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Authors: Ayşe KABATAŞ

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Sturm-Liouville Problems with Polynomially Eigenparameter Dependent Boundary Conditions

Ayşe KABATAŞ *1

Abstract

Sturm-Liouville equation on a finite interval together with boundary conditions arises from the infinitesimal, vertical vibrations of a string with the ends subject to various constraints. The coefficient (also called potential) function in the differential equation is in a close relationship with the density of the string. In this sense, the computation of solutions plays a rather important role in both mathematical and physical fields. In this study, asymptotic behaviors of the solutions for Sturm-Liouville problems associated with polynomially eigenparameter dependent boundary conditions are obtained when the potential function is real valued $\boldsymbol{L^1}$-function on the interval $(\mathbf{0},\mathbf{1})$. Besides, the asymptotic formulae are given for the derivatives of the solutions.

Keywords: Sturm-Liouville problem, spectral parameter, potential function, asymptotics

1. INTRODUCTION

Consider the regular Sturm-Liouville problems denoted by $L := L(q, B_0, B_1)$:

$$u'' + [\lambda - q(x)]u = 0, x \in (0,1) \tag{1}$$

$$B_0(u) := P_{01}(\lambda)u'(0) + P_{00}(\lambda)u(0) = 0, \tag{2}$$

$$B_1(u) := P_{11}(\lambda)u'(1) + P_{10}(\lambda)u(1) = 0. \tag{3}$$

Here, λ is a real spectral parameter, q is a real-valued L^1-function on $(0,1)$ and

$$P_{\xi k}(\lambda) = \sum_{l=0}^{r_{\xi k}} a_{\xi k l}\, \lambda^{r_{\xi k}-l}, r_{\xi 1} = r_{\xi 0} = r_\xi \geq 0,$$

$$a_{\xi 10} = 1, \xi, k = 0,1 \tag{4}$$

are arbitrary polynomials of degree r_ξ with real coefficients such that $P_{\xi 1}(\lambda)$ and $P_{\xi 0}(\lambda)$ have no common zeros for $\xi = 0,1$.

Sturm-Liouville problems have been studied since the fundamental work of Sturm and Liouville in the 19th century [1-6]. These types of problems associated with ordinary differential equations arise in considering physical problems, such as determining the temperature distribution of a heat conducting rod vibration problems of the wire hanging on

* Corresponding author: akabatas@ktu.edu.tr (A. KABATAŞ)
[1] Karadeniz Technical University, Türkiye
ORCID: https://orcid.org/0000-0003-3273-3666

some internal points, wave and diffusion problems and etc. by the method of separation of variables, see [7]. Also, such problems with linear or nonlinear dependence on the spectral parameter in boundary conditions arise in various problems of mathematics as well as in the more contemporary applications of quantum mechanics and acoustic scattering theory and so on [8, 9] and there is an extensive literature for these problems in recent years, see [10-14]. Detailed studies on direct spectral problems for general classes of ordinary differential operators depending nonlinearly on the parameter can be found in various publications, see e.g. [15-19].

The values of the parameter λ for which L has nonzero solutions are called eigenvalues, and the corresponding nontrivial solutions are called eigenfunctions. The derivation of asymptotic formulae for eigenvalues and eigenfunctions of regular Sturm-Liouville problems is of interest in its own right and has a long history. Motivation for studying eigenvalue and eigenfunction asymptotics has come from several different types of problems including theory of equiconvergence of eigenfunction expansions for Sturm-Liouville problems with Fourier Series, inverse spectral theory and theory on reconstructing the potential function from knowledge of spectral data, and the general theory of periodic potentials, see [20].

In the present paper, we determine the asymptotic solutions and their derivatives of the problem (1)-(3) when the potential q is a real-valued member of $L^1(0,1)$. In addition, we give the asymptotic approximations on the derivatives of solutions.

