m-1}x_0, T^{m-1}y_0) = (x_m, y_m)$ to $(T^m x_l, T^m y_l) = (x_{m+l}, y_{m+l})$ of length l and hence $(x_{m+l}, y_{m+l}) \in [(x_m, y_m)]^l_G$ for all $m \in \mathbb{N}$. This confirms that (x_m, y_m) is a $G^* - TWC$ bisequence, which shows that

$$(T^m x_j, T^m y_j) \in E(G) \text{ for } j = 1, 2 \ldots, l \text{ and } m \in \mathbb{N}.$$

Then by using (5.2), we obtain

$$\begin{aligned} d_G(T^m x_j, T^m y_j) \;\; &\leq \;\; \zeta \max\{d_G(T^{m-1}x_j, T^{m-1}y_j), d_G(T^{m-1}x_j, T^m x_j), d_G(T^{m-1}y_j, T^m y_j), \\ &\qquad d_G(T^{m-1}x_j, T^m y_j), d_G(T^{m-1}y_j, T^m x_j)\} \\ &\leq \;\; \zeta d_G(T^{m-1}x_j, T^{m-1}y_j). \end{aligned}$$

If we choose maximum from these terms

$$\{d_G(T^{m-1}x_j, T^m x_j), d_G(T^{m-1}y_j, T^m y_j), d_G(T^{m-1}x_j, T^m y_j), d_G(T^{m-1}y_j, T^m x_j)\},$$

which is a contradiction because first two terms are not bisequence and other two terms are not decreasing bisequences. Continue the same process we get

$$d_G(T^m x_j, T^m y_j) \leq \zeta^m d_G(x_j, y_j).$$

Since the bisequence (x_n, y_n) is a $G^* - TWC$ bisequence. Now, we have to show that (x_n, y_n) is Cauchy bisequence in graphical bipolar metric space (X, Y, d_G) for this we suppose that $P = d_G(x_0, y_1) + d(x_0, y_0)$ and $Q_n = \frac{\zeta^n P}{1-\zeta}$. Then for each positive integer n and p we have

$$\begin{aligned}
d_G(x_n, y_n) &= d_G(T(x_{n-1}), T(y_{n-1})) \\
&\leq \zeta \max\{d_G(x_{n-1}, y_{n-1}), d_G(x_{n-1}, Tx_{n-1}), d_G(y_{n-1}, Ty_{n-1}) \\
&\quad d_G(x_{n-1}, Ty_{n-1}), d_G(y_{n-1}, Tx_{n-1})\} \\
&= \zeta \max\{d_G(x_{n-1}, y_{n-1}), d_G(x_{n-1}, x_n), d_G(y_{n-1}, y_n), \\
&\quad d_G(x_{n-1}, y_n), d_G(y_{n-1}, x_n)\} \\
&\leq \zeta d_G(x_{n-1}, y_{n-1}) \\
&\leq \zeta^2 d_G(x_{n-2}, y_{n-2}) \\
&\quad \vdots \\
&\leq \zeta^n d_G(x_0, y_0),
\end{aligned}$$

and also we have

$$\begin{aligned}
d_G(x_n, y_{n+1}) &= d_G(T(x_{n-1}), T(y_n)) \\
&\leq \zeta \max\{d_G(x_{n-1}, y_n), d_G(x_{n-1}, Tx_{n-1}), d_G(y_n, Ty_n), \\
&\quad d_G(x_n, Ty_n), d_G(y_n, Tx_{n-1})\} \\
&= \zeta \max\{d_G(x_{n-1}, y_n), d_G(x_{n-1}, x_n), d_G(y_n, y_{n+1}), \\
&\quad d_G(x_n, y_{n+1}), d_G(y_n, x_n)\} \\
&\leq \zeta \max d_G(x_{n-1}, y_n) \\
&\leq \zeta^2 d_G(x_{n-2}, y_{n-1}) \\
&\quad \vdots \\
&\leq \zeta^n d_G(x_0, y_1).
\end{aligned}$$

