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On Gamma Ring of Quotients of a Semiprime Gamma Ring

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Abstract

The aim of this paper is to define the gamma rings of quotients of a semiprime gamma ring and investigate some properties of gamma ring of quotients.

Keywords: Gamma Rings, Prime Gamma Rings, Semiprime Gamma Rings, Ring of Quotients.

Yarıasal Gamma Halkalanın Kesirlerinin Gamma Halkaları Üzerine

Özet

Bu makalenin amacı yarıasal bir gamma halkasının kesirli gamma halkalarını tanımlamak ve kesirli gamma halkasının bazı özelliklerini araştırmaktır.

Anahtar Kelimeler: Gamma Halkaları, Asal Gamma Halkaları, Yarıasal Gamma Halkaları, Kesirli Halkalar.

1. Introduction

The notion of a Γ −ring was introduced by Nobusawa in [9]. Let M be an abelian additive group whose elements are denoted by $a, b, c, \cdots$ and Γ another abelian additive group whose elements are $\gamma, \beta, \alpha, \cdots$. Suppose that $a\gamma b$ is defined to be an

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element of M and that $\gamma a \beta$ is defined to be an element of Γ for every a, b, γ and β. If the products satisfy the following three conditions:

(i) $(a+b)\gamma c = a\gamma c + b\gamma c, a(\alpha+\beta)b = a\alpha b + a\beta b, a\gamma(b+c) = a\gamma b + a\gamma c,$

(ii) $(a\gamma b)\beta c = a(\gamma b\beta)c = a\gamma(b\beta c),$

(iii) $a\alpha b = 0$ for all $a, b \in M$ implies $\alpha = 0,$

then M is called a Γ −ring in the sense of Nobusawa.

After this research, in [2], Barnes defined the structure of Γ −rings in some different way from that of Nobusawa as follows:

Let M and Γ be additive abelian groups. If there exits a mapping of $M \times \Gamma \times M$ to M (the image of (a, γ, b) $a, b \in M, \gamma \in \Gamma$ being denoted by $(a\gamma b)$) satisfying for all $a, b, c \in M, \alpha, \beta, \gamma \in \Gamma$:

(i) $(a+b)\gamma c = a\gamma c + b\gamma c, a(\alpha+\beta)b = a\alpha b + a\beta b, a\gamma(b+c) = a\gamma b + a\gamma c,$

(ii) $(a\gamma b)\beta c = a\gamma(b\beta c),$

then M is called a Γ-ring in the sense of Barnes.

In the present paper, the symbol $(\Gamma, M)_N$ stands for M is the Γ −ring in the sense of Nobusawa and the symbol $(\Gamma, M)_B$ stands for M is the Γ-ring in the sense of Barnes. In [7], it is shown that for all $(\Gamma, M)_B$ there exists Γ′ is an additive group such that $(\Gamma', M)_N$ Therefore, if M is Γ-ring in the sense of Barnes, then M is Γ′-ring in the sense of Nobusawa. Thus, meaningful works on Γ-ring in the sense of Nobusawa. Throughout the present paper, M will a Γ-ring in the sense of Nobusawa and the symbol (Γ, M) stands for the $(\Gamma, M)_N$.

Let (Γ, M) be a gamma ring in the sense of Nobusawa. A right (resp. left) ideal of M is an additive subgroup of U such that $U\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$). If U is both a right and left ideal of M, then we say that U is an ideal of M. An ideal P of a gamma ring (Γ, M) is said to be prime if for any ideals A and B of M, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. An ideal P of a gamma ring (Γ, M) is said to be semiprime if for any ideal U of M, $U\Gamma U \subseteq P$ implies $U \subseteq P$. A gamma ring (Γ, M) is said to be semiprime if the zero ideal is semiprime. This definition is given as "A gamma ring (Γ, M) is said to be prime if $a\Gamma b = (0)$ with $a, b \in M$, implies $a = 0$ or $b = 0$ and semiprime $a\Gamma a = (0)$ with $a \in M$, implies $a = 0$" in [4].

In [7], Kyuno defined the notation of gamma ring homomorphism as follows: Let (Γ_1, M_1) and (Γ_2, M_2) be two gamma rings, $\varphi: \Gamma_1 \to \Gamma_2$ and $\theta: M_1 \to M_2$ be two functions. Then an ordered pair (φ, θ) of mappings is called a homomorphism of (Γ_1, M_1) into (Γ_2, M_2) if it satisfies the following properties:

(i) $\theta: M_1 \to M_2$ is group homomorphism,

(ii) $\varphi: \Gamma_1 \to \Gamma_2$ is group homomorphism,

(iii) $\theta(x\alpha y) = \theta(x)\varphi(\alpha)\theta(y)$, for all $x, y \in M, \alpha \in \Gamma$,

(iv) $\varphi(\alpha x\beta) = \varphi(\alpha)\theta(x)\varphi(\beta)$, for all $x \in M, \alpha, \beta \in \Gamma$.

A homomorphism (φ, θ) of a gamma ring (Γ_1, M_1) into a gamma ring (Γ_2, M_2) is called a monomorphism if φ and θ are one-one.

We now turn our attention to the gamma module. Let (Γ, M) be a gamma ring. A commutative additive group N is called a right gamma M-module (or right gamma M-module) if for all $n, n_1, n_2 \in N, m, m_1, m_2 \in M$ and $\alpha, \beta \in \Gamma$,

(i) $n\alpha m \in N$,

(ii) $(n_1 + n_2)\alpha m = n_1\alpha m + n_2\alpha m$,

(iii) $n(\alpha + \beta)m = n\alpha m + n\beta m$,

(iv) $n\alpha(m_1 + m_2) = n\alpha m_1 + n\alpha m_2$.

Let N_1 and N_2 be two right gamma M-modules. Then θ is called a right gamma M-module homomorphism (or right gamma M-module homomorphism) of N_1 into N_2 if it satisfies the following properties:

(i) $\theta: N_1 \to N_2$ is group homomorphism,

(ii) $\theta(x\alpha m) = \theta(x)\alpha m$, for all $x \in N_1, m \in M, \alpha \in \Gamma$.

A great deal work has been done on gamma ring in the sense of Barnes and Nobusawa. The author studied the structure of gamma rings and obtained various generalizations analogous of corresponding parts in ring theory. The study of two-sided rings of quotients was initiated by W. S. Martindale [8] for prime rings and extented for semiprime rings by S. A. Amitsur in [1]. The concept of centroid of a prime gamma ring was defined and researched in [10, 11]. In [12], the authors proved that the generalized centroid of a semiprime gamma ring is a regular gamma ring. We introduced and investigated the rings of quotients of a semiprime gamma ring in [5]. In this paper, we will show that the rings of quotients of a semiprime gamma ring is a gamma ring and we shall prove several properties of gamma ring of quotients.

2. Results

Throughout the present paper, M will a Γ-ring in the sense of Nobusawa and the symbol (Γ, M) stands for the $(\Gamma, M)_N$.

Definition 2.1 Let (Γ, M) be a gamma ring. If there exists $e \in M$ and $\delta \in \Gamma$ such that $e\delta x = x$ for all $x \in M$, then (δ, e) is said to be strong left identity element of (Γ, M). Similarly, if there exists $e \in M$ and $\delta \in \Gamma$ such that $x\delta e = x$ for all $x \in M$, then (δ, e) is said to be strong right identity element of (Γ, M). If (δ, e) is both a right and left strong identity element of (Γ, M), then we say that (δ, e) is an strong identity element of (Γ, M).

Definition 2.2 Let (Γ, M) be a gamma ring. For a subset S of M,

$$r_\Gamma(S) = \{c \in M | S\gamma c = (0), \forall \gamma \in \Gamma\}$$

is called the right annihilator of S. A left annihilator $l_\Gamma(S)$ can be defined similarly.

Lemma 2.3 Let (Γ, M) be a semiprime gamma ring and

$$I(\Gamma, M) = \{U | U \text{ is an ideal of } M \text{ and } l_\Gamma(U) = (0)\}.$$

If $U, V \in I(\Gamma, M)$, then $U\Gamma V \in I(\Gamma, M)$.

Proof. Let $U, V \in I(\Gamma, M)$. Thus U,V are ideals of M and $l_\Gamma(U) = l_\Gamma(V) = (0)$. Clearly, $U\Gamma V$ is an ideal of M. We will show that $l_\Gamma(U\Gamma V) = (0)$. Let x be any element of $l_\Gamma(U\Gamma V)$. That is $x\gamma(U\Gamma V) = (0)$, for all $\gamma \in \Gamma$, and so $(x\gamma U)\Gamma V = (0)$, for all $\gamma \in \Gamma$. Since $l_\Gamma(V) = (0)$, we have $x\gamma U = (0)$, for all $\gamma \in \Gamma$. Again using $l_\Gamma(U) = (0)$, we get $x = 0$. Hence $U\Gamma V \in I(\Gamma, M)$. This completes proof.

