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On Data Dependency and Solutions of Nonlinear Fredholm Integral Equations with the Three-Step Iteration Method

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Banach Fixed Point Theorem; Nonlinear Fredholm Integral Equations; Three-Step Iteration Method; Data Dependency.

Abstract — In this study, the solution of the second type of homogeneous nonlinear Fredholm integral equations is investigated using a three-step iteration algorithm. In other words, it has been shown that the sequences obtained from this algorithm converge to the solution of the mentioned equations. Also, data dependency is obtained for the second type of homogeneous nonlinear Fredholm integral equations and this result is supported by an example.

Subject Classification (2020): 45B05, 47H10.

1. Introduction

The widely used fixed point theory has its origins in the approximation methods of Liouville, Cauchy, Lipschitz, Peano and Picard towards the end of the 19th century (for more detail see [2],[3],[7],[9],[10]). In 1922, Stephan Banach introduced the Banach fixed point theorem, which proved the existence and uniqueness of the fixed point under various conditions [22]. One of the important results obtained is that the sequence obtained by Picard iteration converges to the fixed point [6]. Since iteration methods have wide application areas, many researches have been done on this subject. This process, which started with Picard, has developed and has survived to the present day. There are two main points to consider when defining the iteration method. The first is that the iteration to be defined is faster than existing iteration methods, and the second is that this iteration method is simple. For detailed information about iteration methods frequently used in the literature, you can refer to the following sources: [1], [10]- [20].

In addition to many iteration methods developed in this process, the strong convergence of these methods, convergence equivalence, convergence speed and whether the fixed points of these transformations are data dependent were investigated for certain transformation classes ([4],[8],[10],[11],[20]- [24]).The knowledge of which method converges faster for two iteration methods whose convergence is equivalent is of great importance in applied mathematics. Another

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transformation, called the approximation operator, can be used, which is close to the one used when constructing an iteration. Since this approximation operator has a different fixed point, the questions of how close the fixed point of the transformation and the fixed point of the approximation operator are to each other and how to calculate the distance between them bring up the concept of data dependency of fixed points.

One of the most common uses of fixed point theory, especially in applied mathematics, is the theory of integral equations. It is very important to determine the existence and uniqueness of integral equations. Fixed point theory is one of the most important tools used for this purpose. In our study, Fredholm integral equations, which are used in modeling many current problems, are discussed with the new three-step iteration method developed by Karakaya et al [14]. The reason why we use this iteration algorithm is that it has been proven to be faster than many iteration algorithms such as Picard, Mann, Ishikawa, Noor, SP, CR, Sahu-S and Picard-S [3]. Briefly, our study examines the strong convergence of the sequence obtained from the new three-step iteration method to the solution and the dependence of this solution on the data, under operators corresponding to nonlinear Fredholm integral equations.

2. Known Results

Definition 1. Let (X, d) be a metric space and $T : X \to X$ be a mapping. T is called a Lipschitzian mapping, if there is a $\lambda > 0$ number such that

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in X$ [10].

Definition 2. Let (X, d) be a metric space and $T : X \to X$ be a Lipschitzian mapping. T is called a contraction mapping, if there is at least one $\lambda \in (0,1)$ real number such that

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in X$. λ is called the contraction ratio [10].

Definition 3. Let X be a normed space and $T : X \to X$ be a Lipschitzian mapping. T is called a contraction mapping, if there is at least one $\lambda \in (0,1)$ real number such that

$$\|Tx - Ty\| \leq \lambda \|x - y\|$$

for all $x, y \in X$ [10].

Geometrically, Definition 2 and Definition 3 can be interpreted as Tx and Ty, which are images of any x and y points, are closer together than x and y [10].

Theorem 1. (Banach Fixed Point Theorem) If (X, d) is a complete metric space and $T : X \to X$ is a contraction mapping,

- T has one and only one fixed point $x \in X$.
- For any $x_0 \in X$, iteration sequence $(T^n x_0)$ (ie iteration sequence (x_n) defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$) converges to unique fixed point of T [6].

