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Investigation of Mathematical Thinking Processes of Gifted Students in Different Environments: GeoGebra's Potential

Yavuz İsa AYGÜN[1]**, Keziban ORBAY**[2]**, Funda AYDIN GÜÇ**[3]

Abstract

Mathematical thinking is a higher-order thinking style specific to mathematics that allows the solving of problems. In this context, it is inevitable to consider mathematical thinking in determining giftedness specific to mathematics. How superior mathematical thinking should be measured and the potential of different environments to elicit this mathematical thinking are a matter of debate. In this study, it was investigated how mathematical thinking in gifted students differed between using a paper and a pencil and using dynamic geometry software. Since students' current mathematical thinking processes were examined, this study can be said as a case study. Three gifted students' solutions for given tasks in the paper-and-pencil and GeoGebra environments were compared within the scope of sub-dimensions (specializing, generalizing, conjecturing, and proving) of mathematical thinking. As a result of the study, the work undertaken by the students in the specializing step were seen to be similar in both the P&P and GeoGebra environments. On the other hand, it can be said that GeoGebra had the potential to reveal high-level work at the generalizing step. Different environments seemed to be important in revealing the ability to make assumptions. And it was seen that higher-order thinking skills for proof can be revealed with GeoGebra.

Keywords

Mathematical thinking
Superior abilities
Field-specific giftedness
Gifted students
Dynamic geometry software

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[1] Giresun Directorate of National Education, Şehit İsmail Kefal Anatolian Imam Hatip High School, Giresun, TURKİYE; https://orcid.org/0000-0002-6234-0559

[2] Amasya University, Faculty of Education, Department of Mathematics and Science Education, Amasya, TÜRKİYE; https://orcid.org/0000-0002-7642-4139

[3] **Corresponding Author,** Giresun University, Faculty of Education, Department of Mathematics and Science Education, Giresun, TÜRKİYE; https://orcid.org/0000-0002-3922-017X

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INTRODUCTION

Individuals encounter many problems in their daily lives and use various thinking structures to solve them. The relationship of mathematics with these thinking structures manifests itself as mathematical thinking, which is the use of mathematical techniques and methods in solving problems directly or indirectly (Henderson et al., 2003). Stacey, Burton and Mason (1985) describe mathematical thinking as specializing, generalizing, conjecturing, justifying, and convincing. Specializing can be expressed as concentrating efforts in a particular field of activity by trying special cases, examining examples (Stacey, 2006), and gathering evidence (Stacey et al., 1985). In addition to being a simple process, it is an introductory behavior. Generalizing is the activity of seeking out relationships and patterns (Stacey, 2006). Actions such as matching and ranking; comparing, classifying, and determining similarities and differences; expressing the relationship of two variables mathematically or verbally; and describing all possible situations are involved in generalizing (Mason, Burton & Stacy, 1991). Efforts are made to make inferences about the suggestion obtained from studies in special cases. Conjecturing is the process of predicting. It is the process of finding approximate answers without proper calculation; it is not a random event (Pesen, 2003). It includes examining why an assumption is correct and how it could be corrected if it is wrong (Stacey et al., 1985). Proving (can also be referred to as justifying or convincing) is defined as finding and expressing why something is true (Öztürk, 2013). It is the mental activity that individuals engage in to eliminate doubts about the accuracy of a claim (Harel & Sowder, 2007).

These steps are not hierarchical. They show dynamism based on the use of mathematical thinking. Sometimes, they emerge separately, whereas, at other times, they are intertwined with, and overlap, each other. In this context, the similarity with the mindset employed in the solution processes of the problems we encounter in daily life can be seen. Mathematical thinking is not only a thinking process that paves the way for finding an answer to a problem but also a higher-order thinking process that requires the management of processes that will solve a problem (Polya, 1945). Therefore, mathematical thinking could be said to be a higher-order thinking activity that is specific to mathematics and that is expected to occur in problem-solving processes. In this context, it could be said that it is a distinctive thinking structure in determining giftedness in the field of mathematics.

Giftedness in Mathematics

Jensen Sheffield (1994) lists the characteristics of giftedness specific to the field of mathematics as rapid learning process, disposition for observation skills, inquiry skills, capacity for comprehending extraordinary cause-effect relationships, and creativity. Jensen Sheffield (1994) indicates that students with mathematical giftedness may have higher-level skills than their peers in areas such as understanding, visualizing, and generalizing patterns and relationships; analytical, inductive, and deductive reasoning; reversing this reasoning process; working successfully with mathematical concepts; being persistent in solving relatively difficult problem situations and thinking abstractly. It is stated that mathematically gifted students differ in processes such as determining the relationships between different elements and generalizing mathematical ideas (Gutierrez et al., 2018). Chang (1985), on the other hand, demonstrates that a student's giftedness in mathematics does not mean that they are generally gifted. In this context, it could be said that it is important to employ field-specific talents in the identification of gifted individuals. In this study, it is accepted that giftedness in mathematics is a skill that is specific to the field and that there is a potential waiting to be discovered. Considering that one of the abilities specific to mathematics is mathematical thinking, it is important to investigate how to determine the giftedness of mathematical thinking.