As we know that

$$\begin{aligned}
d_G(x_{n+p}, y_n) &\leq d_G(x_{n+p}, y_{n+1}) + d_G(x_n, y_{n+1}) + d_G(x_n, y_n) \\
&\leq d_G(x_{n+p}, y_{n+1}) + \zeta^n P \\
&\leq d_G(x_{n+p}, y_{n+2}) + d_G(x_{n+1}, y_{n+2}) + d_G(x_{n+1}, y_{n+1}) + \zeta^n P \\
&\leq d_G(x_{n+p}, y_{n+2}) + (\zeta^{n+1} + \zeta^n)P \\
&\quad \vdots \\
&\leq d_G(x_{n+p}, y_{n+p}) + (\zeta^{n+p-1} + \cdots + \zeta^{n+1} + \zeta^n)P \\
&\leq (\zeta^{n+p} + \cdots + \zeta^{n+1} + \zeta^n)P \\
&\leq \zeta^n(1 + \zeta + \zeta^2 + \cdots + \zeta^{p-1} + \zeta^p)P \\
&\leq \zeta^n P \sum_{k=0}^{\infty} \gamma^k = Q_n.
\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}
d_G(x_n, y_{n+p}) &\leq d_G(x_n, y_n) + d_G(x_{n+1}, y_n) + d_G(x_{n+1}, y_{n+p}) \\
&\leq \zeta^n P + d_G(x_{n+1}, y_{n+p}) \\
&\leq \zeta^n P + d_G(x_{n+1}, y_{n+1}) + d_G(x_{n+2}, y_{n+1}) + d_G(x_{n+2}, y_{n+p}) \\
&\leq (\zeta^{n+1} + \zeta^n)P + d_G(x_{n+2}, y_{n+p}) \\
&\vdots \\
&\leq (\zeta^{n+p-1} + \cdots + \zeta^{n+1} + \zeta^n)P + d_G(x_{n+p}, y_{n+p}) \\
&\leq (\zeta^{n+p} + \cdots + \zeta^{n+1} + \zeta^n)P \\
&\leq \zeta^n(1 + \zeta + \zeta^2 + \cdots + \zeta^{p-1} + \zeta^p)P \\
&\leq \zeta^n P \sum_{k=0}^{\infty} \zeta^k = Q_n.
\end{aligned}$$

Let $\epsilon > 0$ and $\zeta \in (0,1)$, there exists $n_0 \in N$ such that $Q_{n_0} = \frac{\zeta^{n_0} P}{1-\zeta} < \frac{\epsilon}{3}$. Then

$$\begin{aligned}
d_G(x_n, y_m) &\leq d_G(x_n, y_{n_0}) + d_G(x_{n_0}, y_{n_0}) + d_G(x_{n_0}, y_m) \\
&\leq 3Q_n < \epsilon,
\end{aligned}$$

which shows that (x_n, y_n) is a Cauchy bisequence. Since (X, Y, d_G) is a $G-$complete graphical bipolar metric space, then bisequence (x_n, y_n) biconvergent to in $X \cap Y$ and from the given condition (i) there exists some $v \in X \cap Y$, $n_0 \in \mathbb{N}$ such that $((x_n, y_n), v) \in E(G)$ and $(v, (x_n, y_n)) \in E(G)$ for all $n > n_0$ and

$$\lim_{n\to\infty} d_G\left((x_n, y_n), v\right) = 0,$$

which confirms that bisequence (x_n, y_n) biconvergent to v. If $((x_n, y_n), v) \in E(G)$ then by using (5.2)

$$\begin{aligned}
d_G\left((x_{n+1}, y_{n+1}), v\right) &= d_G\left((Tx_n, Ty_n), Tv\right) \\
&\leq \zeta \max\{d_G\left((x_n, y_n), v\right), d_G\left((x_n, y_n), T(x_n, y_n)\right), d_G\left(v, Tv\right), \\
&\quad d_G\left((x_n, y_n), Tv\right), d_G\left(v, T(x_n, y_n)\right)\} \\
&\leq \zeta d_G\left((x_n, y_n), v\right),
\end{aligned}$$