2. METHOD

Let $u(x,\lambda)$ be a complex valued solution of the equation (1). If $w(x,\lambda)=\frac{u\prime(x,\lambda)}{u(x,\lambda)}$ transform is applied to (1), we have the Riccati equation

$$w' = -\lambda + q - w^2. \tag{5}$$

We set

$$S(x,\lambda) := Re\{w(x,\lambda)\},$$

$$T(x,\lambda) := Im\{w(x,\lambda)\}$$

where $w(x,\lambda)$ is a complex-valued solution of (5). It is given in [21] that any nontrivial real-valued solution, z, of (1) can be expressed as

$$z(x,\lambda) = c_1 \exp\left(\int_0^x S(t,\lambda)dt\right) \times \cos\left\{c_2 + \int_0^x T(t,\lambda)dt\right\} \tag{6}$$

with

$$z'(x,\lambda) = c_1 S(x,\lambda)\exp\left(\int_0^x S(t,\lambda)dt\right) \times \cos\left\{c_2 + \int_0^x T(t,\lambda)dt\right\} - c_1 T(x,\lambda)\exp\left(\int_0^x S(t,\lambda)dt\right) \times \sin\{c_2 + \int_0^x T(t,\lambda)dt\}. \tag{7}$$

We suppose that there exist functions $A(x)$ and $\eta(\lambda)$ so that

$$\left|\int_x^1 e^{2i\lambda^{\frac{1}{2}}t} q(t)dt\right| \le A(x)\eta(\lambda), \quad x \in [0,1]$$

where

(i) $A(x) := \int_x^1 |q(t)|\, dt$ is decreasing function of x,
(ii) $\eta(\lambda) \to 0$ as $\lambda \to \infty$
(iii) $A(x) \in L^1[0,1]$.

The existence of these functions are established in [21]. We define $F(x,\lambda)$ for completeness as follows:

$$F(x,\lambda) := \begin{cases} \frac{\left|\int_x^1 e^{2i\lambda^{\frac{1}{2}}t} q(t)dt\right|}{\int_x^1 |q(t)|dt}, & \text{if} \int_x^1 |q(t)|\, dt \neq 0, \\ 0, & \text{if} \quad \int_x^1 |q(t)|\, dt = 0 \end{cases}$$

and we set $\eta(\lambda) := sup_{0\le x\le 1} F(x,\lambda)$ ($0 \le F(x,\lambda) \le 1$). So, $\eta(\lambda)$ is well-defined and goes to zero as $\lambda \to \infty$ [21].

Now, we seek a solution of the Riccati equation (5) as

$$w(x,\lambda):=i\lambda^{1/2}+\sum_{k=1}^{\infty}w_k(x,\lambda)$$

and choose w_k so that

$$\begin{aligned} w'_1+2i\lambda^{\frac{1}{2}}w_1 &= q,\\ w'_2+2i\lambda^{\frac{1}{2}}w_2 &= -w_1^2,\\ w'_k+2i\lambda^{\frac{1}{2}}w_k &= -\begin{pmatrix} w_{k-1}^2+2w_{k-1} \\ \sum_{m=1}^{k-2}w_m \end{pmatrix}, \quad k\geq 3. \end{aligned}$$

Solution of above equation

$$w_1(x,\lambda)=-e^{-2i\lambda^{1/2}x}\int_x^1 e^{2i\lambda^{1/2}t}\,q(t)dt,$$

$$w_2(x,\lambda)=e^{-2i\lambda^{\frac{1}{2}}x}\int_x^1 e^{2i\lambda^{\frac{1}{2}}t}\,w_1^2(t,\lambda)dt,$$

and for $k\geq 3$

$$w_k(x,\lambda)=e^{-2i\lambda^{1/2}x}\int_x^1 e^{2i\lambda^{1/2}t}\,[w_{k-1}^2+2w_{k-1}\sum_{m=1}^{k-2}w_m]dt.$$

It is shown in [21] that the series $\sum_{k=1}^{\infty}w_k\,(x,\lambda)$ and $\sum_{k=1}^{\infty}w'_k(x,\lambda)$ are uniformly absolutely convergent for all $\lambda\geq\lambda_0$. The series $i\lambda^{1/2}+\sum_{k=1}^{\infty}w_k\,(x,\lambda)$ is a solution of (5)

$$\begin{aligned} S(x,\lambda) &= Re\sum_{k=1}^{\infty}w_k\,(x,\lambda),\\ T(x,\lambda) &= \lambda^{\frac{1}{2}}+Im\sum_{k=1}^{\infty}w_k\,(x,\lambda). \end{aligned}$$