The Need for a Different Environment in Revealing Superior Mathematical Thinking: Dynamic Geometry Software

Mathematics curricula can be arranged to improve students' mathematical thinking only if it is known how to identify students with mathematical thinking skills. Therefore, it is important to determine the environment wherein mathematical thinking can be revealed through studies so that students with different abilities can be identified. Examining and comparing the action taken by students in different environments while solving a given problem in terms of the steps of specializing, generalizing, conjecturing, and proving will help determine the potential of these environments to reveal their thinking processes. DGS is known to be effective in revealing many mathematical skills. Dede and Karakuş (2014) demonstrated that DGS could provide significant advantages in proofs and that different thinking structures can emerge in the process of proving with DGS. Yıldız (2016) and Baltacı et al. (2016) demonstrate that GeoGebra has an advantage over P&P in studies in revealing different solutions by, and creativity of, gifted students. Edwards and Jones (2006) state that dynamic software is useful not only in improving shape-forming skills but also in exhibiting mathematical thinking skills. Dynamic Geometry Software (DGS) is one of these environments.

DGS is known to be effective in revealing many mathematical skills. Dede and Karakuş (2014) demonstrated that DGS could provide significant advantages in proofs and that different thinking structures could emerge in the process of proving with DGS. GeoGebra, a dynamic software program, is known to be effective in revealing the skills of students by its positive effect at almost every step of the solution process, including establishing relationships between mathematical concepts and implementing different solution strategies (Hıdıroğlu & Bukova-Güzel, 2014). It is known that GeoGebra has a positive effect on making mathematical predictions (Baltacı, Yıldız & Kösa, 2015). Sarracco (2005) states that geometric constructions made with DGS help discover geometric relationships. Kondratieva (2013) indicates that DGS allows creating complex constructions and making generalizations from simple simulations. Stylianides and Stylianides (2005) and Köse, Uygan and Özen (2012) state that students have the opportunity to discover and verify by using the drag feature of the DGS. Yavuzsoy-Köse, Tanışlı, Özdemir-Erdoğan, and Yüzügüllü-Ada (2012) showed that DGS is more effective than the paper-and-pencil (P&P) method in detecting, discovering, and verifying geometric relationships. Yıldız (2016) and Baltacı, Yıldız, Kaymaz and Aytekin (2016) demonstrate that GeoGebra has an advantage over P&P in studies in revealing different solutions by, and creativity of, gifted students. Edwards and Jones (2006) state that dynamic software is useful not only in improving shape-forming skills but also in exhibiting mathematical thinking skills. Studies demonstrate that mathematical thinking can differ in different environments and different student groups. On the other hand, they emphasize that it is insufficient to measure giftedness with tests and in one dimension (Tarhan & Kılıç, 2014). In this context, it is important to investigate the potential of different environments in revealing these differences in children who have different thinking structures and higher-order thinking skills. A study to this end will fill this gap in the literature and present concrete evidence about how mathematical thinking in gifted students differs in different environments (P&P vs. DGS). In addition, the potential of GeoGebra in revealing superior mathematical thinking skills will be discussed. Thus, it will shed light on the learning environments to be designed to develop mathematical thinking skills. In this context, the problem to be addressed in this study is as follows: "*What is GeoGebra's role in determining superior mathematical thinking ability for giftedness students?*"

METHOD

In this study, mathematical thinking processes were analyzed in different environments. In this context, the qualitative research design was adopted because it implies an intensive, holistic description and analysis of the phenomenon or social unit. Research designs describe a process in which boundaries are determined (Yıldırım & Şimşek, 2013). The focus of each qualitative research design is different. The focus of the case study is to try to describe an event as it exists (Leymun, Odabaşı & Yurdakul, 2017). Case studies differ from other designs in terms of examining a single unit or a limited system, making intensive descriptions, and interpreting depending on the context (Hancock & Algozzine, 2006). In the case study design, it is aimed to comprehensively examine a situation in its natural environment and to determine its components (Subaşı & Okumuş, 2017). There is no intervention of the researcher in the process, environment and ongoing actions (Yin, 1994). In this study, the mathematical thinking skills of students diagnosed as gifted were examined and compared in different environments. One-on-one interviews were required with the students and an in-depth analysis was required to reveal their current mathematical thinking processes. In this context, it can be said that the study is a case study.

Participants

The study was carried out with three 7th graders diagnosed as gifted—two female and one male. The students were selected on a voluntary basis among those attending a Science and Art Center, an institution that undertakes the education of gifted and talented students in Turkey during the academic year 2016–2017. While these students were in the 2nd grade of primary school, they were diagnosed as gifted by using the WISC_R intelligence test. They were entitled to enroll at the Science and Art Center based on the quota determined that year.

Collection of Data

The data of the study was obtained in two step clinical interviews. The first is clinical interviews conducted in paper-and-pencil solutions of the activities, and the second is clinical interviews in solutions of the same activities with DGS.

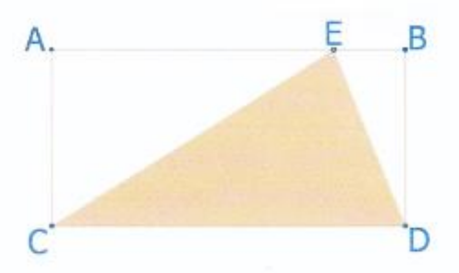
Activities

Researchers designed six activities containing tasks that could be solved with both P&P and DGS and could reveal the mathematical thinking processes of participants. GeoGebra was taken as preferred DGS environment to reveal mathematical thinking in the present study.