for all $n > n_0$, which shows that

$$\lim_{n\to\infty} d_G\left((x_{n+1}, y_{n+1}), v\right) = 0.$$

If $(v, (x_n, y_n)) \in E(G)$, similar argument that we used above

$$\lim_{n\to\infty} d_G\left(v, (x_{n+1}, y_{n+1})\right) = 0,$$

hence bisequence (x_n, y_n) biconvergent to v and Tv. □

Theorem 5.3. *If all the hypothesis embodied in Theorem 5.2 are fulfilled, and further assume that the quadruple (X, Y, d_G, T) satisfies the property (M^*), then T concedes a fixed point.*

Proof. From the Theorem 5.2 provide that $T - P_b S$ (x_n, y_n) with initial value $x_0 \in X$ biconverges to both v and Tv. Since $v \in X \cap Y$ and $Tv \in T(X \cap Y)$, by hypothesis we obtain $v = Tv$ and hence T concedes a fixed point. □

Definition 5.4. Let G be a graph containing all the loops associated with graphical bipolar metric space (X, Y, d_G). A contravariant mapping $T : (X, Y, d_G) \rightleftarrows (X, Y, d_G)$ is said to be $G^*_\zeta-$contraction (on $X \cup Y$) on graphical bipolar metric space (X, Y, d_G) such that:

(i): $(x, y) \in E(G)$, then we have

$$(Tx, Ty) \in E(G), \text{ for all } x \in X, \ y \in Y. \tag{5.3}$$

(ii): there exists $\zeta \in (0,1)$ for all $x \in X$, $y \in Y$ with $(x, y) \in E(G)$ implies

$$d_G(Ty, Tx) \leq \zeta \max\{d_G(x, y), d_G(x, Tx), d_G(Ty, y), d_G(x, Ty), d_G(y, Tx)\}. \tag{5.4}$$

Theorem 5.5. *Let $T : (X, Y, d_G) \rightleftarrows (X, Y, d_G)$ be a graphical $G^*_\zeta-$contraction on a $G-$complete graphical bipolar metric space (X, Y, d_G). If the following conditions hold:*

(i): G satisfies the property (P^*),
(ii): there exist $x_0 \in X$, $y_0 \in Y$ with $(Tx_0, Ty_0) \in [(x_0, y_0)]^l_G$ for some $l \in \mathbb{N}$.

Then, there exists $v \in X \cap Y$ such that $T - P_bS$ (x_m, y_m) with initial values $x_0 \in X$, $y_0 \in Y$ is $G^* - TWC$ and biconverges to both v and Tv.

Proof. Let $x_0 \in X$ such that $(Tx_0, Ty_0) \in [(x_0, y_0)]^l_G$ for some $l \in \mathbb{N}$. By taking x_0 be initial values of $T - P_bS$ (x_m, y_m), $\exists$ a path $\{(x_j, y_j)\}^l_{j=0}$, such that $y_n = Tx_n$ and $x_{n+1} = Ty_n$ where $(x_j, y_j) \in E(G)$ for $j = 0, 1, 2 \ldots, l$. By using (5.3), we have $(Tx_{j-1}, Ty_j) \in E(G)$ for $j = 1, 2 \ldots, l$. This implies that $\{(Tx_j, Ty_j)\}^l_{j=0}$ is a path from $x_2 = T^2x_1 = T^3x_0$ to $y_2 = T^2y_1 = T^3y_0$ having length l such that $(x_2, y_2) \in [(x_1, y_1)]^l_G$. Continuing this procedure we conclude that $\{(T^mx_j, T^my_j)\}^l_{j=0}$ is a path from $(T^mx_0, T^my_0) = (x_m, y_m)$ to $(T^{m+l}x_0, T^{m+l}y_0) = (x_{m+l}, y_{m+l})$ of length l and hence $(x_{m+1}, y_{m+1}) \in [(x_m, y_m)]^l_G$ for all $m \in \mathbb{N}$. This confirms that (x_m, y_m) is a $G^* - TWC$ bisequence, which shows that

$$(T^mx_j, T^my_j) \in E(G) \text{ for } j = 1,, 2 \ldots, l \text{ and } m \in \mathbb{N}.$$