In [22], the asymptotic approximations for $S(x,\lambda)$ and $T(x,\lambda)$ are given as

$$S(x,\lambda)=-\sin\left(2\lambda^{\frac{1}{2}}x+\xi_x\right)+O\big(\eta^2(\lambda)\big) \tag{8}$$

and

$$T(x,\lambda)=\lambda^{1/2}-\cos\big(2\lambda^{1/2}x+\xi_x\big)+O\big(\eta^2(\lambda)\big) \tag{9}$$

where

$$\sin\xi_x:=\int_x^1 q\,(t)\cos(2\lambda^{\frac{1}{2}}t)dt,$$

$$\cos\xi_x:=\int_x^1 q\,(t)\sin(2\lambda^{1/2}t)dt.$$

Also, it is determined in [23] that

$$\int_0^x S\,(t,\lambda)dt=\frac{1}{2}\lambda^{-\frac{1}{2}}\left\{\cos\left(2\lambda^{\frac{1}{2}}x+\xi_x\right)-\cos\xi_0\right\}+O\left(\lambda^{-1/2}\eta^2(\lambda)\right) \tag{10}$$

$$\int_0^x T\,(t,\lambda)dt=\lambda^{\frac{1}{2}}x-\frac{1}{2}\lambda^{-\frac{1}{2}}\times\left\{\sin\left(2\lambda^{\frac{1}{2}}x+\xi_x\right)-\sin\xi_0+\int_0^x q\,(t)dt\right\}+O\left(\lambda^{-1/2}\eta^2(\lambda)\right). \tag{11}$$

3. RESULTS

We define two solutions, $u_-(x,\lambda)$ and $u_+(x,\lambda)$, of equation (1) with initial conditions

$$u_-(0,\lambda)=P_{01}(\lambda),u'_-(0,\lambda)=-P_{00}(\lambda), \tag{12}$$

$$u_+(1,\lambda)=P_{11}(\lambda),\mathrm{u}'_+(1,\lambda)=-\mathrm{P}_{10}(\lambda). \tag{13}$$

Theorem 1. The solutions, $u_-(x,\lambda)$ and $u_+(x,\lambda)$ satisfy the following equalities for $q\in L^1(0,1)$, respectively.

(i)

$$u_-(x,\lambda)=\frac{P_{01}(\lambda)}{cos[tan^{-1}F_0(\lambda)]}exp\Big(\int_0^x S\,(t,\lambda)dt\Big)\times cos\Big[tan^{-1}F_0(\lambda)+\int_0^x T\,(t,\lambda)dt\Big] \tag{14}$$

where

$$F_0(\lambda)=\frac{P_{01}(\lambda)S(0,\lambda)+P_{00}(\lambda)}{P_{01}(\lambda)T(0,\lambda)}, \tag{15}$$

(ii)

$$u_+(x,\lambda) = \frac{P_{11}(\lambda)}{cos[tan^{-1}F_1(\lambda)]} \times exp\left(-\int_x^1 S\,(t,\lambda)dt\right) \times cos\left[tan^{-1}F_1(\lambda) - \int_x^1 T\,(t,\lambda)dt\right]$$

where

$$F_1(\lambda) = \frac{P_{11}(\lambda)S(1,\lambda)+P_{10}(\lambda)}{P_{11}(\lambda)T(1,\lambda)}.$$

Proof. (i) Using (6), (7) and (12) we find

$$u_-(0,\lambda) = c_1\cos c_2 = P_{01}(\lambda),$$

$$u'_-(0,\lambda) = c_1S(0,\lambda)\cos c_2 - c_1T(0,\lambda)\sin c_2 = -P_{00}(\lambda).$$

So,

$$c_1 = \frac{P_{01}(\lambda)}{\cos c_2} \qquad (16)$$

and

$$c_2 = \tan^{-1}F_0(\lambda). \qquad (17)$$

The proof is completed by substituting the values (16) and (17) into (6).