The following three-step iteration algorithm, defined by Karakaya et al. in 2017, has been shown to be faster than many iteration algorithms such as Picard, Mann, Ishikawa, Noor, SP, S, CR and Picard-S [14]:

Definition 4. The iteration method

$$\left.\begin{array}{l} x_0 \in X \\ x_{n+1} = Ty_n \\ y_n = (1-\beta_n)z_n + \beta_n T z_n \\ z_n = Tx_n \end{array}\right\} \tag{1}$$

is called the three-step iteration method, where X is a Banach space, $T : X \to X$ is an operator and $\{\beta_n\}_{n=0}^{\infty} \subset [0,1]$ is a sequence satisfying certain conditions [14].

Definition 5. The integral equations in the form of

$$x(t) = \lambda \int_a^b k\big(t, s, x(s)\big)ds, \tag{2}$$

where $k(t, s, x)$ is the known function defined over the region

$$D = \{(t, s, x) \in \mathbb{R}^3 : a \leq t, s \leq b, \quad -\infty < x < \infty\}$$

and $x(t)$ an unknown function whose solution is desired, and λ is any numerical parameter, are called the second type of nonlinear Fredholm integral equations. Here, k is called the kernel of the integral equation [2].

Lemma 1. $C([a, b], \|\cdot\|_\infty)$ is the space of all continuous functions in the interval $[a, b]$ defined by

$$d(x, y) = \sup_{t \in [a,b]} \|x(t) - y(t)\|_\infty.$$

Now, let the theorem be expressed which gives the existence and uniqueness conditions of the second type of nonlinear Fredholm integral equations:

Theorem 2. Consider the operator $T : C([a, b], \|\cdot\|_\infty) \to C([a, b], \|\cdot\|_\infty)$ defined by

$$Tx(t) = \lambda \int_a^b k\big(t, s, x(s)\big)ds. \tag{3}$$

$k(t, s, x)$ is continuous over the region

$$D = \{(t, s, x) \in \mathbb{R}^3 : a \leq t, s \leq b, \quad -\infty < x < \infty\}$$

and if $L > 0$ exists such that

$$|k(t, s, x_1) - k(t, s, x_2)| \leq L|x_1 - x_2|$$

for

$$\forall (t, s, x_1), (t, s, x_2) \in D_r = \{(t, s, x) \in \mathbb{R}^3 : a \leq t, s \leq b, \ \ |x| \leq r \ (r > 0)\},$$

there is only one solution $x^*(t)$ of equation (2) in $C([a, b])$ when $|\lambda| < \lambda_0$. Here

$$\lambda_0 = \min\left\{\frac{1}{L(b-a)}, \frac{r}{rL(b-a)+L_0}\right\}$$

and

$$L_0 = \max_{t,s\in[a,b]}\left\{\int_a^b |k(t,s,0)|ds\right\}.$$

The sequence $(x_n(t))$ defined as

$$x_n(t) = \lambda\int_a^b k\big(t,s,x_{n-1}(s)\big)ds\ , \quad n = 1,2,\ldots$$

converges smoothly to the function $x^*(t)$ for any initial function $x_0 \in \{C([a,b]),\ \|x\|_\infty \leq r\}$ [2].

Definition 6. Let $A_1, A_2 : Y \to Y$ be operators. If $\|A_1x - A_2x\| \leq \varepsilon$ for each $x \in Y$ and constant $\varepsilon > 0$, then A_2 is called the approximation operator of A_1 [24].

Lemma 2. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two non-negative real sequences satisfting the following condition:

$$a_{n+1} \leq (1-\mu_n)a_n + b_n\ ,$$

where $\mu_n \in (0,1)$ for each $n \geq n_0, \sum_{n=0}^{\infty}\mu_n = \infty$ and $\frac{b_n}{\mu_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty}a_n = 0$ [25].

Lemma 3. Let $\{a_n\}_{n=0}^{\infty}$ be a non-negative real sequence and there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ satisfting the following condition:

$$a_{n+1} \leq (1-\mu_n)a_n + \mu_n\gamma_n\ ,$$

where $\mu_n \in (0,1)$ such that $\sum_{n=0}^{\infty}\mu_n = \infty$ and $\gamma_n \geq 0$. Then the following inequality holds:

$$0 \leq \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty}\gamma_n$$

[24].