Let (Γ, M) be a semiprime gamma ring. Consider the set

$$\mathcal{L} = \{(f;U) | U \in I(\Gamma,M), f: U \to M \text{ is a right } \Gamma\ M - \text{module homomorphism}\}.$$

We define "$(f;U) \simeq (g;V) \Leftrightarrow$ There exists $W \in I(\Gamma, M)$ such that $W \subseteq U \cap V$ and $f = g$ on W." We can readily check that "$\simeq$" is an equivalence relation. We let $\{f;U\}$ denote the equivalence class determined by $(f;U) \in \mathcal{L}$. Let Q_r be the set of all equivalence classes. We now define addition of Q_r as follow:

$$\{f;U\} + \{g;V\} = \{f + g; V\Gamma U\}, \text{ for all } \{f;U\}, \{g;V\} \in Q_r.$$

We will show that addition is well defined. By Lemma 2.3, we see that $V\Gamma U \in I(\Gamma, M)$. For all $\{f_1;U_1\}, \{f_2;U_2\}, \{g_1;V_1\}, \{g_2;V_2\} \in Q_r$, we have

$$(\{f_1;U_1\}, \{g_1;V_1\}) = (\{f_2;U_2\}, \{g_2;V_2\}).$$

That is,

$$\{f_1;U_1\} = \{f_2;U_2\} \text{ and } \{g_1;V_1\} = \{g_2;V_2\}$$

i.e.,

$$(f_1;U_1) \simeq (f_2;U_2) \text{ and } (g_1;V_1) \simeq (g_2;V_2).$$

Hence there exists $W_1, W_2 \in I(\Gamma, M)$ such that $W_1 \subseteq U_1 \cap U_2, W_2 \subseteq V_1 \cap V_2$ and $f_1 = f_2$ on $W_1, g_1 = g_2 \text{ on } W_2$.

Setting $W = W_2\Gamma W_1$. By Lemma 2.3, we have $W \in I(\Gamma, M)$. We will show that $W \subseteq V_1\Gamma U_1 \cap V_2\Gamma U_2$. Let w be any element of W. Then $w = w_2\gamma w_1$, where $w_2 \in W_2, \gamma \in \Gamma, w_1 \in W_1$. That is $w_2 \in W_2 \subseteq V_1 \cap V_2$ and $w_1 \in W_1 \subseteq U_1 \cap U_2$. Thus $w \in V_1\Gamma U_1 \text{ and } w \in V_2\Gamma U_2$, i.e., $w \in V_1\Gamma U_1 \cap V_2\Gamma U_2$.

On the other hand, since W_1, W_2 are ideals of M, we have $w \in W = W_2 \Gamma W_1 \subseteq W_2$ and $w \in W = W_2 \Gamma W_1 \subseteq W_1$. Since $f_1 = f_2$ on W_1 and $g_1 = g_2$ on W_2, we have

$$(f_1 + g_1)(w) = f_1(w) + g_1(w) = f_2(w) + g_2(w) = (f_2 + g_2)(w).$$

That is $f_1 + g_1 = f_2 + g_2$ on W, and so

$$\{f_1 + g_1; V_1 \Gamma U_1\} = \{f_2 + g_2; V_2 \Gamma U_2\}.$$

We will prove that Q_r is abelian additive group.

i) For all $\{f;U\}, \{g;V\}, \{h;W\} \in Q_r$,

$$(\{f;U\} + \{g;V\}) + \{h;W\} = \{f + g; V\Gamma U\} + \{h;W\} = \{(f+g) + h; W\Gamma(V\Gamma U)\}$$

$$= \{f + (g+h); (W\Gamma V)\Gamma U\}$$

$$= \{f;U\} + \{g + h; W\Gamma V\}$$

$$= \{f;U\} + (\{g;V\} + \{h;W\}).$$

ii) Now, let $0: M \to M, 0(x) = 0$. First of all we note that $\{0; M\} \in Q_r$. Indeed, clearly M is an ideal of M. For $x \in l_\Gamma(M)$, we have $x\gamma M = (0)$, for all $\gamma \in \Gamma$. That is $x\gamma m = 0$, for all $\gamma \in \Gamma, m \in M$. Replacing m by x in last equation, we have $x\gamma x = 0$, for all $\gamma \in \Gamma$ and so, $x\Gamma x = (0)$. By the semiprimeness of (Γ, M), we have $x = 0$, i.e. l_Γ (M)=(0). Consequently, $M \in I(\Gamma, M)$.

One easily checks that $0: M \to M$ is a right gamma M-module homomorphism. Hence $\{0; M\} \in Q_r$. We will prove that,

$$\{f;U\} + \{0;M\} = \{f + 0; M\Gamma U\} = \{f;U\}.$$

Indeed, let $W = M\Gamma U$. Clearly, $W \subseteq M\Gamma U \cap U$. Then

$$(f + 0)(w) = f(w) + 0(w) = f(w), for\ all\ w \in W.$$

That is $f + 0 = f$ on W. Moreover, by Lemma 2.3, we have $W \in I(\Gamma, M)$. In similar fasion,

$$\{0; M\} + \{f; U\} = \{f; U\}.$$

Hence $\{0; M\}$ is the identity element of Q_r.

iii) For any $\{f; U\} \in Q_r$, we will show that $\{-f; U\} \in Q_r$. One easily checks that $(-f): U \to M$ is a right Γ M-module homomorphism and $U \in I(\Gamma, M)$. Also

$$\{f; U\} + \{-f; U\} = \{f + (-f); U\Gamma U\} = \{0; M\}.$$

Indeed, let $W = U\Gamma U \ and \ w \in W$. We get

$$(f + (-f))(w) = f(w) + (-f)(w) = f(w) - f(w) = 0 = 0(w).$$

Moreover, $W \subseteq U\Gamma U \cap M$ and $W \in I(\Gamma, M)$. Hence $\{-f; U\}$ is the inverse of $\{f; U\}$.

iv) For all $\{f; U\}, \{g; V\} \in Q_r$,

$$\{f; U\} + \{g; V\} = \{f + g; V\Gamma U\} \qquad (2.1)$$

and

$$\{g; V\} + \{f; U\} = \{g + f; U\Gamma V\}. \qquad (2.2)$$

We will prove that (2.1) and (2.2) are equal. Let $W = U\Gamma V\Gamma V\Gamma U$ and $w \in W$. Since

$$(f + g)(w) = f(w) + g(w) = g(w) + f(w) = (g + f)(w),$$

we have $f + g = g + f \ on \ W$.

Moreover, clearly $W \subseteq U\Gamma V\Gamma V\Gamma U$. By Lemma 2.3, we have $W \in I(\Gamma, M)$. Hence Q_r is an abelian group.

In the same way, let (M, Γ) be a semiprime gamma ring,

$$I(M, \Gamma) = \{\Omega | \Omega \ is \ an \ ideal \ of \ \Gamma \ and \ r_M(\Omega) = (0)\}$$

and

$$\mathcal{F} = \{(\tau; \Omega) | \Omega \in I(M, \Gamma), \tau: \Omega \to \Gamma \ is \ a \ left \ M \ \Gamma - module \ homomorphism\}.$$

Lemma 2.4 Let (M, Γ) be a semiprime gamma ring. If $\Omega, \Lambda \in I(M, \Gamma)$, then $\Omega M \Lambda \in I(M, \Gamma)$.

Proof. The procedures in Lemma 2.3 can be exactly applied in set $I(M,\Gamma)$ and the same results are obtained. This completes proof.