Then by using 5.4, we obtain

$$\begin{aligned} d_G(T^mx_j, T^my_j) &\leq \zeta \max\{d_G(T^mx_j, T^{m-1}y_j), d_G(T^{m-1}x_j, T^mx_j), d_G(T^{m-1}y_j, T^my_j), \\ &\quad d_G(T^{m-1}x_j, T^my_j), d_G(T^{m-1}y_j, T^mx_j)\} \\ &\leq \zeta d_G(T^mx_j, T^{m-1}y_j). \end{aligned} \tag{5.5}$$

If we choose maximum from

$$\{d_G(T^{m-1}x_j, T^mx_j), d_G(T^{m-1}y_j, T^my_j), d_G(T^{m-1}x_j, T^my_j)\},$$

which is contradiction because first two terms are not bisequences and third term is decreasing but not as suggested by contravarient mapping (first decrease term x and then decrease y in next step). Similarly, we get

$$\begin{aligned} d_G(T^mx_j, T^{m-1}y_j) &\leq \zeta \max\{d_G(T^{m-1}x_j, T^{m-1}y_j), d_G(T^{m-1}x_j, T^mx_j), d_G(T^{m-2}y_j, T^{m-1}y_j), \\ &\quad d_G(T^{m-1}x_j, T^{m-2}y_j), d_G(T^{m-2}y_j, T^{m-1}x_j)\} \\ &\leq \zeta d_G(T^{m-1}x_j, T^{m-1}y_j). \end{aligned} \tag{5.6}$$

If we choose maximum from

$$\{d_G(T^{m-1}x_j, T^mx_j), d_G(T^{m-2}y_j, T^{m-1}y_j), d_G(T^{m-1}x_j, T^{m-2})\},$$

which is contradiction because first two terms are not bisequences and third term is not decreasing bisequence as contravarient mapping. From equation 5.5 and 5.6 we can write

$$d_G(T^mx_j, T^my_j) \leq \zeta^2 d_G(T^{m-1}x_j, T^{m-1}y_j),$$

therefore, it is easy to see that

$$d_G(T^mx_j, T^my_j) \leq \zeta^{2m} d_G(x_j, y_j).$$

The remaining part of the proof follows from that of Theorem 5.2 and Theorem 5.3. $\square$

Example 5.6. Let $X = \{1,3,5,7\}$, $Y = \{5,7,9,11\}$ and $d_G : X \times Y \to \mathbb{R}^+$ be defined as:

d_G	1	3	5	7	9	11
1	0	$\frac{17}{18}$	$\frac{16}{17}$	$\frac{15}{16}$	$\frac{14}{15}$	$\frac{13}{14}$
3	$\frac{17}{18}$	0	$\frac{12}{13}$	$\frac{11}{12}$	$\frac{10}{11}$	$\frac{9}{10}$
5	$\frac{16}{17}$	$\frac{12}{13}$	0	$\frac{8}{9}$	$\frac{7}{8}$	$\frac{6}{7}$
7	$\frac{15}{16}$	$\frac{11}{12}$	$\frac{8}{9}$	0	$\frac{5}{6}$	$\frac{4}{5}$
9	$\frac{14}{15}$	$\frac{10}{11}$	$\frac{7}{8}$	$\frac{5}{6}$	0	$\frac{3}{4}$
11	$\frac{13}{14}$	$\frac{9}{10}$	$\frac{6}{7}$	$\frac{4}{5}$	$\frac{3}{4}$	0

for all $x \in X$ and $y \in Y$. It is easy to see that (X, Y, d_G) is graphical bipolar metric space with vertex set $V(G) = X \cup Y$ and edges set as shown in Figure 4:

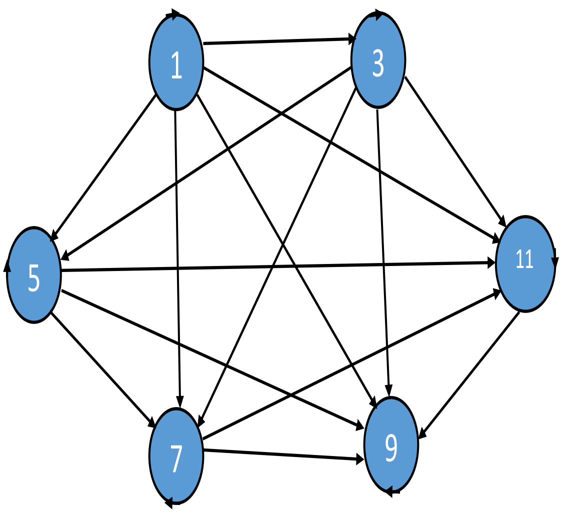

Figure 4. Graph associated with Example 5.6

$$
\begin{aligned}
E(G) \quad = \quad & \{(1,3),(1,5),(1,7),(1,9),(1,11), \\
& (3,5),(3,7),(3,9),(3,11),(5,7), \\
& (5,9),(5,11),(7,9),(7,11),(9,11)\}.
\end{aligned}
$$

Define a covariant mapping

$$T : (X, Y, d_G) \rightrightarrows (X, Y, d_G),$$

with $T(X) \subseteq X$, $T(Y) \subseteq Y$

$$Tx = \begin{cases} 1, & \text{when } x = 3, \\ 7, & \text{when } x = \{5,7\}, \end{cases}$$

and

$$Ty = \begin{cases} 9, & \text{when } y = 11, \\ 11, & \text{when } y = 9, \end{cases}$$

for all $x \in X$, $y \in Y$.
Case (i): When $x = 3$ and $y = 11$, then

$$\begin{aligned} d_G(T3, T11) &\leq \zeta \max\{d_G(3,11), d_G(3,T3), d_G(11,T11), d_G(3,T11), d_G(11,T3)\} \\ d_G(1,9) &\leq \zeta \max\{d_G(3,11), d_G(3,1), d_G(11,9), d_G(3,9), d_G(11,1)\} \\ \frac{14}{15} &\leq \zeta \max\{\frac{9}{10}, \frac{17}{18}, \frac{3}{4}, \frac{10}{11}, \frac{13}{14}\} \\ \frac{14}{15} &\leq \zeta(\frac{17}{18}). \end{aligned}$$

Case (ii): For $x = 3$ and $y = 9$, we have

$$\begin{aligned} d_G(T3, T9) &\leq \zeta \max\{d_G(3,9), d_G(3,T3), d_G(9,T9), d_G(3,T9), d_G(9,T3)\} \\ d_G(1,11) &\leq \zeta \max\{d_G(3,9), d_G(3,1), d_G(9,11), d_G(3,11), d_G(9,1)\} \\ \frac{13}{14} &\leq \zeta \max\{\frac{10}{11}, \frac{17}{18}, \frac{3}{4}, \frac{9}{10}, \frac{13}{14}\} \\ \frac{13}{14} &\leq \zeta(\frac{17}{18}). \end{aligned}$$

Case (iii): If $x = 5$ and $y = 11$, then

$$\begin{aligned} d_G(T5, T11) &\leq \zeta \max\{d_G(5,11), d_G(5,T5), d_G(11,T11), d_G(5,T11), d_G(11,T5)\} \\ d_G(7,9) &\leq \zeta \max\{d_G(5,11), d_G(5,7), d_G(11,9), d_G(5,9), d_G(11,7)\} \\ \frac{5}{6} &\leq \zeta \max\{\frac{6}{7}, \frac{8}{9}, \frac{3}{4}, \frac{7}{8}, \frac{4}{5}\} \\ \frac{5}{6} &\leq \zeta(\frac{8}{9}). \end{aligned}$$