3. Main Results

Theorem 3. Let $T : C([a,b],\|\cdot\|_\infty) \to C([a,b],\|\cdot\|_\infty)$ be an operator and $\{\beta_n\}_{n=0}^{\infty} \subset [0,1]$ be a sequence satisfying certain conditions. In this case, the integral equation given by equation (2) has a unique solution in the form of $x^* \in C[a,b]$ and the sequence $\{x_n\}_{n=0}^{\infty}$ obtained from the iteration algorithm given by equation (1) converges to this solution.

Proof Consider the sequence $\{x_n\}_{n=0}^{\infty}$ obtained from the iteration algorithm given by equation (1) constructed with the operator $T : C([a,b],\|\cdot\|_\infty) \to C([a,b],\|\cdot\|_\infty)$. It will be shown that for $n \to \infty$ is $x_n \to x^*$. Using equation (1), equation (2) and conditions of Theorem 2, we are obtained the following inequality:

$$|x_{n+1}(t) - x^*(t)| = |Ty_n(t) - Tx^*(t)|$$

$$= \left|\lambda \int_a^b k(t,s,y_n(s))ds - \lambda \int_a^b k(t,s,x^*(s))ds\right|$$
$$= |\lambda| \left|\int_a^b k(t,s,y_n(s)) - k(t,s,x^*(s))ds\right|$$
$$\leq |\lambda| \int_a^b \left|k(t,s,y_n(s)) - k(t,s,x^*(s))\right|ds$$
$$\leq |\lambda| L \int_a^b |y_n(s) - x^*(s)|ds$$
$$\leq |\lambda| L(b-a)\|y_n - x^*\|_\infty$$
$$\leq \lambda_0 L(b-a)\|y_n - x^*\|_\infty \,. \tag{4}$$

Similarly, by making the necessary calculations, the following inequalities are obtained:

$$\begin{aligned}\|y_n - x^*\|_\infty &= \|(1-\beta_n)z_n + \beta_n T z_n - T x^*\|_\infty \\ &= \|z_n - x^* + \beta_n(Tz_n - z_n)\|_\infty \\ &\leq \|z_n - x^*\|_\infty \,,\end{aligned} \tag{5}$$

$$\begin{aligned}|z_n - x^*| &= |Tx_n - Tx^*| \\ &= \left|\lambda \int_a^b k(t,s,x_n(s))ds - \lambda \int_a^b k(t,s,x^*(s))ds\right| \\ &= |\lambda| \left|\int_a^b k(t,s,x_n(s)) - k(t,s,x^*(s))ds\right| \\ &\leq |\lambda| \int_a^b \left|k(t,s,x_n(s)) - k(t,s,x^*(s))\right|ds \\ &\leq |\lambda| L \int_a^b |x_n(s) - x^*(s)|ds \\ &\leq |\lambda| L(b-a)\|x_n - x^*\|_\infty \\ &\leq \lambda_0 L(b-a)\|x_n - x^*\|_\infty \,.\end{aligned} \tag{6}$$

Then, taking the supremum of both sides of inequality (6),

$$\|z_n - x^*\|_\infty \leq \lambda_0 L(b-a)\|x_n - x^*\|_\infty \tag{7}$$

is obtained. If inequality (7) and inequality (5) are written in inequality (4),

$$\|x_{n+1}(t) - x^*(t)\|_\infty \leq \lambda_0^2 L^2 (b-a)^2 \|x_n - x^*\|_\infty$$

And by applying induction to the last inequality, the following inequality is obtained:

$$\begin{aligned}\|x_{n+1}(t) - x^*(t)\|_\infty &\leq \alpha^2 \|x_n - x^*\|_\infty \\ &\leq \alpha^4 \|x_{n-1} - x^*\|_\infty \\ &\leq \alpha^6 \|x_{n-2} - x^*\|_\infty \\ &\;\;\vdots \\ &\leq \alpha^{2(n+1)} \|x_0 - x^*\|_\infty\end{aligned}$$

then

$$\|x_{n+1}(t) - x^*(t)\|_\infty \leq \alpha^{2(n+1)} \|x_0 - x^*\|_\infty \,,$$

is found. Thus, since $0 < \alpha < 1$,

$$\lim_{n\to\infty} \|x_{n+1}(t) - x^*(t)\|_\infty = 0 \,.$$

So, the proof is completed.