(ii) From (6), (7) and (13) it can be written

$$u_+(1,\lambda) = c_1\exp\left(\int_0^1 S\,(t,\lambda)dt\right)\cos\left[c_2 + \int_0^1 T\,(t,\lambda)dt\right] = P_{11}(\lambda),$$

$$u'_+(1,\lambda) = c_1\exp\left(\int_0^1 S\,(t,\lambda)dt\right)$$

$$\times\left\{\begin{array}{c} S(1,\lambda)\cos\left[c_2 + \int_0^1 T\,(t,\lambda)dt\right] - \\ T(1,\lambda)\sin\left[c_2 + \int_0^1 T\,(t,\lambda)dt\right] \end{array}\right\} = -P_{10}(\lambda).$$

Thus, we obtain

$$c_1 = \frac{P_{11}(\lambda)}{\exp\left(\int_0^1 S(t,\lambda)dt\right)\cos\left[c_2+\int_0^1 T(t,\lambda)dt\right]}$$

and

$$c_2 = \tan^{-1}F_1(\lambda) - \int_0^1 T\,(t,\lambda)dt.$$

For the proof, these values of c_1 and c_2 are used in (6).

Now, asymptotic approximations will be given for the solutions, $u_-(x,\lambda)$ and $u_+(x,\lambda)$.

Theorem 2. Let $q(x)$ be a real-valued L^1-function on $(0,1)$. As $\lambda \to \infty$, we have the following asymptotic approximations for the solutions of (1) with the initial conditions (12) and (13), respectively.

(i)

$$u_-(x,\lambda) = \lambda^{r_0}\cos\left(\lambda^{\frac{1}{2}}x\right) - \lambda^{r_0-\frac{1}{2}}\left[a_{000} - \frac{1}{2}\int_0^x q\,(t)dt\right]\sin\left(\lambda^{\frac{1}{2}}x\right) + O\left(\lambda^{r_0-\frac{1}{2}}\eta(\lambda)\right), \qquad (18)$$

(ii)

$$u_+(x,\lambda) = \lambda^{r_1}\cos\left[\lambda^{\frac{1}{2}}(1-x)\right] + \lambda^{r_1-\frac{1}{2}}\left[a_{100} + \frac{1}{2}\int_x^1 q\,(t)dt\right]\sin\left[\lambda^{\frac{1}{2}}(1-x)\right] + O\left(\lambda^{r_1-\frac{1}{2}}\eta(\lambda)\right). \qquad (19)$$

Proof. (i) We evaluate the terms in (14) as $\lambda \to \infty$. Together with (4), (8), (9) and (15) we obtain

$$F_0(\lambda) = \frac{S(0,\lambda)}{T(0,\lambda)} + \frac{P_{00}(\lambda)}{P_{01}(\lambda)T(0,\lambda)} = \frac{O(\eta(\lambda))}{\lambda^{\frac{1}{2}}\left[1+O\left(\lambda^{-\frac{1}{2}}\eta(\lambda)\right)\right]} + \frac{a_{000}\lambda^{r_0}+O(\lambda^{r_0-1})}{\lambda^{r_0+\frac{1}{2}}+O(\lambda^{r_0}\eta(\lambda))}$$

$$= O\left(\lambda^{-\frac{1}{2}}\eta(\lambda)\right)\left[1 + O\left(\lambda^{-\frac{1}{2}}\eta(\lambda)\right)\right] + \left[a_{000}\lambda^{-\frac{1}{2}} + O\left(\lambda^{-\frac{3}{2}}\right)\right]\left[1 + O\left(\lambda^{-\frac{1}{2}}\eta(\lambda)\right)\right]$$

$$= a_{000}\lambda^{-\frac{1}{2}} + O\left(\lambda^{-\frac{1}{2}}\eta(\lambda)\right). \qquad (20)$$

It is clear from (20) that

$$tan^{-1}F_0(\lambda)=a_{000}\lambda^{-\frac{1}{2}}+O(\lambda^{-\frac{1}{2}}\eta(\lambda)).$$

So,

$$\cos[tan^{-1}F_0(\lambda)]=1-\frac{1}{2}(a_{000})^2\lambda^{-1}+O(\lambda^{-1}\eta(\lambda)), \qquad (21)$$

$$\sin[tan^{-1}F_0(\lambda)]=a_{000}\lambda^{-\frac{1}{2}}+O\left(\lambda^{-\frac{1}{2}}\eta(\lambda)\right). \qquad (22)$$