Case (iv): If $x = 5$ and $y = 9$, then

$$\begin{aligned} d_G(T5, T9) &\leq \zeta \max\{d_G(5,9), d_G(5,T5), d_G(9,T9), d_G(5,T9), d_G(9,T5)\} \\ d_G(7,11) &\leq \zeta \max\{d_G(5,9), d_G(5,7), d_G(9,11), d_G(5,11), d_G(9,7)\} \\ \frac{4}{5} &\leq \zeta \max\{\frac{7}{8}, \frac{8}{9}, \frac{3}{4}, \frac{6}{7}, \frac{5}{6}\} \\ \frac{4}{5} &\leq \zeta(\frac{8}{9}). \end{aligned}$$

Hence, $G_\zeta-$contraction is satisfied for $\zeta = \frac{221}{222}$. All the properties of the Theorem 5.2 are satisfied, and T concedes a fixed point for all $x \in X$ and $y \in Y$, which is $7 \in X \cap Y$.

Example 5.7. If we take all the terms and conditions of the example 4.9, then we have
Case (i): When $x = 0$ and $y = 2$, then

$$\begin{aligned} d_G(T0, T2) &\leq \zeta \max\{d_G(0,2), d_G(0,T0), d_G(2,T2), d_G(0,T2), d_G(2,T0)\} \\ d_G(1,3) &\leq \zeta \max\{d_G(0,2), d_G(0,1), d_G(2,3), d_G(0,3), d_G(2,1)\} \\ 8 &\leq \zeta \max\{11, 10, 9, 7, 12\} \\ 8 &\leq \zeta(12). \end{aligned}$$

Case (ii): For $x = 1$ and $y = 2$, we have

$$\begin{aligned} d_G(T1, T2) &\leq \zeta \max\{d_G(1,2), d_G(1,T1), d_G(2,T2), d_G(1,T2), d_G(2,T1)\} \\ d_G(2,3) &\leq \zeta \max\{d_G(1,2), d_G(1,2), d_G(2,3), d_G(1,3), d_G(2,2)\} \\ 9 &\leq \zeta \max\{12, 12, 9, 8, 0\} \\ 9 &\leq \zeta(12), \end{aligned}$$

Hence, graphical $G_\zeta-$contraction is satisfied for $\zeta = \frac{5}{6}$. All the properties of the Theorem 5.2 are satisfied and T concedes a fixed point for all $x \in X$ and $y \in Y$ that is $2 \in X \cap Y$.

References

[1] H. Ahmad, M. Younis and M. E. Köksal, *Double controlled partial metric type spaces and convergence results*, J. Math. **2021**, 7008737, 11 pages.

[2] N. Chuensupantharat, P. Kumam, V. Chauhan, D. Singh and R. Menon, *Graphic contraction mappings via graphical b-metric spaces with applications*, Bull. Malays. Math. Sci. Soc. **42** 3149–3165, 2019.

[3] L. B. Ćirić, A generalization of *Banach's contraction principle*, Proc. Amer. Math. Soc. **45** 267–273, 1974.

[4] S. Çetin, U. Gürdal, *Characterization of bipolar ultrametric spaces and fixed point theorems*, Hacet. J. Math. Stat. **52** (1), 185-196, 2023.

[5] A.K. Dubey, U. Mishra and W.H. Lim, *Some new fixed point theorems for generalized contraction involving rational expressions in complex valued $b-$metric spaces*, Nonlinear Funct. Anal. Appl. **24** (3), 477–483, 2019.

[6] R. George, S. Radenovic, K.P. Reshma and S. Shukla, *Rectangular b-metric spaces and contraction principle*, J. Nonlinear Sci. Appl. **8** (6), 1005–1013, 2015.

[7] A. F. de Hierro and N. Shahzad, *From graphical metric spaces to fixed point theory in binary related distance spaces*, Filomat, **31**, 3209–3231, 2017.

[8] N. Hussain, M. Arshad and A. Shoaib, *Common fixed point results for α-ψ-contractions on a metric space endowed with graph*, J. Inequal. Appl. **(1)**, 136, 2014.

[9] N. Hussain, J. Ahmad and M.A. Kutbi MA, *Fixed point theorems for generalized Mizoguchi-Takahashi graphic contractions*, J. Funct. Spaces, **2016**, 6514920, 7 pp.