Now, let us examine the data dependency of the solution of the integral equation given by equation (2) using the iteration algorithm given in equation (1). On the other hand, for data dependency, consider the integral equation

$$u(t) = \lambda_1 \int_a^b h(t,s,u(s))ds\,, \tag{8}$$

where $h(t,s,u)$ is a continuous function given over the region

$$D = \{(t,s,u) \in \mathbb{R}^3 : a \le t,s \le b\,,\quad -\infty < u < \infty\}\,,$$

and λ_1 is a parameter. If equation (8) is written with the operator $S : C([a,b],\|\cdot\|_\infty) \to C([a,b],\|\cdot\|_\infty)$,

$$S(u(t)) = \lambda_1 \int_a^b h(t,s,u(s))ds \tag{9}$$

is obtained. If the iteration algorithm given in equation (1) is reconstructed with operators $T(3)$ and $S(9)$, respectively,

$$\left.\begin{aligned} x_{n+1}(t) &= \lambda \int_a^b k(t,s,y_n(s))ds \\ y_n(t) &= (1-\beta_n)z_n(t) + \beta_n\left[\lambda \int_a^b k(t,s,z_n(s))ds\right] \\ z_n(t) &= \lambda \int_a^b k(t,s,x_n(s))ds \end{aligned}\right\} \tag{10}$$

and

$$\left.\begin{aligned} u_{n+1}(t) &= \lambda_1 \int_a^b h(t,s,v_n(s))ds \\ v_n(t) &= (1-\beta_n)w_n(t) + \beta_n\left[\lambda_1 \int_a^b h(t,s,w_n(s))ds\right] \\ w_n(t) &= \lambda_1 \int_a^b h(t,s,u_n(s))ds \end{aligned}\right\} \tag{11}$$

iteration algorithms can be written.

Theorem 4. Let the sequence $\{\beta_n\}_{n=0}^\infty \subset [0,1]$ satisfy the condition $\beta_n \ge \frac{1}{2}$ for each $n \in \mathbb{N}$. Consider the sequence $\{x_n\}_{n=0}^\infty$ obtained from equation (10) and the sequence $\{u_n\}_{n=0}^\infty$ obtained from equation (11). Let the solutions of equation (2) and equation (8) be x^* and u^*, respectively, with the conditions of Theorem 2. Let the constant ε exists such that $\|k(t,s,p(s)) - h(t,s,p(s))\|_\infty \le \varepsilon$ for each $a \le t,s \le b$ and $-\infty < p < \infty$. $k(t,s,p)$ and $h(t,s,p)$ are continuous functions given over the region

$$A = \{(t,s,p) \in \mathbb{R}^3 : a \le t,s \le b,\ -\infty < p < \infty\}.$$

λ and λ_1 are parameters.

If $x_n \to x^*$ and $u_n \to u^*$ as $n \to \infty$, then the inequality

$$\|x^* - u^*\| \le \frac{3\varepsilon\lambda_{\max}(b-a)}{1-\lambda_{\max}(b-a)L}$$

is valid, with $\lambda_{\max} = \max\{|\lambda|,|\lambda_1|\}$.

Proof With the hypotheses of the theorem, the following inequality is obtained:

$$\|x_{n+1} - u_{n+1}\|_\infty = \left\|\lambda \int_a^b k(t,s,y_n(s))ds - \lambda_1 \int_a^b h(t,s,v_n(s))ds\right\|_\infty$$

$$\leq \left\| \lambda_{\max} \int_a^b \begin{pmatrix} k\big(t,s,y_n(s)\big) - k\big(t,s,v_n(s)\big) \\ +k\big(t,s,v_n(s)\big) - h\big(t,s,v_n(s)\big) \end{pmatrix} ds \right\|_\infty$$
$$\leq \lambda_{\max} \begin{pmatrix} L\int_a^b \|y_n(s) - v_n(s)\|_\infty ds \\ +\int_a^b \left\|k\big(t,s,v_n(s)\big) - h\big(t,s,v_n(s)\big)\right\|_\infty ds \end{pmatrix}$$
$$\leq \lambda_{\max}(b-a)(L\|y_n - v_n\|_\infty + \varepsilon)$$
$$\leq \lambda_{\max}(b-a)L\|y_n - v_n\|_\infty + \lambda_{\max}(b-a)\varepsilon. \tag{12}$$