The activities were examined by an expert in the field of mathematics education in terms of the suitability of the problems and tasks for the purpose. The corrections suggested by the expert were made. Pilot implementation of the activities in both P&P and GeoGebra environments was carried out with a student who was diagnosed as gifted but who would not participate in the study. The solution processes were discussed with the student, and the changes to be made in the problems and tasks in the activities were determined. Incomprehensible sentences and tasks with low potential to reveal the targeted process were edited.

We can exemplify the potential of activities to reveal mathematical thinking skills with an activity. For this purpose, the second activity is given below.

Activity 2:



The long side of the rectangle above is 10 units and the short side is 6 units.

a) Accordingly, find the area of triangle CED.
b) What is the pattern/relationship between the area of the triangle and the area of the rectangle in the figure? Explain.
c) Can you show with a formula or express in words the relationship/pattern between the areas of all triangles and rectangles that are like this (the base is on one side of the rectangle, the vertex is on the opposite side of the rectangle)?
d) How can you show that your formula is absolutely correct? How can you explain that it is valid in all cases?

This activity includes a context that aims to enable the student to make a general judgment by inferring from specific situations. The aim was to reveal students' mathematical thinking skills regarding specializing with the first sub-question, generalization with the second sub-question, conjecturing with the third sub-question, and persuasion/proof processes with the fourth sub-question. In this context, it can be said that the activity allows revealing all components of mathematical thinking skills. All other activities similarly include four sub-questions for four sub-processes.

Clinical Interviews

To determine the mathematical thinking processes of the students, clinical interviews were conducted about the problem-solving processes in the activity. Before the interview, the researchers discussed the situations that might arise and determined the interview questions that would reveal the thinking structure of the students. The sample questions were as follows: "Can you explain the steps you followed for a solution with reasons?" or "How did you reach this result?" or "Why did you give up on this solution?" The interviews were recorded on video to prevent any data loss. A similar procedure was used for clinical interviews for both P&P and GeoGebra solutions.

Implementation Process

The implementation process carried out within the scope of the study is given in Figure 1.