We define

$$"(\tau;\Omega) \cong (\sigma;\Lambda) \Leftrightarrow \textit{There exists } \Pi \in I(M,\Gamma) \textit{ such that } \Pi \subseteq \Omega \cap \Lambda \textit{ and } \tau = \sigma \textit{ on } \Pi."$$

Clearly, " $\cong$ " is an equivalence relation. Let $\{\tau;\Omega\}$ denote the equivalence class determined by $(\tau;\Omega) \in \mathcal{F}$ and Δ_l denote the set of all equivalence classes. We then define addition of Δ_l as follow:

$$\{\tau;\Omega\} + \{\sigma;\Lambda\} = \{\tau + \sigma; \Lambda M \Omega\}.$$

We will prove that addition is well defined. By Lemma 2.4, we have $\Lambda M \Omega \in I(M,\Gamma)$. For all $\{\tau_1;\Omega_1\}, \{\tau_2;\Omega_2\}, \{\sigma_1;\Lambda_1\}, \{\sigma_2;\Lambda_2\} \in \Delta_l$, we get

$$(\{\tau_1;\Omega_1\}, \{\sigma_1;\Lambda_1\}) = (\{\tau_2;\Omega_2\}, \{\sigma_2;\Lambda_2\})$$

and so

$$\{\tau_1;\Omega_1\} = \{\tau_2;\Omega_2\} \textit{ and } \{\sigma_1;\Lambda_1\} = \{\sigma_2;\Lambda_2\}.$$

That is

$$(\tau_1;\Omega_1) \cong (\tau_2;\Omega_2) \textit{ and } (\sigma_1;\Lambda_1) \cong (\sigma_2;\Lambda_2).$$

Hence there exists $\Pi_1, \Pi_2 \in I(M,\Gamma)$ such that $\Pi_1 \subseteq \Omega_1 \cap \Omega_2, \Pi_2 \subseteq \Lambda_1 \cap \Lambda_2$ and $\tau_1 = \tau_2$ on Π_1, $\sigma_1 = \sigma_2$ on Π_2. Setting $\Pi = \Pi_2 M \Pi_1$. By Lemma 2.4, we have $\Pi \in I(M,\Gamma)$. We will show that $\Pi \subseteq \Lambda_1 M \Omega_1 \cap \Lambda_2 M \Omega_2$. Let α be any element of Π. Then $\alpha = \pi_2 m \pi_1$, where $\pi_2 \in \Pi_2, m \in M, \pi_1 \in \Pi_1$. That is, $\pi_2 \in \Pi_2 \subseteq \Lambda_1 \cap \Lambda_2$ and $\pi_1 \in \Pi_1 \subseteq \Omega_1 \cap \Omega_2$. We conclude that $\alpha \in \Lambda_1 M \Omega_1$ and $\alpha \in \Lambda_2 M \Omega_2$, i.e., $\alpha \in \Lambda_1 M \Omega_1 \cap \Lambda_2 M \Omega_2$.

Let α be any element of Π. Using Π_1, Π_2 are ideals of Γ, we have $\alpha \in \Pi = \Pi_2 M \Pi_1 \subseteq \Pi_2$ and $\alpha \in \Pi = \Pi_2 M \Pi_1 \subseteq \Pi_1$. Thus

$$(\tau_1 + \sigma_1)(\alpha) = \tau_1(\alpha) + \sigma_1(\alpha) = \tau_2(\alpha) + \sigma_2(\alpha) = (\tau_2 + \sigma_2)(\alpha).$$

Then $\tau_1 + \sigma_1 = \tau_2 + \sigma_2$ on Π. That is

$$\{\tau_1 + \sigma_1; \Lambda_1 M \Omega_1\} = \{\tau_2 + \sigma_2; \Lambda_2 M \Omega_2\}.$$

We will prove that Δ_l is abelian additive group.

i) For all $\{\tau;\Omega\},\{\sigma;\Lambda\},\{\delta;\Sigma\} \in \Delta_l$,

$$\begin{aligned}(\{\tau;\Omega\}+\{\sigma;\Lambda\})+\{\delta;\Sigma\} &= \{\tau+\sigma;\Lambda M\Omega\}+\{\delta;\Sigma\} = \{(\tau+\sigma)+\delta;\Sigma M(\Lambda M\Omega)\} \\ &= \{\tau+(\sigma+\delta);(\Sigma M\Lambda)M\Omega\} = \{\tau;\Omega\}+\{\sigma+\delta;\Sigma M\Lambda\} \\ &= \{\tau;\Omega\}+(\{\sigma;\Lambda\}+\{\delta;\Sigma\}).\end{aligned}$$

ii) Let $0:\Gamma\to\Gamma, 0(\gamma)=0$. We will show that $\{0;\Gamma\}\in\Delta_l$. Indeed, clearly Γ is an ideal of Γ. For $\alpha\in r_M(\Gamma)$, we have $\Gamma m\alpha=(0)$, for all $m\in M$. Thus $\gamma m\alpha=0$, for all $\gamma\in\Gamma, m\in M$. Replacing γ by α in last equation, we get $\alpha m\alpha=0$, for all $m\in M$. That is $\alpha M\alpha=(0)$. By the semiprimeness of (M,Γ), we obtain that $\alpha=0$, i.e., $r_M(\Gamma)=(0)$. Consequently, $\Gamma\in I(M,\Gamma)$.

One easily checks that $0:\Gamma\to\Gamma$ is a left M Γ-module homomorphism. Therefore $\{0;\Gamma\}\in\Delta_l$. Also,

$$\{\tau;\Omega\}+\{0;\Gamma\}=\{\tau+0;\Gamma M\Omega\}=\{\tau;\Omega\}.$$

Indeed, let $\Pi=\Gamma M\Omega$. Clearly, $\Pi\subseteq\Gamma M\Omega\cap\Omega$. Hence

$$(\tau+0)(\alpha)=\tau(\alpha)+0(\alpha)=\tau(\alpha), for\ all\ \alpha\in\Pi$$

and so, $\tau+0=\tau\ on\ \Pi$. Also, by Lemma 2.4, we have $\Pi\in I(M,\Gamma)$. In similar fasion,

$$\{0;\Gamma\}+\{\tau;\Omega\}=\{\tau;\Omega\}.$$

We arrive at $\{0;\Gamma\}$ is the identity element of Δ_l.

iii) For any $\{\tau;\Omega\}\in\Delta_l$, we will prove that $\{-\tau;\Omega\}\in\Delta_l$. One easily checks that $(-\tau):\Omega\to\Gamma$ is a left M Γ-module homomorphism and $\Omega\in I(M,\Gamma)$. Also

$$\{\tau;\Omega\}+\{-\tau;\Omega\}=\{\tau+(-\tau);\Omega M\Omega\}=\{0;\Gamma\}.$$

Indeed, let $\Pi=\Omega M\Omega\ and\ \alpha\in\Pi$. We get

$$(\tau+(-\tau))(\alpha)=\tau(\alpha)+(-\tau)(\alpha)=\tau(\alpha)-\tau(\alpha)=0=0(\alpha).$$

Also, $\Pi \subseteq \Omega M\Omega \cap \Gamma$ and $\Pi \in I(M,\Gamma)$. Hence $\{-\tau;\Omega\}$ is the inverse of $\{\tau;\Omega\}$.

iv) For all $\{\tau;\Omega\},\{\sigma;\Lambda\} \in \Delta_l$,

$$\{\tau;\Omega\} + \{\sigma;\Lambda\} = \{\tau + \sigma; \Lambda M\Omega\} \tag{2.3}$$

and

$$\{\sigma;\Lambda\} + \{\tau;\Omega\} = \{\sigma + \tau; \Omega M\Lambda\}. \tag{2.4}$$

We will prove that (2.3) and (2.4) are equal. Let $\Pi = \Lambda M\Omega M\Omega M\Lambda$ and $\alpha \in \Pi$. We get

$$(\tau+\sigma)(\alpha) = \tau(\alpha) + \sigma(\alpha) = \sigma(\alpha) + \tau(\alpha) = (\sigma+\tau)(\alpha)$$

and so, $\tau + \sigma = \sigma + \tau \; on \; \Pi$.

Moreover, clearly $\Pi \subseteq \Lambda M\Omega M\Omega M\Lambda$. By Lemma 2.4, we get $\Pi \in I(M,\Gamma)$. Hence Δ_l is abelian additive group.

Let $\tau: \Omega \to \Gamma$ be a left M Γ-module homomorphism. Define $\hat{\tau}: M\Omega M \to M$ defined by $\hat{\tau}(m\gamma n) = m\tau(\gamma)n \; for \; all \; m,n \in M, \gamma \in \Omega$.

We now define multiplication of equivalence classes as follow:

$$\{f;U\}\{\tau;\Omega\}\{g;V\} = \{f\hat{\tau}g; V\Omega U\}, for\; all\; \{f;U\},\{g;V\} \in Q_r, \{\tau;\Omega\} \in \Delta_l.$$

We first prove that multiplication is well defined. Let $\{f_1;U_1\},\{f_2;U_2\},\{g_1;V_1\},\{g_2;V_2\} \in Q_r$ and $\{\tau_1;\Omega_1\},\{\tau_2;\Omega_2\} \in \Delta_l$. Suppose

$$(\{f_1;U_1\},\{\tau_1;\Omega_1\},\{g_1;V_1\}) = (\{f_2;U_2\},\{\tau_2;\Omega_2\},\{g_2;V_2\}).$$

We get

$$\{f_1;U_1\} = \{f_2;U_2\}, \{\tau_1;\Omega_1\} = \{\tau_2;\Omega_2\}, \{g_1;V_1\} = \{g_2;V_2\}.$$

Then

$$(f_1;U_1) \simeq (f_2;U_2), (\tau_1;\Omega_1) \cong (\tau_2;\Omega_2), (g_1;V_1) \simeq (g_2;V_2),$$

i.e., there exists $W_1, W_2 \in I(\Gamma, M)$ such that $W_1 \subseteq U_1 \cap U_2, W_2 \subseteq V_1 \cap V_2, f_1 = f_2$ on W_1, $g_1 = g_2$ on W_2 and $\Pi \in I(M,\Gamma)$ such that $\Pi \subseteq \Omega_1 \cap \Omega_2, \tau_1 = \tau_2$ on Π.