Similarly,

$$\|y_n - v_n\|_\infty \leq (1-\beta_n)\|z_n - w_n\|_\infty + \beta_n \left\| \lambda \int_a^b k(t,s,z_n)ds - \lambda_1 \int_a^b h(t,s,w_n)ds \right\|_\infty$$
$$\leq (1-\beta_n)\|z_n - w_n\|_\infty + \beta_n \left\| \lambda_{\max} \int_a^b k(t,s,z_n) - h(t,s,w_n)ds \right\|_\infty$$
$$\leq (1-\beta_n)\|z_n - w_n\|_\infty + \beta_n\lambda_{\max} \int_a^b \|k(t,s,z_n) - h(t,s,w_n)\|_\infty ds$$
$$\leq (1-\beta_n)\|z_n - w_n\|_\infty + \beta_n\lambda_{\max} \int_a^b \left\| \begin{matrix} k(t,s,z_n) - k(t,s,w_n) \\ +k(t,s,w_n) - h(t,s,w_n) \end{matrix} \right\|_\infty ds$$
$$\leq (1-\beta_n)\|z_n - w_n\|_\infty + \beta_n\lambda_{\max} \left(L\int_a^b \|z_n - w_n\|_\infty ds + \int_a^b \varepsilon ds \right)$$
$$\leq (1-\beta_n)\|z_n - w_n\|_\infty + \beta_n\lambda_{\max}(b-a)(L\|z_n - w_n\|_\infty + \varepsilon)$$
$$\leq [1-\beta_n + \beta_n\lambda_{\max}(b-a)L]\|z_n - w_n\|_\infty + \beta_n\lambda_{\max}(b-a)\varepsilon \tag{13}$$

and

$$\|z_n - w_n\|_\infty = \left\| \lambda \int_a^b k(t,s,x_n)ds - \lambda_1 \int_a^b h(t,s,u_n)ds \right\|_\infty$$
$$\leq \lambda_{\max} \left\| \int_a^b k(t,s,x_n) - h(t,s,u_n)ds \right\|_\infty$$
$$\leq \lambda_{\max} \int_a^b \left\| \begin{matrix} k(t,s,x_n) - k(t,s,u_n) \\ +k(t,s,u_n) - h(t,s,u_n) \end{matrix} \right\|_\infty ds$$
$$\leq \lambda_{\max} \left(L\int_a^b \|x_n - u_n\|_\infty ds + \int_a^b \varepsilon ds \right)$$
$$\leq \lambda_{\max}(b-a)(L\|x_n - u_n\|_\infty + \varepsilon)$$
$$\leq \lambda_{\max}(b-a)L\|x_n - u_n\|_\infty + \lambda_{\max}(b-a)\varepsilon$$
$$\leq \|x_n - u_n\|_\infty + \lambda_{\max}(b-a)\varepsilon \tag{14}$$

are found. If inequality (14) is written in inequality (13),

$$\|y_n - v_n\|_\infty \leq [1-\beta_n + \beta_n\lambda_{\max}(b-a)L]\|z_n - w_n\|_\infty + \beta_n\lambda_{\max}(b-a)\varepsilon$$
$$\leq [1-\beta_n + \beta_n\lambda_{\max}(b-a)L][\|x_n - u_n\|_\infty + \lambda_{\max}(b-a)\varepsilon] + \beta_n\lambda_{\max}(b-a)\varepsilon$$
$$\leq [1-\beta_n + \beta_n\lambda_{\max}(b-a)L]\|x_n - u_n\|_\infty + [\lambda_{\max}(b-a)\varepsilon] + \beta_n\lambda_{\max}(b-a)\varepsilon$$
$$\leq [1-\beta_n + \beta_n\lambda_{\max}(b-a)L]\|x_n - u_n\|_\infty + \varepsilon\lambda_{\max}(b-a)(1+\beta_n)$$