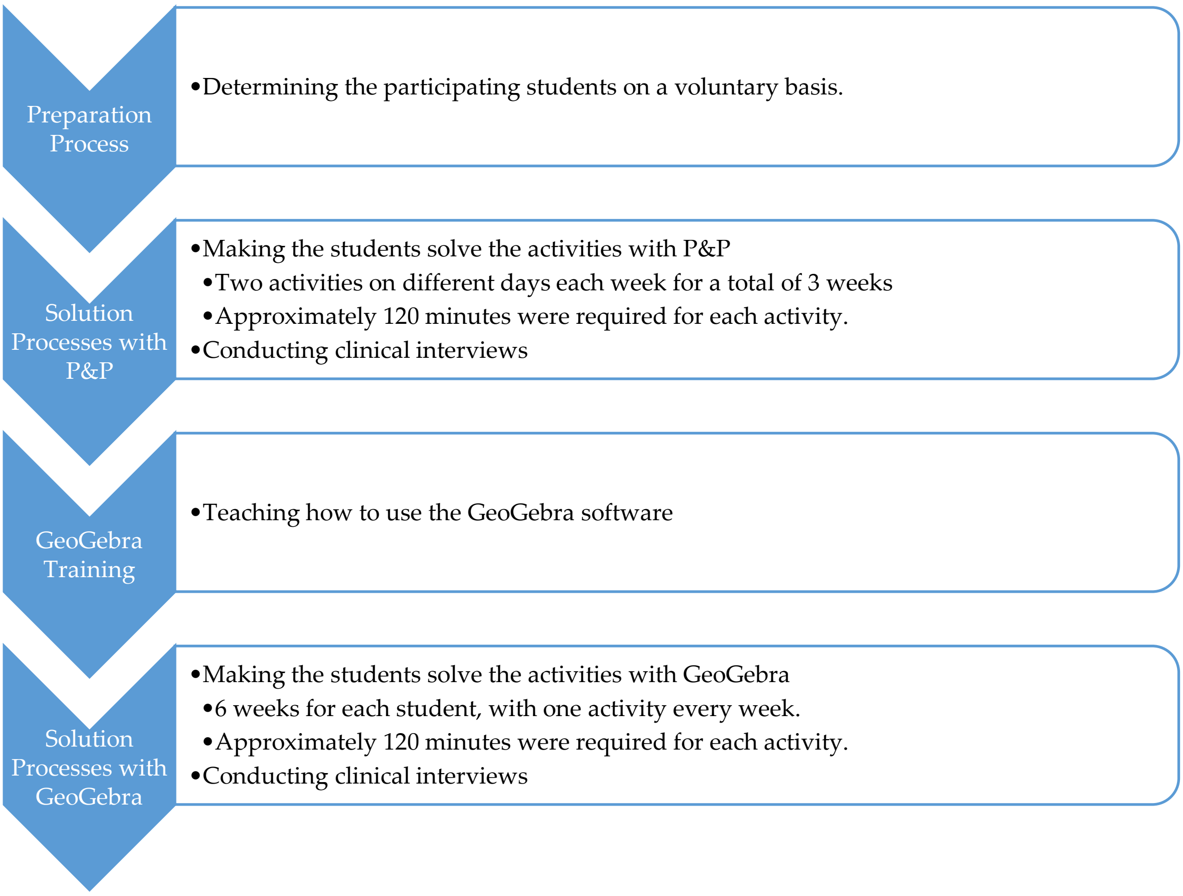


Figure 1. Implementation process

Before the implementation of the activities in the GeoGebra, the interface of GeoGebra was customized. Necessary restrictive changes were made to the software according to the features specific to the problem situations in the activities. The functions that made it possible to find results without allowing any mathematical thinking skill to be observed were disabled. For example, to observe the mathematical thinking process for calculating distances, it was necessary to disable counting based on isometric lengths. In this context, the coordinate system tray was disabled for the relevant activity. Under these circumstances, the students were asked to solve the same activities with the help of GeoGebra.

Data Analysis

To reveal mathematical thinking in detail, definitions in the literature were examined and the process was discussed in four steps: specializing, generalizing, conjecturing, and proving. In the pilot implementation process, it was observed that the students carried out different work for the same step in different activities. According to the pilot study, sub-steps/codes and descriptions were determined. Eventually, it was deduced that mathematical thinking comprises nine sub-steps belonging to the four steps, as indicated in Table 1. The sub-steps were considered codes to determine the mathematical thinking process.

Table 1. Mathematical thinking process, sub-steps, and descriptors

Mathematical thinking process	Sub-steps /codes	Descriptors
Specializing	S1	• Examining unique examples specific to the given situation through a correct process
Generalizing	G1	• Testing different special cases • Conducting correct work on the trial and error process
	G2	• Determining the relationship specific to the given situation accurately and completely
Conjecturing	C1	• Estimating and expressing the insight realized
	C2	• Testing, being at the beginning of the proof or making sure that the assumption is not wrong
	C3	• Expressing/formulating the assumption verbally or algebraically
Proving	P1	• Heuristic verification: sensing the correctness of the situation with his feelings and expressing it
	P2	• Making inductive explanation • Testing the accuracy of the assumption with examples
	P3	• Making generalizations with special changes instead of special examples • Structuring the proof with correct reasoning

As with the components of the mathematical thinking process, there was no hierarchy among the subcomponents. That is to say, P3 in the proving step can be seen before P1 appears in that the behavior of making generalizations with variables could be observed before that of attempting to demonstrate the assumption with simple drawings. However, P3 represents a higher level of behavior than P1. The data obtained from the main work were subjected to descriptive analysis within this theoretical framework. To ensure the reliability of the analysis, all work conducted by a student in all activities was examined by two researchers who were field experts. This way, efforts were made to ensure analysis reliability.

Research Ethics

As the authors of the research, we declare that scientific and ethical rules are followed in this article and that the article does not require the permission of ethical committee for the reason that the data used in this study belong to before 2020.

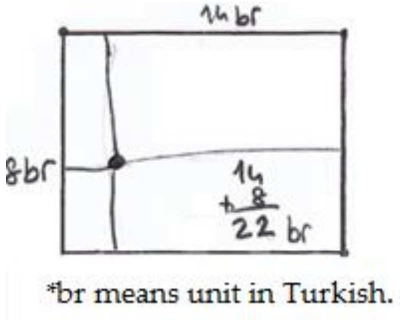
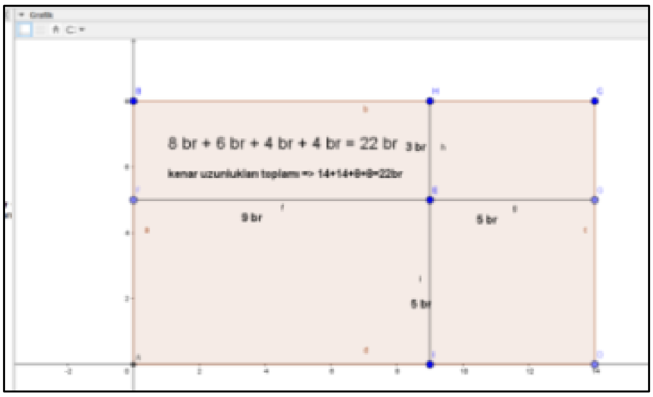
RESULTS

In this section, results from Activity 1 were discussed in detail for third gifted student (GS3) (selected randomly), and then, the process that emerged in all six activities was evaluated as a whole for three students.

Works Carried Out at the Specializing Step

In Activity 1, to observe the specializing process, students were given the following task: "Find the sum of the distances of a randomly chosen point from the four sides of the rectangle, whose short side is 8 units and long side is 14 units." The answers given by GS3 to this task are given in Table 2.

Table 2. GS3 responses to the first task in P&P and GeoGebra environments

P&P process	
GeoGebra process	

GS3 combined a point on the paper with the lines drawn toward the sides and sensed that the sum of these was equal to the sides, which, in turn, contributed to achieving evidence for this situation. When asked to explain what process he followed, he said, "*It's the same line with the sides*." He demonstrated S1 with this approach. Using GeoGebra, GS3 selected a random point and summed up its distances to the sides, thus demonstrating S1. GS3 explained his work, saying, "*I took a random point E, and I calculated its distances to the sides as 22.*" It can be seen that GS3 exhibited S1 in GeoGebra. For all activities, the situation of the students exhibiting S1 in P&P and GeoGebra environments is given in Figure 2.