Let $W = W_2 \Pi W_1$. We will show that $W \in I(\Gamma, M)$. Clearly, W is an ideal of M. If $x \in l_\Gamma(W), then\ x\gamma W = (0)$, for all $\gamma \in \Gamma$, i.e., $x\gamma(W_2 \Pi W_1) = (0)$, for all $\gamma \in \Gamma$. Since Π is an ideal of Γ, we get

$$(x\gamma W_2)\Pi M\Gamma W_1 \subseteq x\gamma(W_2 \Pi W_1) = (0).$$

That is

$$(x\gamma W_2)\Pi M\Gamma W_1 = (0).$$

By $l_\Gamma(W_1) = (0)$, we have

$$(x\gamma W_2)\Pi M = (0), for\ all\ \gamma \in \Gamma.$$

Again, since Π is an ideal of Γ, we obtain that

$$(x\gamma W_2)\Gamma M\Pi M \subseteq (x\gamma W_2)\Pi M = (0)$$

and so

$$(x\gamma W_2)\Gamma M\Pi M = (0), for\ all\ \gamma \in \Gamma.$$

Using the last equation, we see that

$$M\Pi M\Gamma(x\gamma W_2)\Gamma M\Pi M\Gamma(x\gamma W_2) = (0).$$

Since (Γ, M) is semiprime gamma ring, we get

$$M\Pi M\Gamma(x\gamma W_2) = (0), for\ all\ \gamma \in \Gamma.$$

Since W_2 is an ideal of M, we find that

$$M\Pi M\Gamma x\gamma W_2\Gamma M \subseteq M\Pi M\Gamma(x\gamma W_2) = (0).$$

That is

$$M\Pi M\Gamma x\gamma W_2\Gamma M = (0), for\ all\ \gamma \in \Gamma.$$

Since (Γ, M) is gamma ring, we get

$$\Pi M\Gamma x\gamma W_2\Gamma = (0), for\ all\ \gamma \in \Gamma.$$

By $r_M(\Pi) = (0)$, we have

$$\Gamma x\gamma W_2\Gamma = (0), for\ all\ \gamma \in \Gamma.$$

Using the above equation, we see that

$$(x\gamma W_2\Gamma x\gamma W_2)\Gamma(x\gamma W_2\Gamma x\gamma W_2) = (0).$$

Again using the semiprimenessly of (Γ, M), we get $x\gamma W_2 = (0)$, for all $\gamma \in \Gamma$. Then $x \in l_\Gamma(W_2)$, and so $x = 0$ by $l_\Gamma(W_2) = (0)$. Thus we must have $l_\Gamma(W) = (0)$, and so $W \in I(\Gamma, M)$.

Now we show that $W \subseteq V_1\Omega_1U_1 \cap V_2\Omega_2U_2$. Let w be any element of W. Thus $w = w_2\gamma w_1$, where $w_2 \in W_2, \gamma \in \Pi, w_1 \in W_1$. Therefore, we get $w_2 \in W_2 \subseteq V_1 \cap V_2$, $\gamma \in \Pi \subseteq \Omega_1 \cap \Omega_2$ and $w_1 \in W_1 \subseteq U_1 \cap U_2$. Hence $w \in V_1\Omega_1U_1$ and $w \in V_2\Omega_2U_2$, i.e., $w \in V_1\Omega_1U_1 \cap V_2\Omega_2U_2$.

On the other hand, we will show that $f_1\widehat{\tau_1}g_1 = f_2\widehat{\tau_2}g_2$ on W. For any $w \in W$, taking w by $w_2\gamma w_1$ where $w_2 \in W_2, \gamma \in \Pi, w_1 \in W_1$, we have

$$(f_1\widehat{\tau_1}g_1)(w) = (f_1\widehat{\tau_1}g_1)(w_2\gamma w_1) = f_1(\widehat{\tau_1}(g_1(w_2\gamma w_1))) = f_1(\widehat{\tau_1}(g_1(w_2)\gamma w_1)).$$

Since $g_1 = g_2\ on\ W_2$, we find that

$$f_1(\widehat{\tau_1}(g_1(w_2)\gamma w_1)) = f_1(\widehat{\tau_1}(g_2(w_2)\gamma w_1)) = f_1(g_2(w_2)\tau_1(\gamma)w_1).$$

Using $\tau_1 = \tau_2$ on Π, we obtain that

$$f_1(g_2(w_2)\tau_1(\gamma)w_1) = f_1(g_2(w_2)\tau_2(\gamma)w_1).$$

Since $g_2(w_2)\tau_2(\gamma)w_1 \in W_1$ and $f_1 = f_2$ on W_1, we have

$$f_1(g_2(w_2)\tau_2(\gamma)w_1) = f_2(g_2(w_2)\tau_2(\gamma)w_1) = f_2(\widehat{\tau_2}(g_2(w_2)\gamma w_1)) = (f_2\widehat{\tau_2}g_2)(w).$$

This implies that $f_1\widehat{\tau_1}g_1 = f_2\widehat{\tau_2}g_2$ on W. Hence

$$\{f_1\widehat{\tau_1}g_1; V_1\Omega_1U_1\} = \{f_2\widehat{\tau_2}g_2; V_2\Omega_2U_2\}$$

and so, the multiplication is well defined.

We now show that (Δ_l, Q_r) is a gamma ring.

a) i) For all $\{f; U\}, \{g; V\}, \{h; K\} \in Q_r\ and\ \{\tau; \Omega\} \in \Delta_l$,

$$(\{f;U\}+\{g;V\})\{\tau;\Omega\}\{h;K\} = \{f+g;V\Gamma U\}\{\tau;\Omega\}\{h;K\}$$

$$= \{(f+g)\hat{\tau}h;K\Omega(V\Gamma U)\}. \qquad (2.5)$$

Also,

$$\{f;U\}\{\tau;\Omega\}\{h;K\}+\{g;V\}\{\tau;\Omega\}\{h;K\} = \{f\hat{\tau}h;K\Omega U\}+\{g\hat{\tau}h;K\Omega V\}$$

$$= \{f\hat{\tau}h+g\hat{\tau}h;(K\Omega V)\Gamma(K\Omega U)\}. \quad (2.6)$$

We will prove that (2.5) and (2.6) are equal. Choosing $W=(K\Omega V)\Gamma(K\Omega U)$. Clearly, $W\in I(\Gamma,M)$ and $W\subseteq((K\Omega V)\Gamma(K\Omega U))\cap K\Omega(V\Gamma U)$. Moreover, we get

$$((f+g)\hat{\tau}h)(w)=(f\hat{\tau}h+g\hat{\tau}h)(w), for\ all\ w\in W.$$

ii) For all $\{f;U\},\{g;V\}\in Q_r and$ $\{\tau;\Omega\},\{\sigma;\Lambda\}\in\Delta_l$,

$$\{f;U\}(\{\tau;\Omega\}+\{\sigma;\Lambda\})\{g;V\}=\{f;U\}\{\tau+\sigma;\Lambda M\Omega\}\{g;V\}$$

$$= \{f(\widehat{\tau+\sigma})g;V(\Lambda M\Omega)U\} \qquad (2.7)$$

and

$$\{f;U\}\{\tau;\Omega\}\{g;V\}+\{f;U\}\{\sigma;\Lambda\}\{g;V\}=\{f\tau g;V\Omega U\}+\{f\sigma g;V\Lambda U\}$$

$$= \{f\hat{\tau}g+f\hat{\sigma}g;(V\Lambda U)\Gamma(V\Omega U)\}. \qquad (2.8)$$