is obtained. If the last inequality is written in inequality (12),

$$\|x_{n+1} - u_{n+1}\|_\infty \leq \|y_n - v_n\|_\infty + \lambda_{\max}(b-a)\varepsilon$$
$$\leq [1-\beta_n + \beta_n\lambda_{\max}(b-a)L]\|x_n - u_n\|_\infty + \varepsilon\lambda_{\max}(b-a)(1+\beta_n) + \lambda_{\max}(b-a)\varepsilon$$

$$\leq [1-\beta_n+\beta_n\lambda_{\max}(b-a)L]\|x_n-u_n\|_\infty+\varepsilon\lambda_{\max}(b-a)(2+\beta_n)$$
$$\leq \{1-\beta_n(1-\lambda_{\max}(b-a)L)\}\|x_n-u_n\|_\infty+\varepsilon\lambda_{\max}(b-a)(2+\beta_n)$$
$$\leq \{1-\beta_n(1-\lambda_{\max}(b-a)L)\}\|x_n-u_n\|_\infty+\beta_n(1-\lambda_{\max}(b-a)L)\frac{3\varepsilon\lambda_{\max}(b-a)}{1-\lambda_{\max}(b-a)L} \quad (15)$$

is found. If is chosen a_n , μ_n , γ_n as follows in inequality (15), satisfies the conditions of Lemma 3.

$$a_n=\|x_n-u_n\|_\infty,$$
$$\mu_n=\beta_n(1-\lambda_{\max}(b-a)L)\in(0,1),$$
$$\gamma_n=\frac{3\varepsilon\lambda_{\max}(b-a)}{1-\lambda_{\max}(b-a)L}\geq 0.$$

$\beta_n\geq\frac{1}{2}$ requires $\sum_{n=0}^{\infty}\beta_n=\infty$ for each $n\in\mathbb{N}$. Then,

$$0\leq \limsup_{n\to\infty}\|x_n-u_n\|_\infty\leq \limsup_{n\to\infty}\gamma_n=\limsup_{n\to\infty}\frac{3\varepsilon\lambda_{\max}(b-a)}{1-\lambda_{\max}(b-a)L}$$

is obtained. Since $x_n\to x^*$ and $u_n\to u^*$as $n\to\infty$,

$$\|x^*-u^*\|_\infty\leq\frac{3\varepsilon\lambda_{\max}(b-a)}{1-\lambda_{\max}(b-a)L} \quad (16)$$

is found.

Example 1.

$$x(t)=\frac{17}{64}\int_0^1\frac{1}{1+x^2(s)}ds$$

where $k(t,s,x)=\frac{1}{1+x^2(s)}$ is a continuous function given over the region

$$D=\{(t,s,x): 0\leq t,s\leq 1,\quad -\infty<x<\infty\}.$$

The partial derivative of $k(t,s,x)$:

$$\frac{\partial k}{\partial x}=-\frac{2x}{(1+x^2)^2}$$

is bounded over the region D.

$$\left|\frac{\partial k}{\partial x}\right|=\left|-\frac{2x}{(1+x^2)^2}\right|\leq 1,\quad (t,s,x)\in D$$

In this case, $k(t,s,x)$ satisfies the Lipschitz condition with the coefficient $L=1$.

$$\lambda=\frac{17}{64},\qquad a=0,\qquad b=1,\qquad \alpha=|\lambda|L(b-a)=\frac{17}{64}<1\,.$$

The equation in question has only one continuous solution x^* on [0,1].

Let's define the following algorithm with the operator

$$Tx_n(t)=\frac{17}{64}\int_0^1\frac{1}{1+x_n^2(s)}ds$$

for the solution:

$$x_{n+1}(t) = Ty_n(t) = \frac{17}{64}\int_0^1 \frac{1}{1+y_n^2(s)}ds$$
$$y_n(t) = (1-\beta_n)z_n(t) + \beta_n Tz_n(t) = (1-\beta_n)z_n(t) + \beta_n\left(\frac{17}{64}\int_0^1 \frac{1}{1+z_n^2(s)}ds\right)$$
$$z_n(t) = Tx_n(t) = \frac{17}{64}\int_0^1 \frac{1}{1+x_n^2(s)}ds.$$