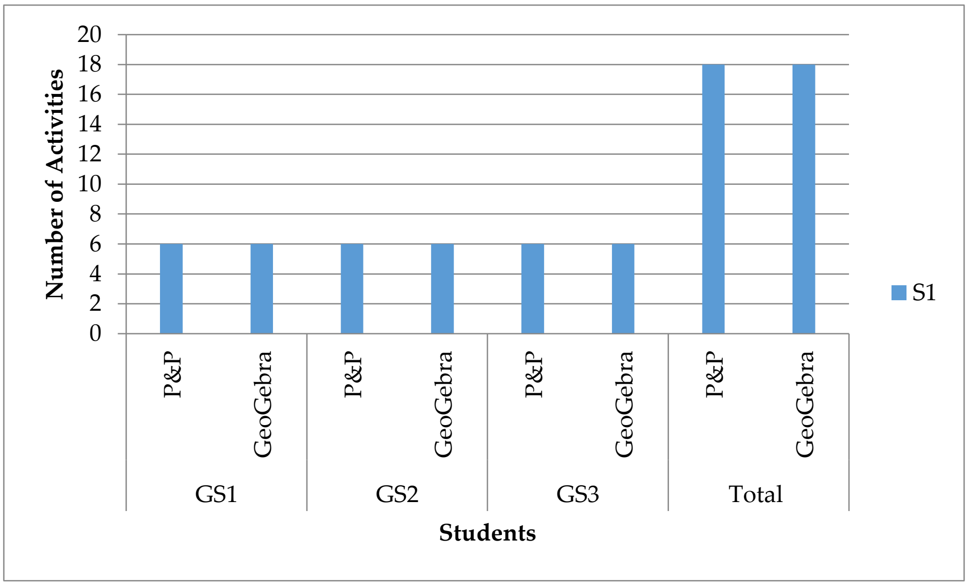


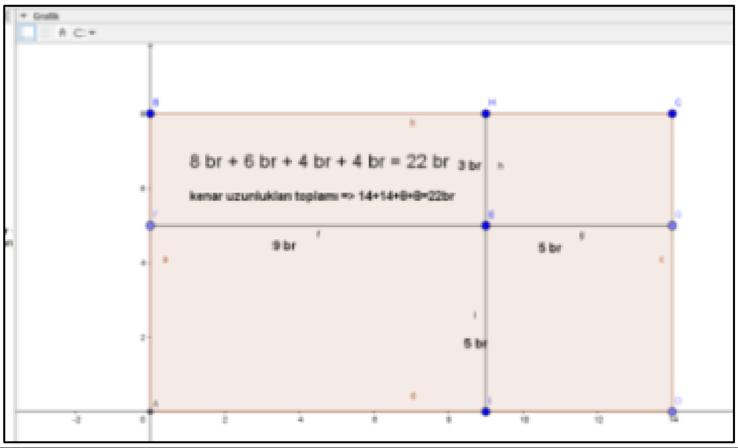
Figure 2. Overall observation of the specializing step

In the Figure 2 which is about specializing step, it was observed that all students undertook the work to find the answer to the problem situation in the activity given in both P&P and GeoGebra environments, worked on the example for the original situation given, and gathered evidence appropriate to the situation. In the specializing step, it was seen that S1 was exhibited by all students in both environments (in six activities for all students).

Works Carried Out in the Generalizing Step

To observe the generalizing process, the students were given the following task: "When you review the process of finding the result above, is there a relationship between the sum of the distances of other randomly selected points inside the rectangle from the sides and the lengths of the sides of the rectangle? How would you explain this relationship?" The answers given by GS3 to this task are given in Table 3.

Table 3. GS3 responses to the second task in P&P and GeoGebra environments

P&P process	The sum of one short side and one long side equals the sum of the shortest distances of any selected point to all sides. That is half the sum of all the sides of the rectangle. *All responses are translated English by researchers.
GeoGebra process	

GS3 stated that the line segments indicating the distances of any point to the sides would be equal to the short and long sides when examined horizontally and vertically, regardless of where the point was chosen, and tested different special cases, thus exhibiting G1 and G2 in the same process. In GeoGebra, he intersected two perpendicular lines inside the rectangle he created with two pairs of parallel lines, and he took this intersection point as a randomly selected point. The dialog between the researcher and GS3 showing that the correct attempts he made to find the relationship and exhibited G1 and G2 is as follows:

R: Is there a relationship or pattern?

GS3: Teacher, the distances of this point (pointing to the intersection point) to the sides are these parts (showing the line segments of the point to the sides)… For example, this is the same length as the side it is parallel to, and this applies to this one as well. Therefore, regardless of where we select the point (moving the point by dragging it), it will be the same as the length of the sides.

GS3 exhibited the G1 and G2 codes of the generalizing step by using similar explanations in the P&P and GeoGebra processes. The status of exhibiting G1 and G2 in P&P and GeoGebra environments by the students for all activities is given in Figure 3.