We will show that (2.7) and (2.8) are equal. Let $W=(V\Lambda U)\Gamma(V\Omega U)$. By Lemma 2.3, we get $W\in I(\Gamma,M)$. Since

$$W=(V\Lambda U)\Gamma(V\Omega U)=V\Lambda(U\Gamma V)\Omega U\subseteq V(\Lambda M\Omega)U,$$

we have

$$W\subseteq V(\Lambda M\Omega)U\cap(V\Lambda U)\Gamma(V\Omega U).$$

Also, we get $f(\widehat{\tau+\sigma})g=f\hat{\tau}g+f\hat{\sigma}g\ on\ W.$

iii) For all $\{f;U\},\{g;V\},\{h;K\}\in Q_r and$ $\{\tau;\Omega\}\in\Delta_l$,

$$\{f;U\}\{\tau;\Omega\}(\{g;V\}+\{h;K\})=\{f;U\}\{\tau;\Omega\}\{g+h;K\Gamma V\}$$

$$= \{f\hat{\tau}(g+h);(K\Gamma V)\Omega U\}$$

$$= \{f\hat{\tau}g + f\hat{\tau}h; (K\Omega U)\Gamma(V\Omega U)\}$$

$$= \{f\hat{\tau}g; V\Omega U\} + \{f\hat{\tau}h; K\Omega U\}$$

$$= \{f; U\}\{\tau; \Omega\}\{g; V\} + \{f; U\}\{\tau; \Omega\}\{h; K\}.$$

Setting $W = (K\Omega U)\Gamma(V\Omega U)$. Using the same arguments in the proof (ii), we find the required result.

b) i) For all $\{f; U\}, \{g; V\}, \{h; K\} \in Q_r$ and $\{\tau; \Omega\}, \{\sigma; \Lambda\} \in \Delta_l$,

$$(\{f; U\}\{\tau; \Omega\}\{g; V\})\{\sigma; \Lambda\}\{h; K\} = \{f\hat{\tau}g; V\Omega U\}\{\sigma; \Lambda\}\{h; K\}$$

$$= \{(f\hat{\tau}g)\hat{\sigma}h; K\Lambda(V\Omega U)\}$$

$$= \{f\hat{\tau}(g\hat{\sigma}h); (K\Lambda V)\Omega U\}$$

$$= \{f; U\}\{\tau; \Omega\}\{g\hat{\sigma}h; K\Lambda V\}$$

$$= \{f; U\}\{\tau; \Omega\}(\{g; V\}\{\sigma; \Lambda\}\{h; K\}).$$

ii) For all $\{f; U\}, \{g; V\}, \{h; K\} \in Q_r$ and $\{\tau; \Omega\}, \{\sigma; \Lambda\} \in \Delta_l$,

$$(\{f; U\}\{\tau; \Omega\}\{g; V\})\{\sigma; \Lambda\}\{h; K\} = \{f\hat{\tau}g; V\Omega U\}\{\sigma; \Lambda\}\{h; K\}$$

$$= \{(f\hat{\tau}g)\hat{\sigma}h; K\Lambda(V\Omega U)\}$$

$$= \{f(\hat{\tau}g\hat{\sigma})h; K(\Lambda V\Omega)U\}$$

$$= \{f; U\}\{\hat{\tau}g\hat{\sigma}; \Lambda V\Omega\}\{h; K\}$$

$$= \{f; U\}(\{\tau; \Omega\}\{g; V\}\{\sigma; \Lambda\})\{h; K\}.$$

c) Let $\{f; U\}\{\tau; \Omega\}\{g; V\} = \{0; M\}$ for all $\{f; U\}, \{g; V\} \in Q_r$ and $\{\tau; \Omega\} \in \Delta_l$. Replacing $\{f; U\}, \{g; V\}$ by $\{I_M; M\}$ where $I_M: M \to M$ is an identity right Γ M-module homomorphism, we obtain that

$$\{I_M; M\}\{\tau; \Omega\}\{I_M; M\} = \{0; M\}.$$

That is

$$\{I_M\hat{\tau}I_M; M\Omega M\} = \{0; M\},$$

i.e.,

$$(I_M \hat{\tau} I_M ; M\Omega M) \simeq (0; M).$$

Hence, there exists $W \in I(\Gamma, M)$ such that $W \subseteq (M\Omega M) \cap M$ and $I_M \hat{\tau} I_M = 0$ on W, i.e., $I_M \hat{\tau} I_M (w) = 0(w)$, for all $w \in W$. Since $W \subseteq M\Omega M$, we get $w \in W \subseteq M\Omega M$.

Thus, $w = m_1 \gamma m_2 \in M\Omega M, where\ m_1, m_2 \in M, \gamma \in \Omega$. We have

$$I_M \hat{\tau} I_M (w) = I_M \hat{\tau} I_M (m_1 \gamma m_2) = 0(m_1 \gamma m_2)$$

and so

$$m_1 \tau(\gamma) m_2 = 0, for\ all\ m_1, m_2 \in M\ and\ \gamma \in \Omega.$$

Using (Γ, M) is gamma ring, we get $\tau(\gamma) = 0, for\ all\ \gamma \in \Omega, i.e., \tau = 0$. Hence we have $\{\tau; \Omega\} = \{0; \Gamma\}$.

Thus, we shown that (Δ_l, Q_r) is a gamma ring. We shall denote the gamma ring costructed above by (Δ_l, Q_r) and we call the two sided right gamma ring of quotients of (Γ, M).

Similarly, using the following operations, the two sided left gamma ring of quotients of (M, Γ) may be defined:

$$\{\tau; \Omega\}\{f; U\}\{\sigma; \Lambda\} = \{\tau \hat{f} \sigma; \Lambda U \Omega\}$$

where $\hat{f}: \Gamma U \Gamma \to \Gamma, \hat{f}(\gamma m \beta) = \gamma f(m) \beta$.

We will prove that multiplication is well defined. For all $\{\tau_1; \Omega_1\}, \{\tau_2; \Omega_2\}, \{\sigma_1; \Lambda_1\}, \{\sigma_2; \Lambda_2\} \in \Delta_l$ and$\{f_1; U_1\}, \{f_2; U_2\} \in Q_r$, we get

$$(\{\tau_1; \Omega_1\}, \{f_1; U_1\}, \{\sigma_1; \Lambda_1\}) = (\{\tau_2; \Omega_2\}, \{f_2; U_2\}, \{\sigma_2; \Lambda_2\}).$$

This implies that

$$\{\tau_1; \Omega_1\} = \{\tau_2; \Omega_2\}, \{f_1; U_1\} = \{f_2; U_2\}, \{\sigma_1; \Lambda_1\} = \{\sigma_2; \Lambda_2\},$$

and so

$$(\tau_1; \Omega_1) \cong (\tau_2; \Omega_2), (f_1; U_1) \simeq (f_2; U_2), (\sigma_1; \Lambda_1) \cong (\sigma_2; \Lambda_2).$$

Hence there exists $\Pi_1, \Pi_2 \in I(M,\Gamma)$ such that $\Pi_1 \subseteq \Omega_1 \cap \Omega_2, \Pi_2 \subseteq \Lambda_1 \cap \Lambda_2$ and $\tau_1 = \tau_2$ on $\Pi_1, \sigma_1 = \sigma_2$ on Π_2 and $W \in I(\Gamma, M)$ such that $W \subseteq U_1 \cap U_2, f_1 = f_2$ on W.

Setting $\Pi = \Pi_2 W \Pi_1$. We will prove that $\Pi \in I(M,\Gamma)$. Clearly, Π is an ideal of Γ. If $\alpha \in r_M(\Pi)$, then $\Pi m\alpha = (0)$, for all $m \in M$. That is $(\Pi_2 W \Pi_1)m\alpha = (0)$, for all $m \in M$. Since W is an ideal of M, we have

$$\Pi_2 M\Gamma W(\Pi_1 m\alpha) \subseteq (\Pi_2 W \Pi_1)m\alpha = (0),$$

and so

$$\Pi_2 M\Gamma W(\Pi_1 m\alpha) = (0), for\ all\ m \in M.$$

Using $r_M(\Pi_2) = (0)$, we get

$$\Gamma W(\Pi_1 m\alpha) = (0), for\ all\ m \in M.$$

Again, since W is an ideal of M, we see that

$$\Gamma W\Gamma M(\Pi_1 m\alpha) \subseteq \Gamma W(\Pi_1 m\alpha) = (0).$$

That is

$$\Gamma W\Gamma M(\Pi_1 m\alpha) = (0), for\ all\ m \in M.$$

By the above equation, we find that

$$(\Pi_1 m\alpha)M\Gamma W\Gamma M(\Pi_1 m\alpha)M\Gamma W\Gamma = (0).$$

Since (M,Γ) is semiprime gamma ring, we get

$$(\Pi_1 m\alpha)M\Gamma W\Gamma = (0), for\ all\ m \in M.$$

Since Π_1 is an ideal of Γ, we have

$$\Gamma M(\Pi_1 m\alpha)M\Gamma W\Gamma \subseteq (\Pi_1 m\alpha)M\Gamma W\Gamma = (0),$$

i.e.,

$$\Gamma M(\Pi_1 m\alpha)M\Gamma W\Gamma = (0), for\ all\ m \in M.$$

Using (M,Γ) is gamma ring, we obtain

$$M(\Pi_1 m\alpha)M\Gamma W = (0), for\ all\ m \in M.$$

By $l_\Gamma(W) = (0)$, we see that

$$M(\Pi_2 m\alpha)M = (0), for\ all\ m \in M.$$

By the last equation, we see that

$$(\Pi_1 m\alpha M \Pi_1 m\alpha)M(\Pi_1 m\alpha M \Pi_1 m\alpha) = (0), for\ all\ m \in M.$$

Again using the semiprimenessly of (M,Γ), we arrive at $\Pi_1 m\alpha = (0)$, for all $m \in M$. Then $\alpha \in r_M(\Pi_1)$ and so, α=0 by $r_M(\Pi_1) = (0)$. Thus we must have $r_M(\Pi) = (0)$ and so, $\Pi \in I(M,\Gamma)$.