On the other hand, let's consider the integral equation

$$u(t) = \frac{65}{256}\int_0^1 \frac{s}{1+u^2(s)}ds\,,$$

where

$$h(t,s,u) = \frac{s}{1+u^2(s)}$$

is a continuous function given over the region

$$G = \{(t,s,u) : 0 \le t,s \le 1\,,\quad -\infty < u < \infty\}\,.$$

The partial derivative of $h(t,s,u)$:

$$\frac{\partial h}{\partial u} = -\frac{2su}{(1+u^2)^2}$$

is bounded over the region G.

$$\left|\frac{\partial h}{\partial u}\right| = \left|-\frac{2su}{(1+u^2)^2}\right| \le 1\,,\ \ (t,s,u) \in G$$

In this case, $h(t,s,u)$ satisfies the Lipschitz condition with the coefficient $L = 1$.

$$\lambda_1 = \frac{65}{256}\,,\ \ a = 0\,,\ \ b = 1\,,\ \ \alpha = |\lambda_1|L(b-a) = \frac{65}{256} < 1\,.$$

The equation in question has only one continuous solution u^*on [0,1].

Let's define the following algorithm with the operator

$$S\big(u_n(t)\big) = \frac{65}{256}\int_0^1 \frac{s}{1+u^2(s)}ds$$

for the solution:

$$u_{n+1}(t) = Sv_n(t) = \frac{65}{256}\int_0^1 \frac{s}{1+v^2(s)}ds$$
$$v_n(t) = (1-\beta_n)w_n(t) + \beta_n Sw_n(t) = (1-\beta_n)w_n(t) + \beta_n\left(\frac{65}{256}\int_0^1 \frac{s}{1+w^2(s)}ds\right)$$
$$w_n(t) = Su_n(t) = \frac{65}{256}\int_0^1 \frac{s}{1+u^2(s)}ds.$$

Thus,

$$\lambda_{\max} = \max\{|\lambda|, |\lambda_1|\} = \max\left\{\frac{17}{64}, \frac{65}{256}\right\} = \frac{17}{64}$$

is found. Let the constant ε exists such that

$$\begin{aligned}\left\|k\big(t,s,p(s)\big)-h\big(t,s,p(s)\big)\right\|_{\infty} &= \left\|\frac{1}{1+p^2}-\frac{s}{1+p^2}\right\|_{\infty}\\ &\leq \left\|\frac{1}{1+p^2}\right\|_{\infty}\\ &\leq 1=\varepsilon\end{aligned}$$

for each $(t,s,p)\in A$. So, all the conditions of Theorem 4 are satisfied. Therefore, inequation (16) is valid. If the found values are written in the inequation (16),

$$\|x^*-u^*\|_{\infty}\leq \frac{3\varepsilon\lambda_{\max}(b-a)}{1-\lambda_{\max}(b-a)L}=\frac{3.1.\frac{17}{64}.1}{1-\frac{17}{64}.1.1}=\frac{51}{47}\simeq 1.085$$

is obtained. Indeed, $x^*=\frac{1}{4}$ and $u^*=\frac{1}{8}$ are found. So,

$$\|x^*-u^*\|_{\infty}=\left\|\frac{1}{4}-\frac{1}{8}\right\|_{\infty}=\frac{1}{8}=0.125\leq 1.085$$

is found. Thus, the theorem is supported by this example.

4. Conclusion

Many real-life problems are expressed non-linearly. In the modeling of these problems, nonlinear integral equations are mostly used. Fixed point theory is very important for solving these integral equations. The basic idea here is to construct algorithms called iterations by including the equation in an operator class under certain conditions, and to determine the appropriate conditions for the sequence obtained from this iteration to converge to the fixed point of the operator, in other words, to the solution of the equation.In this study, the solution of the second type of homogeneous nonlinear Fredholm integral equations is investigated using a three-step iteration algorithm. In other words, the aim of this study is to show that the sequence obtained from equation (1) iteration method converges strongly to the solution of equation (2). It has been shown that the sequences obtained from this algorithm converge to the solution of the mentioned equations. In addition, data dependence was obtained for the second type of homogeneous nonlinear Fredholm integral equations and this result was supported by an example. Interested researchers can reconstruct the newly described three-step iteration method for more general transformation classes and apply it to many types of integral equations to examine the results of strong convergence and data dependence.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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