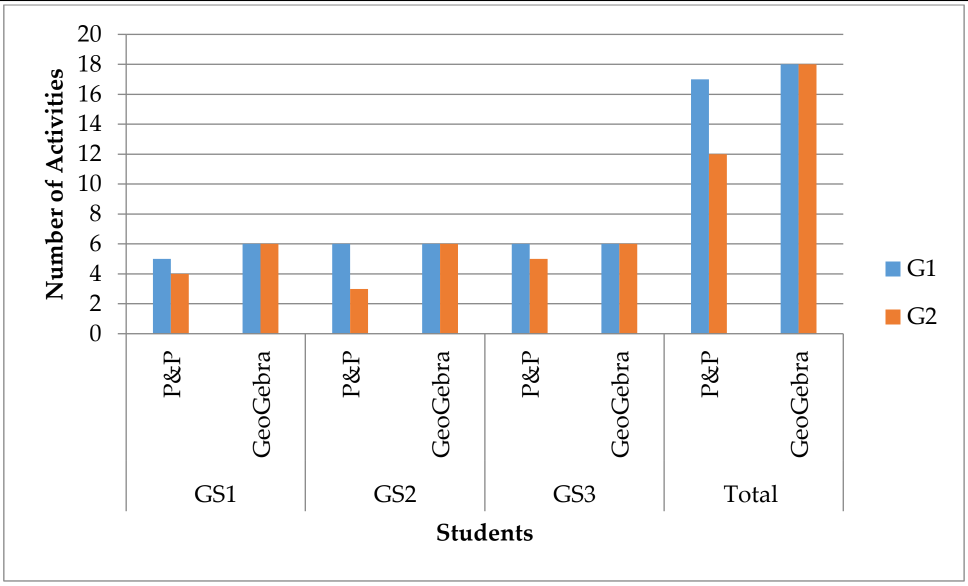


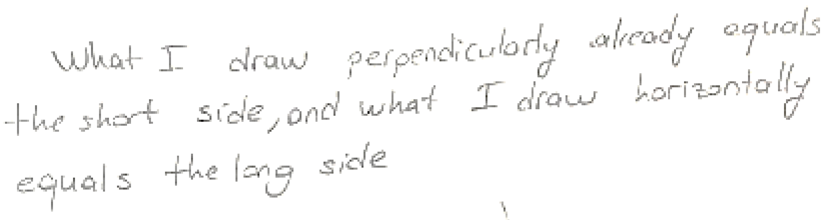
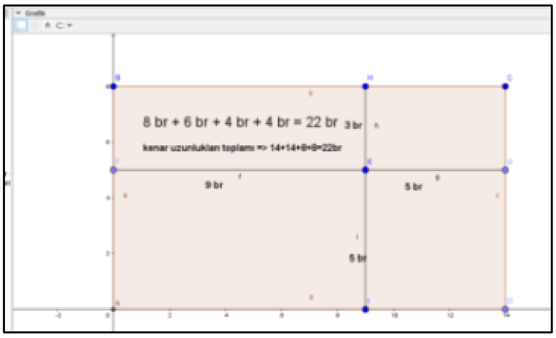
Figure 3. Overall observation of the generalizing step

When Figure 3 was analyzed, it was seen that only GS1 could not exhibit G1 in the P&P environment in an activity. Upon testing different situations for all students in all other activities, it was seen that there was no differentiation in P&P and GeoGebra environments in G1, which contains the correct traces of the trial–error process. In the G2 step, which contains the pattern of the given situation, was observed in more activities in the solution processes in GeoGebra. At this step, it can be said that the work conducted with GeoGebra differed positively. When six activities and three students were assessed in general, the emergence of the codes at the generalizing step in 18 activities, the generalizing process can be said to have differed positively in the GeoGebra environment. It seems that this differentiation was particularly noteworthy for G2.

Works Carried Out in the Conjecturing Step

To observe the conjecturing process, the students were given the following task: "Can you make a verbal or mathematical expression for the sum of the distances of a point inside any rectangle with different lengths from the sides? For example, can you express it with a formula or using words?" The answers given by GS3 to this task are given in Table 4.

Table 4. GS3 responses to the third task in P&P and GeoGebra environments

<table>
<tr><td>P&P process</td><td>
</td></tr>
<tr><td>GeoGebra process</td><td>
</td></tr>
</table>

In the conjecturing step, GS3 repeated his answer in the generalizing step in P&P. When his answer was examined in the context of conjecturing, the student was seen exhibiting C1 by expressing the insight realized by them, exhibiting C2 by checking the accuracy of this situation, and exhibiting C3 by verbally formulating his assumption. The clinical interview conducted with GS3 to obtain detailed information about the process in GeoGebra is as follows:

R: If the side lengths of the rectangle had changed, could we have made a comment about it?

GS3: It would be the same again; the distances to the sides would give the side again.

R: Can you write this down algebraically?

GS3: It gives the lengths of the sides, teacher, whatever the sides are.

GS3 exhibited C1 by expressing the predictions they realized, C2 by checking the accuracy of this situation, and C3 by formulating an expression of their assumption verbally, even if he could not express it algebraically, as seen in the dialog about the process in GeoGebra, similar to P&P. The status of students exhibiting C1, C2, and C3 in P&P and GeoGebra environments for all activities is specified in Figure 4.