We show that $\Pi \subseteq \Lambda_1 U_1 \Omega_1 \cap \Lambda_2 U_2 \Omega_2$. Let α be any element of Π. That is, $\alpha = \pi_2 w \pi_1$, where $\pi_1 \in \Pi_1, w \in W, \pi_2 \in \Pi_2$. Hence we have $\pi_1 \in \Pi_1 \subseteq \Omega_1 \cap \Omega_2, \pi_2 \in \Pi_2 \subseteq \Lambda_1 \cap \Lambda_2$ and $w \in W \subseteq U_1 \cap U_2$. Then $\alpha \in \Lambda_1 U_1 \Omega_1\ and\ \alpha \in \Lambda_2 U_2 \Omega_2, and\ so\ \alpha \in \Lambda_1 U_1 \Omega_1 \cap \Lambda_2 U_2 \Omega_2$.

We will prove that $\tau_1 \widehat{f_1} \sigma_1 = \tau_2 \widehat{f_2} \sigma_2\ on\ \Pi$. Let $\alpha \in \Pi$. Replacing α by $\pi_2 w \pi_1$, where $\pi_1 \in \Pi_1, w \in W, \pi_2 \in \Pi_2$. This implies that

$$(\tau_1 \widehat{f_1} \sigma_1)(\alpha) = (\tau_1 \widehat{f_1} \sigma_1)(\pi_2 w \pi_1) = \tau_1(\widehat{f_1}(\sigma_1(\pi_2 w \pi_1))) = \tau_1(\widehat{f_1}(\sigma_1(\pi_2) w \pi_1)).$$

Since $\sigma_1 = \sigma_2\ on\ \Pi_2$, we get

$$\tau_1(\widehat{f_1}(\sigma_1(\pi_2) w \pi_1)) = \tau_1(\widehat{f_1}(\sigma_2(\pi_2) w \pi_1)) = \tau_1(\sigma_2(\pi_2) f_1(w) \pi_1).$$

Using $f_1 = f_2$ on W, we have

$$\tau_1(\sigma_2(\pi_2) f_1(w) \pi_1) = \tau_1(\sigma_2(\pi_2) f_2(w) \pi_1).$$

Since $\sigma_2(\pi_2) f_2(w) \pi_1 \in \Pi_1$ and $\tau_1 = \tau_2$ on Π_1, we have

$$\tau_1(\sigma_2(\pi_2) f_2(w) \pi_1) = \tau_2(\sigma_2(\pi_2) f_2(w) \pi_1) = \tau_2(\widehat{f_2}(\sigma_2(\pi_2) w \pi_1) = (\tau_2 \widehat{f_2} \sigma_2)(\alpha).$$

Hence $\tau_1 \widehat{f_1} \sigma_1 = \tau_2 \widehat{f_2} \sigma_2$ on Π. Therefore

$$\{\tau_1 \widehat{f_1} \sigma_1; \Lambda_1 U_1 \Omega_1\} = \{\tau_2 \widehat{f_2} \sigma_2; \Lambda_2 U_2 \Omega_2\}.$$

We conclude that the multiplication is well defined. Also, for all $\{\tau;\Omega\},\{\sigma;\Lambda\}\in\Delta_l$, $\{f;U\}\in Q_r$ we obtain that $\{\tau;\Omega\}\{f;U\}\{\sigma;\Lambda\}\in\Delta_l$.

We now show that (Q_r,Δ_l) is a gamma ring.

a) i) For all $\{f;U\}\in Q_r$ and $\{\tau;\Omega\},\{\sigma;\Lambda\},\{\delta;\Sigma\}\in\Delta_l$,

$$(\{\tau;\Omega\}+\{\sigma;\Lambda\})\{f;U\}\{\delta;\Sigma\} = \{\tau+\sigma;\Lambda M\Omega\}\{f;U\}\{\delta;\Sigma\}$$

$$=\{(\tau+\sigma)\hat{f}\delta;\Sigma U(\Lambda M\Omega)\}. \tag{2.9}$$

Also,

$$\{\tau;\Omega\}\{f;U\}\{\delta;\Sigma\}+\{\sigma;\Lambda\}\{f;U\}\{\delta;\Sigma\} = \{\tau\hat{f}\delta;\Sigma U\Omega\}+\{\sigma\hat{f}\delta;\Sigma U\Lambda\}$$

$$=\{\tau\hat{f}\delta+\sigma\hat{f}\delta;(\Sigma U\Lambda)M(\Sigma U\Omega)\}. \tag{2.10}$$

We will show that (2.9) and (2.10) are equal. Setting $\Pi=(\Sigma U\Lambda)M(\Sigma U\Omega)$. Clearly, $\Pi\in I(M,\Gamma)$ and $\Pi\subseteq((\Sigma U\Lambda)M(\Sigma U\Omega))\cap\Sigma U(\Lambda M\Omega)$. Then we have

$$((\tau+\sigma)\hat{f}\delta)(\alpha)=(\tau\hat{f}\delta+\sigma\hat{f}\delta)(\alpha), for\ all\ \alpha\in\Pi.$$

ii) For all $\{f;U\},\{g;V\}\in Q_r$ and $\{\tau;\Omega\},\{\sigma;\Lambda\}\in\Delta_l$,

$$\{\tau;\Omega\}(\{f;U\}+\{g;V\})\{\sigma;\Lambda\} = \{\tau;\Omega\}\{f+g;V\Gamma U\}\{\sigma;\Lambda\}$$

$$=\{\tau(\widehat{f+g})\sigma;\Lambda(V\Gamma U)\Omega\} \tag{2.11}$$

and

$$\{\tau;\Omega\}\{f;U\}\{\sigma;\Lambda\}+\{\tau;\Omega\}\{g;V\}\{\sigma;\Lambda\} = \{\tau\hat{f}\sigma;\Lambda U\Omega\}+\{\tau\hat{g}\sigma;\Lambda V\Omega\}$$

$$=\{\tau\hat{f}\sigma+\tau\hat{g}\sigma;(\Lambda V\Omega)M(\Lambda U\Omega)\}. \tag{2.12}$$

We will show that (2.11) and (2.12) are equal. Let $\Pi=(\Lambda V\Omega)M(\Lambda U\Omega)$. By Lemma 2.4, we get $\Pi\in I(M,\Gamma)$. Moreover

$$\Pi=(\Lambda V\Omega)M(\Lambda U\Omega)=\Lambda V(\Omega M\Lambda)U\Omega\subseteq\Lambda(V\Gamma U)\Omega.$$

Thus

$$\Pi\subseteq\Lambda(V\Gamma U)\Omega\cap(\Lambda V\Omega)M(\Lambda U\Omega).$$

Also, we have $\tau(\widehat{f+g})\sigma=\tau\hat{f}\sigma+\tau\hat{g}\sigma\ on\ \Pi.$

iii) For all $\{f;U\} \in Q_r$ and $\{\tau;\Omega\},\{\sigma;\Lambda\},\{\delta;\Sigma\} \in \Delta_l$,

$$\{\tau;\Omega\}\{f;U\}(\{\sigma;\Lambda\}+\{\delta;\Sigma\}) = \{\tau;\Omega\}\{f;U\}\{\sigma+\delta;\Sigma M\Lambda\}$$

$$= \{\tau\hat{f}(\sigma+\delta);(\Sigma M\Lambda)U\Omega\}$$

$$= \{\tau\hat{f}\sigma+\tau\hat{f}\delta;(\Sigma U\Omega)M(\Lambda U\Omega)\}$$

$$= \{\tau\hat{f}\sigma;\Lambda U\Omega\}+\{\tau\hat{f}\delta;\Sigma U\Omega\}$$

$$= \{\tau;\Omega\}\{f;U\}\{\sigma;\Lambda\}+\{\tau;\Omega\}\{f;U\}\{\delta;\Sigma\}.$$