[10] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc. **136**, 1359–1373, 2008.

[11] G. N. V. Kishore, R. P. Agarwal, B. Srinuvasa Rao and R. V. N. Srinuvasa Rao, *Caristi type cyclic contraction and common fixed point theorems in bipolar metric spaces with applications*, Fixed Point Theory Appl. **2018** (1), 1-13.

[12] N. Saleem, H. Ahmad, H. Aydi and Y. Ulrich Gaba, *On Some Coincidence Best Proximity Point Results*, Journal of Mathematics, **2021**, 8005469, 19 pages

[13] Z. Mostefaoui, M. Bousselsal and J.K. Kim, *Some results in fixed point theory concerning rectangular $b-$metric spaces*, Nonlinear Funct. Anal. Appl., **24** (1), 49–59, 2019.

[14] A. Mutlu and U. Gürdal, *Bipolar metric spaces and some fixed point theorems*, J. Nonlinear Sci. Appl. **9** (9), 5362-5373, 2016.

[15] A. Mutlu, K. Ozkan and U. Gurdal, *Coupled fixed point theorems on bipolar metric spaces*, European journal of pure and applied Mathematics, **10** (4), 655-667, 2017.

[16] A. Mutlu, K. Ozkan and U. Gurdal, *Fixed point theorems for multivaled mappings on bipolar metric spaces*, Fixed Point Theory, **21** (1), 271-280, 2020.

[17] S. Radenović, K. Zoto, N. Dedović, V. Šešum-Cavic and A.H. Ansari, *Bhaskar-Guo-Lakshmikantam-Ćirić type results via new functions with applications to integral equations*, Applied Mathematics and Computation, **357** 75–87, 2019.

[18] S. Radenović, T. Došenović, T. Aleksić-Lampert and Z. Golubović, *A note on some recent fixed point results for cyclic contractions in b-metric spaces and an application to integral equations,* Appl. Math. Comput. **273**, 155–164, 2016.

[19] S. Shukla, S. Radenović and C. Vetro, *Graphical metric space: a generalized setting in fixed point theory*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas, **111**, 641–655, 2017.

[20] D. Singh, V. Chauhan and I. Altun, *Common fixed point of a power graphic (F, ψ)-contraction pair on partial $b-$metric spaces with application*, Nonlinear analysis: Modelling and control, **22** (5), 662–678, 2017.

[21] D. Singh, V, Joshi and J.K. Kim, *Existence of solution to Bessel-type boundary value problem Viam G–l cyclic $F-$contractive mapping with graphical verification*, Nonlinear Funct. Anal. Appl. **23** (2), 205–224, 2018.

[22] M. Younis, D. Singh and A. Goyal, *A novel approach of graphical rectangular $b-$metric spaces with an application to the vibrations of a vertical heavy hanging cable*, J. Fixed Point Theory Appl. **21**, 33, 2019.

[23] M. Younis and D. Bahuguna, *A unique approach to graph-based metric spaces with an application to rocket ascension*, Comp. Appl. Math. **42**, 44, 2023.

[24] M. Younis, D. Singh, M. Asadi and V. Joshi, *Results on contractions of Reich type in graphical $b-$metric spaces with applications*, Filomat, **33** (17), 5723–5735, 2019.

[25] M. Younis, D. Singh, D. Gopal, A. Goyal and M. S. Rathore, *On applications of generalized $F-$contraction to differential equations*, Nonlinear Funct. Anal. Appl. **24** (1) 155–174, 2019.

[26] M. Younis, D. Singh, I. Altun and V. Chauhan, *Graphical structure of extended $b-$metric spaces: an application to the transverse oscillations of a homogeneous bar*, Int. J. Nonlinear Sci. Numer. Simul. **23** (7-8), 1239–1252, 2021.

[27] G. N. V. Kishore, K. P. R. Rao, H. Isık, B. Srinuvasa Rao and A. Sombabu, *Covarian mappings and coupled fixed point results in bipolar metric spaces*, Int. J. Nonlinear Anal. Appl. **12** (1), 1-15, 2021.