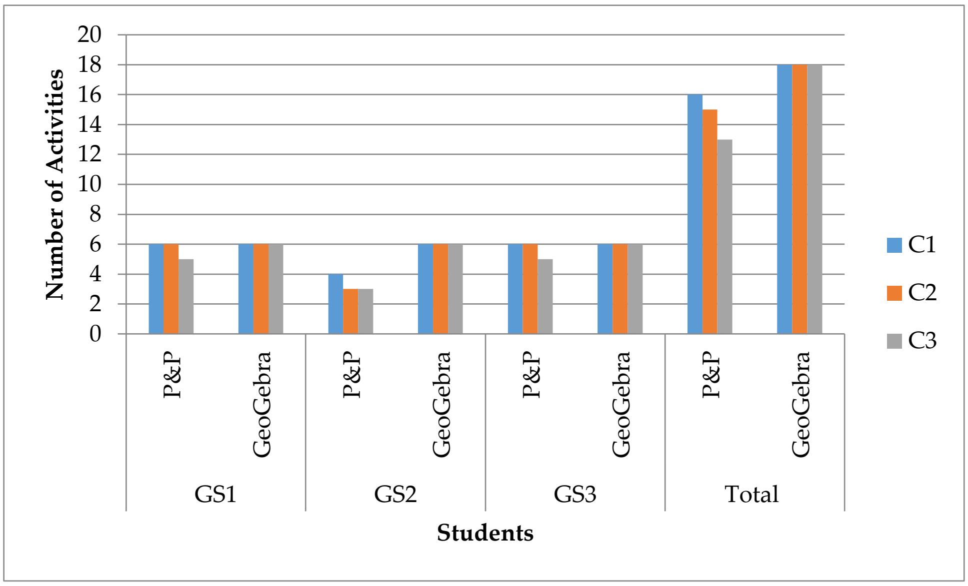


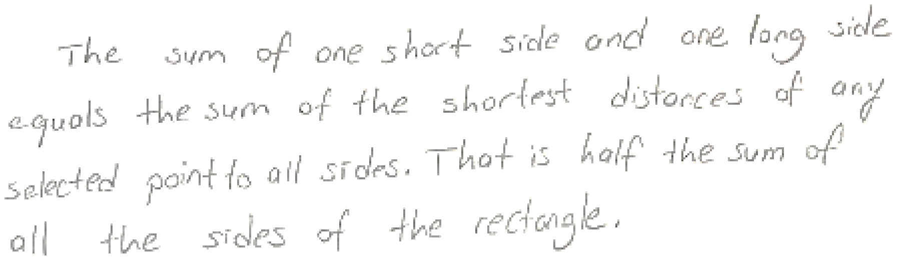
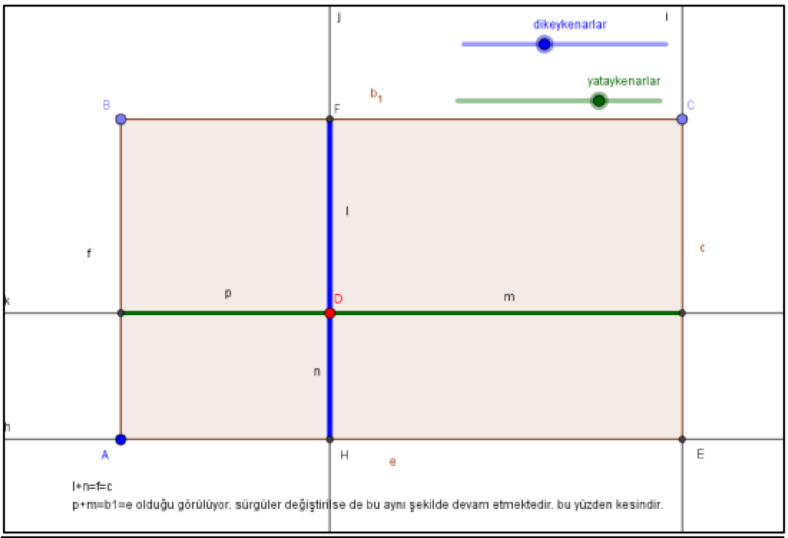
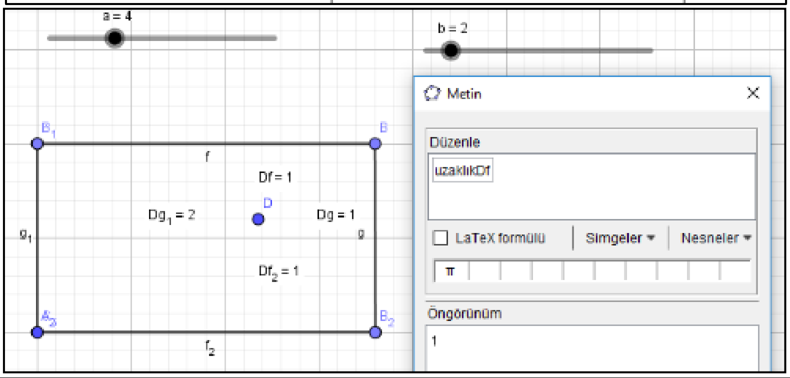
Figure 4. Overall observation of the conjecturing step