Let $\Pi = (\Sigma U\Omega)M(\Lambda U\Omega)$. Using the same arguments in the proof (ii), we find the required result.

b) i) For all $\{f;U\},\{g;V\} \in Q_r$ and $\{\tau;\Omega\},\{\sigma;\Lambda\},\{\delta;\Sigma\} \in \Delta_l$,

$$(\{\tau;\Omega\}\{f;U\}\{\sigma;\Lambda\})\{g;V\}\{\delta;\Sigma\} = \{\tau\hat{f}\sigma;\Lambda U\Omega\}\{g;V\}\{\delta;\Sigma\}$$

$$= \{(\tau\hat{f}\sigma)\hat{g}\delta;\Sigma V(\Lambda U\Omega)\}$$

$$= \{\tau\hat{f}(\sigma\hat{g}\delta);(\Sigma V\Lambda)U\Omega\}$$

$$= \{\tau;\Omega\}\{f;U\}\{\sigma\hat{g}\delta;\Sigma V\Lambda\}$$

$$= \{\tau;\Omega\}\{f;U\}(\{\sigma;\Lambda\}\{g;V\}\{\delta;\Sigma\}).$$

ii) For all $\{f;U\},\{g;V\} \in Q_r$ and $\{\tau;\Omega\},\{\sigma;\Lambda\},\{\delta;\Sigma\} \in \Delta_l$,

$$(\{\tau;\Omega\}\{f;U\}\{\sigma;\Lambda\})\{g;V\}\{\delta;\Sigma\} = \{\tau\hat{f}\sigma;\Lambda U\Omega\}\{g;V\}\{\delta;\Sigma\}$$

$$= \{(\tau\hat{f}\sigma)\hat{g}\delta;\Sigma V(\Lambda U\Omega)\}$$

$$= \{\tau(\hat{f}\sigma\hat{g})\delta;\Sigma(V\Lambda U)\Omega\}$$

$$= \{\tau;\Omega\}\{\hat{f}\sigma\hat{g};V\Lambda U\}\{\delta;\Sigma\}$$

$$= \{\tau;\Omega\}(\{f;U\}\{\sigma;\Lambda\}\{g;V\})\{\delta;\Sigma\}.$$

c) Let $\{\tau;\Omega\}\{f;U\}\{\sigma;\Lambda\} = \{0;\Gamma\}$ for all $\{f;U\} \in Q_r$ and $\{\tau;\Omega\},\{\sigma;\Lambda\} \in \Delta_l$. Replacing $\{\tau;\Omega\},\{\sigma;\Lambda\}$ by $\{I_\Gamma;\Gamma\}$, where $I_\Gamma:\Gamma \to \Gamma$ is an identity left M Γ-module homomorphism, we get

$$\{I_\Gamma;\Gamma\}\{f;U\}\{I_\Gamma;\Gamma\}=\{0;\Gamma\},$$

and so

$$\{I_\Gamma \hat{f} I_\Gamma;\Gamma U\Gamma\}=\{0;\Gamma\}.$$

Thus

$$(I_\Gamma \hat{f} I_\Gamma;\Gamma U\Gamma)\cong(0;\Gamma).$$

Therefore, there exists $\Pi\in I(M,\Gamma)$ such that $\Pi\subseteq(\Gamma U\Gamma)\cap\Gamma$ and $I_\Gamma \hat{f} I_\Gamma=0$ on Π, i.e., $I_\Gamma \hat{f} I_\Gamma(\alpha)=0(\alpha)$, for all $\alpha\in\Pi$. By $\Pi\subseteq\Gamma U\Gamma$, we get $\alpha\in\Pi\subseteq\Gamma U\Gamma$. That is $\alpha=\gamma_1 u\gamma_2\in\Gamma U\Gamma$, where $\gamma_1,\gamma_2\in\Gamma, u\in U$. We obtain that

$$I_\Gamma \hat{f} I_\Gamma(\alpha)=I_\Gamma \hat{f} I_\Gamma(\gamma_1 u\gamma_2)=0(\gamma_1 u\gamma_2),$$

and so

$$\gamma_1 f(u)\gamma_2=0, for\ all\ \gamma_1,\gamma_2\in\Gamma, u\in U.$$

Since (M,Γ) is gamma ring, we have $f(u)=0$, for all $u\in U$, i.e., $f=0$. We conclude that $\{f;U\}=\{0;M\}$.

Hence we proved that (Q_r,Δ_l) is a gamma ring. We shall denote the gamma ring costructed above by (Q_r,Δ_l) and we call the two sided left gamma ring of quotients of (M,Γ).

In what follows, we will see several properties of two sided right gamma ring of quotients. Firstly, we observe the following important remarks.

Remark 2.5 Let (Γ,M) be a gamma ring, $\varepsilon\in\Gamma, e\in M$. If (ε,e) is the strong right identity element of (Γ,M), then (e,ε) is the strong left identity element of (M,Γ).

Proof. Assume that (ε,e) is the strong right identity element of (Γ,M). Thus we get $x\varepsilon e=x$ for all $x\in M$. For any $\gamma\in\Gamma, x,y\in M$, we have

$$x(\varepsilon e\gamma-\gamma)y=x(\varepsilon e\gamma)y-x\gamma y=(x\varepsilon e)\gamma y-x\gamma y=x\gamma y-x\gamma y=0$$

and so

$$M(\varepsilon e\gamma - \gamma)M = (0).$$

Since (Γ, M) is gamma ring, we get $\varepsilon e\gamma = \gamma, for\ all\ \gamma \in \Gamma$. This completes proof.

Using the similar arguments as above, we can prove the following remark:

Remark 2.6 Let (Γ, M) be a gamma ring, $\varepsilon \in \Gamma, e \in M$. If (ε, e) is the strong left identity element of (Γ, M), then (e, ε) is the strong right identity element of (M, Γ).

Let (ε, e) is the strong right identity element of (Γ, M). For a fixed element a in M, consider a mapping $\lambda_{a\varepsilon}: M \to M$ defined by $\lambda_{a\varepsilon}(x) = a\varepsilon x$ for all $x \in M$. It is easy to prove that the mapping $\lambda_{a\varepsilon}$ is a right Γ M-module homomorphism. For all $m \in M$,

$$\lambda_{(a+b)\varepsilon}(m) = (a+b)\varepsilon m = a\varepsilon m + b\varepsilon m = \lambda_{a\varepsilon}(m) + \lambda_{b\varepsilon}(m) = (\lambda_{a\varepsilon} + \lambda_{b\varepsilon})(m),$$

and so

$$\lambda_{(a+b)\varepsilon} = \lambda_{a\varepsilon} + \lambda_{b\varepsilon}.$$

Now, we consider a mapping $\mu_{e\beta}: \Gamma \to \Gamma$ defined by $\mu_{e\beta}(\alpha) = \alpha e\beta$ for all $\alpha \in \Gamma$. $\mu_{e\beta}$ is a left M Γ-module homomorphism. Using arguments as above, we can prove that

$$\mu_{e(\beta+\gamma)} = \mu_{e\beta} + \mu_{e\gamma}.$$

Let's define

$$\wp = \{\lambda_{a\varepsilon} | a \in M\} \ \ and \ \ \mho = \{\mu_{e\beta} | \beta \in \Gamma\}$$

$\wp$ and $\mho$ are additive groups. Defining the mappings

$$\wp \times \mho \times \wp \to \wp, (\lambda_{x\varepsilon}, \mu_{e\gamma}, \lambda_{y\varepsilon}) \mapsto \lambda_{x\varepsilon e\gamma y\varepsilon} = \lambda_{x\gamma y\varepsilon}$$

and

$$\mho \times \wp \times \mho \to \mho, (\mu_{e\gamma}, \lambda_{x\varepsilon}, \mu_{e\beta}) \mapsto \mu_{e\gamma x\varepsilon e\beta} = \mu_{e\gamma x\beta}.$$

It can be shown that $(\mho, \wp)$ is a gamma ring.

Theorem 2.7 Let (Γ, M) be a semiprime gamma ring with strong right identity element. Then (Γ, M) is a subring of (Δ_l, Q_r).