Using (4), (11), (21) and (22) gives

$$\frac{P_{01}(\lambda)}{\cos[tan^{-1}F_0(\lambda)]}=\frac{\lambda^{r_0}+O(\lambda^{r_0-1})}{1-\frac{1}{2}(a_{000})^2\lambda^{-1}+O(\lambda^{-1}\eta(\lambda))}=\lambda^{r_0}+O(\lambda^{r_0-1}) \qquad (23)$$

and

$$\cos\left[tan^{-1}F_0(\lambda)+\int_0^x T\,(t,\lambda)dt\right]=\cos\left(\lambda^{\frac{1}{2}}x\right)-\lambda^{-\frac{1}{2}}\left[a_{000}-\frac{1}{2}\int_0^x q\,(t)dt\right]\sin\left(\lambda^{\frac{1}{2}}x\right)+O\left(\lambda^{-\frac{1}{2}}\eta(\lambda)\right). \qquad (24)$$

Finally; (10), (23) and (24) are replaced in (14) and the proof is done.
The proof of (ii) is similar.

We have also some approximations for the derivatives of the solutions, $u_-(x,\lambda)$ and $u_+(x,\lambda)$ of L.

Lemma 1. As $\lambda\to\infty$, we have
(i)

$$u'_-(x,\lambda)=-\lambda^{r_0+\frac{1}{2}}\sin\left(\lambda^{\frac{1}{2}}x\right)+\lambda^{r_0}\left[a_{000}-\frac{1}{2}\left(\int_0^x q\,(t)dt\right)\right]\cos\left(\lambda^{\frac{1}{2}}x\right)+O\left(\lambda^{r_0}\eta(\lambda)\right) \qquad (25)$$

(ii)

$$u'_+(x,\lambda)=\lambda^{r_1+\frac{1}{2}}\sin\left[\lambda^{\frac{1}{2}}(1-x)\right]-\lambda^{r_1}\times\left[a_{100}+\frac{1}{2}\left(\int_x^1 q\,(t)dt\right)\right]\cos\left[\lambda^{\frac{1}{2}}(1-x)\right]+O\left(\lambda^{r_1}\eta(\lambda)\right)$$

Proof. (i) The equality (7) is used for the proof. With the initial conditions (12) we have obtained the values of c_1 and c_2 as in (16) and (17). If these values are replaced in (7), it is simply derived that

$$u'_-(x,\lambda)=\frac{P_{01}(\lambda)}{\cos[tan^{-1}F_0(\lambda)]}\exp\left(\int_0^x S\,(t,\lambda)dt\right)\{S(x,\lambda)\cos\left[tan^{-1}F_0(\lambda)+\int_0^x T\,(t,\lambda)dt\right]-T(x,\lambda)\sin\left[tan^{-1}F_0(\lambda)+\int_0^x T\,(t,\lambda)dt\right]\}. \qquad (26)$$

We get the asymptotic approximation of (26) by using the results (10), (11), (21), (22) and (23). This gives the equality (25).

(ii) The proof is similar to part (i).

4. CONCLUSIONS AND DISCUSSION

Wang and the others' work [24] has motivated the author to determine the asymptotic formulae for the solutions. In [24], $q(t)$ is assumed to be L^2-function on (0,1) and the solutions $u_-(x,\lambda)$ and $u_+(x,\lambda)$ are derived asymptotically with error term of exponential type, that is,

$$u_-(x,\lambda)=\lambda^{r_0}\left(cos\,\rho x+O\left(\frac{e^{\tau x}}{\rho}\right)\right),$$

$$u_+(x,\lambda)=\lambda^{r_1}\left(cos\,\rho(1-x)+O\left(\frac{e^{\tau(1-x)}}{\rho}\right)\right)$$

where $\lambda=\rho^2$, $\tau=|Im\,\rho|$.

In this paper, the given results with (18) and (19) for the solutions of the problem L appear to be consistent those in [24]. Here, we only assume that $q(t) \in L^1[0,1]$ and find that

$$u_-(x,\lambda) = \cdots + O(\lambda^{r_0-\frac{1}{2}}\eta(\lambda)),$$

$$u_+(x,\lambda) = \cdots + O(\lambda^{r_1-\frac{1}{2}}\eta(\lambda))$$

where $\eta(\lambda) \to 0$ as $\lambda \to \infty$. Besides, we use a similar approach to Harris and obtain more precise asymptotics.

In future studies, the eigenvalues and corresponding eigenfunctions of the problem can be reconsidered under different restrictive conditions on the potential function q.

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