Proof. Let $\theta_\varepsilon : M \to Q_r$ and $\phi_e:\Gamma \to \Delta_l$ be as defined below.

$$\theta_\varepsilon(a) = \{\lambda_{a\varepsilon}; M\}, for\ all\ a \in M$$

and

$$\phi_\varepsilon(\beta) = \{\mu_{\varepsilon\beta}; \Gamma\}, for\ all\ \beta \in \Gamma.$$

We will prove that $(\phi_\varepsilon, \theta_\varepsilon)$ is a gamma ring monomorphism. It is clear that θ_ε is well defined. For all $a, b \in M$, we get

$$\theta_\varepsilon(a+b) = \{\lambda_{(a+b)\varepsilon}; M\} = \{\lambda_{a\varepsilon+b\varepsilon}; M\}, \quad (2.13)$$

$$\theta_\varepsilon(a) + \theta_\varepsilon(b) = \{\lambda_{a\varepsilon}; M\} + \{\lambda_{b\varepsilon}; M\} = \{\lambda_{a\varepsilon} + \lambda_{b\varepsilon}; M\Gamma M\}. \quad (2.14)$$

We show that (2.13) and (2.14) are equivalent. Setting $W = M\Gamma M$. Once easily checks that $W \subseteq M\Gamma M \cap M$. Moreover, using $\lambda_{(a+b)\varepsilon} = \lambda_{a\varepsilon} + \lambda_{b\varepsilon}$, we get

$$\lambda_{(a+b)\varepsilon} = \lambda_{a\varepsilon} + \lambda_{b\varepsilon} \text{ on W.}$$

Hence θ_ε is a group homomorphism. Also, if $\theta_\varepsilon(a) = \theta_\varepsilon(b)$, then $\{\lambda_{a\varepsilon}; M\} = \{\lambda_{b\varepsilon}; M\}$, i.e., $(\lambda_{a\varepsilon}; M) \simeq (\lambda_{b\varepsilon}; M)$. Thus there exists $W \in I(\Gamma, M)$ such that $W \subseteq M \cap M$ and $\lambda_{a\varepsilon} = \lambda_{b\varepsilon}$ on W, i.e., $a\varepsilon w = b\varepsilon w$, for all $w \in W$. That is $(a-b)\varepsilon W = (0)$. Since W is a ideal of M, we have $(a-b)\varepsilon M\Gamma W = (0)$, and so $(a-b)\varepsilon M \subset l_\Gamma(W)$. By $l_\Gamma(W) = (0)$, we get $(a-b)\varepsilon M = (0)$. That is $(a-b)\varepsilon m = 0$, for all $m \in M$. Replacing m by e, we get $(a-b)\varepsilon e = 0$. Hence $a = b$. This implies that θ_ε is one-one, and so θ_ε is a group monomorphism.

In similar fasion, we can shown that ϕ_ε is a group monomorphism. For all $\beta \in \Gamma, m, n \in M$

$$\theta_\varepsilon(m\beta n) = \left\{\lambda_{(m\beta n)\varepsilon}; M\right\}, \quad (2.15)$$

and

$$\theta_\varepsilon(m)\phi_\varepsilon(\beta)\theta_\varepsilon(n) = \{\lambda_{m\varepsilon}; M\}\{\mu_{\varepsilon\beta}; \Gamma\}\{\lambda_{n\varepsilon}; M\} = \left\{\lambda_{m\varepsilon}\widehat{\mu_{\varepsilon\beta}}\lambda_{n\varepsilon}; M\Gamma M\right\}. \quad (2.16)$$

We will show that (2.15) and (2.16) are equal. We have $W = M\Gamma M$. Clearly $W \subseteq M\Gamma M \cap M$ and $W \in I(\Gamma, M)$ by Lemma 2.3 Using $\wp \times \mho \times \wp \to \wp, (\lambda_{x\varepsilon}, \mu_{\varepsilon\gamma}, \lambda_{y\varepsilon}) \mapsto \lambda_{x\varepsilon\varepsilon\gamma y\varepsilon} = \lambda_{x\gamma y\varepsilon}$, we get

$$\lambda_{m\varepsilon}\widehat{\mu_{\varepsilon\beta}}\lambda_{n\varepsilon} = \lambda_{m\varepsilon\varepsilon\beta n\varepsilon} = \lambda_{(m\beta n)\varepsilon}$$

and so $\lambda_{m\varepsilon}\widehat{\mu_{\varepsilon\beta}}\lambda_{n\varepsilon} = \lambda_{(m\beta n)\varepsilon}$ on W. Appliying the same argument as used in the above, we see that

$$\phi_\varepsilon(\alpha m\gamma) = \phi_\varepsilon(\alpha)\theta_\varepsilon(m)\phi_\varepsilon(\gamma), for\ all\ \alpha, \gamma \in \Gamma, m \in M.$$

This implies that $(\phi_\varepsilon, \theta_\varepsilon)$ is a gamma ring monomorphism, and so (Γ, M) is a subring of (Δ_l, Q_r). This completes proof.

We now prove some properties of (Δ_l, Q_r) in the following theorem.

Theorem 2.8 Let (Γ, M) be a semiprime gamma ring with strong left identity element.

i) If $U \in I(\Gamma, M)$ and $f: U \to M$ is a right Γ M-module homomorphism, then there exists an element $q \in Q_r$ such that $f(u) = q\varepsilon u\ for\ all\ u \in U$.

ii) There exists $U \in I(\Gamma, M)$ such that $q\varepsilon U \subseteq M\ for\ all\ q \in Q_r$.

iii) Then $q\varepsilon U = (0)\ for\ all\ q \in Q_r$ and $U \in I(\Gamma, M)$ if and only if $q = 0$.

Proof.

i) Let $U \in I(\Gamma, M), f: U \to M$ be a right Γ M-module homomorphism and $q = \{f; U\}$. Since M can be embedded in Q_r, we have $u = \{\lambda_{u\varepsilon}; M\}$ such that $\lambda_{u\varepsilon}: M \to M, x \mapsto u\varepsilon x\ for\ all\ u \in U$. We get

$$q\varepsilon u = \{f; U\}\{\mu_{\varepsilon\varepsilon}; \Gamma\}\{\lambda_{u\varepsilon}; M\} = \{f\widehat{\mu_{\varepsilon\varepsilon}}\lambda_{u\varepsilon}; M\Gamma U\} \qquad (2.17)$$

And

$$f(u) = \{\lambda_{f(u)\varepsilon}; M\}. \qquad (2.18)$$

We will prove that (2.17) and (2.18) are equivalent. Choose $W = M\Gamma U$. It is a direct compution to verify that $W \in I(\Gamma, M)$ *and* $W \subseteq M \cap M\Gamma U$. For $w \in W$ and $w = m\gamma v, m \in M, \gamma \in \Gamma, v \in U$, we have

$$(f\widehat{\mu_{e\varepsilon}}\lambda_{u\varepsilon})(w) = (f\widehat{\mu_{e\varepsilon}}\lambda_{u\varepsilon})(m\gamma v) = (f\widehat{\mu_{e\varepsilon}})(u\varepsilon m\gamma v)$$

$$= f(u\varepsilon m\mu_{e\varepsilon}(\gamma)v) = f(u\varepsilon m\gamma e\varepsilon v)$$

$$= f(u\varepsilon m\gamma v) = f(u)\varepsilon m\gamma v$$

$$= f(u)\varepsilon w = \lambda_{f(u)\varepsilon}(w).$$

Thus $f\widehat{\mu_{e\varepsilon}}\lambda_{u\varepsilon} = \lambda_{f(u)\varepsilon}$ on W. Hence there exists an element $q \in Q_r$ such that $f(u) = q\varepsilon u \text{ for all } u \in U.$

ii) For any $q \in Q_r$ there exists $U \in I(\Gamma, M)$ such that $q = \{f; U\}$ and $f: U \to M$ is a right Γ M-module homomorphism. By Theorem 2.8 (i), we obtain $f(u) = q\varepsilon u$ for all $u \in U$. This shows that $q\varepsilon U \subseteq M \text{ for all } q \in Q_r$.

iii) Suppose $q\varepsilon U = (0)$ for all $q \in Q_r$ and $U \in I(\Gamma, M)$. Since $q \in Q_r$, we get $q = \{f; U\}$. We have $q\varepsilon u = 0, \text{for all } u \in U$. By Theorem 2.8 (i), $f(u) = q\varepsilon u = 0$, for all $u \in U$. That is $f(u) = 0$, for all $u \in U$. Hence $q = \{0; U\} = \{0; M\} = 0_{Q_r}$.

Conversely, let $q = 0$. Then $q = \{f; U\} = \{0; M\}$, i.e., $f(u) = 0$ for all $u \in U$. Again using Theorem 2.8 (i), we get $f(u) = q\varepsilon u = 0$, for all $u \in U$. That is $q\varepsilon U = (0)$. This completes proof.

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