When Figure 4 was examined, it was seen that C1, which contains the prediction and expression of the noticed pattern, was exhibited in more activities in GeoGebra than those exhibited in P&P by only one student. The C1 behaviors of other students were seen as not differing in different ways. It was seen that for all activities in GeoGebra, the students exhibited the C2 step, which contains the stages of checking patterns claimed verbally or visually and returning to the beginning if it is wrong. It is worth noting that the GeoGebra work differed for a student in three activities. In this context, it can be said that the P&P and GeoGebra environments led to differentiation in terms of a student revealing C2. Upon examining Figure 4, noteworthy, C3, which contains the behavior of expressing the found pattern algebraically or verbally, was exhibited in all activities in GeoGebra and did not apply to P&P. In this context, it can be said that P&P and GeoGebra environments differed in favor of GeoGebra in terms of revealing C3. When six activities and three students were assessed in general, the emergence of the codes at the conjecturing step in 18 activities, it can be said that the conjecturing process differed in the GeoGebra environment. It is noteworthy that the differentiation increased in favor of GeoGebra, particularly as one moved toward the advanced behavior of conjecturing.

Work Carried Out in the Proving Step

To observe the proving process, the students were given the task: "If we think about all rectangles, how would you show the accuracy of your generalization above for the distances of a chosen point from the sides? How can you be sure that this relationship is valid for all rectangles?" The answers given by GS3 to this task are given in Table 5.

Table 5. GS3 responses to the fourth task in P&P and GeoGebra environments

P&P process	
GeoGebra process	

GS3 explained the pattern with an inductive explanation in the generalizing step in the P&P process and exhibited the P2 behavior of the proving step. However, he did not engage in work regarding P1 and P3. GS3 assigned a variable to the distance to the sides in GeoGebra. He showed the special changes by adding the code of the sum of the variables they had assigned in the code entry screen of GeoGebra as objects to the screen. This way, he performed formal proving for their generalizations. Thus, he exhibited P3. However, he did not engage in work regarding P1 and P2. The status of the students exhibiting P1, P2, and P3 in P&P and GeoGebra environments for all activities is given in Figure 5.

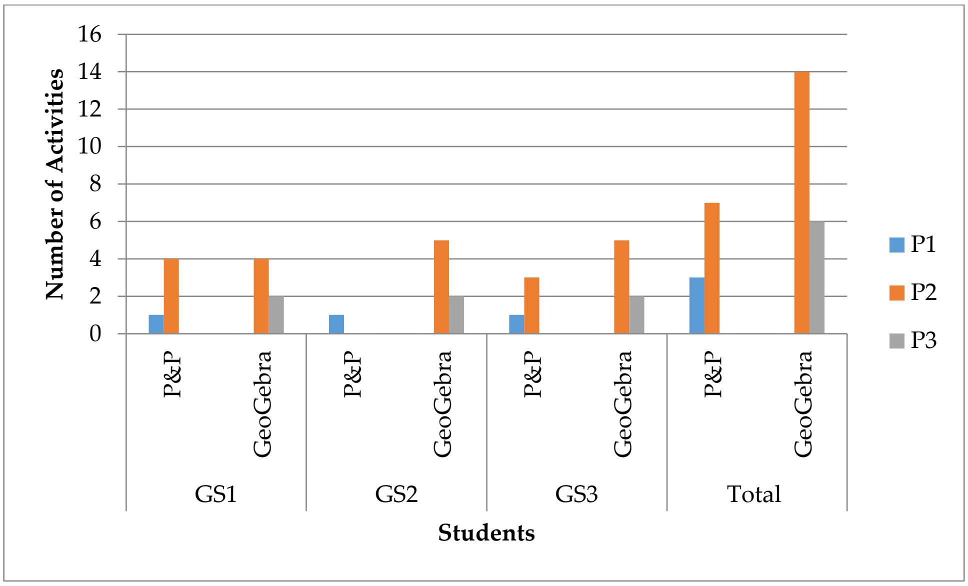


Figure 5. Overall observation of the proving step

Upon examining Figure 5, the behaviors involving P1, which contains attempts to demonstrate the claim with simple drawings not containing any strong evidence, were seen only in one activity each in solutions with P&P. This behavior was never observed in GeoGebra. This indicates that the exhibition of P1 in both environments was limited. In addition, it was seen that there were different situations for each student in the P2 step, which contains the behavior of demonstrating accuracy of the assumption by putting forth one or more special cases of quantitative evaluations. GS1 exhibited P2 in an equal number of activities in both settings, while GS2 did not exhibit any behavior regarding this code in any activity during P&P, but exhibited P2 in five activities in GeoGebra. GS3, on the other hand, exhibited P2 in more activities in GeoGebra. It can be said that GeoGebra ensured differentiation between GS2 and GS3 in terms of displaying P2. When Figure 5 was examined, P3, in which the students were expected to reach generalizations with special changes, was not seen to emerge in any activity in the P&P environment. This code emerged in two activities for each student in GeoGebra. It can be said that GeoGebra ensured differentiation for P3. When six activities and three students were assessed in general, the emergence of the codes at the proving step in 18 activities, it is noteworthy that the differentiation increased in favor of GeoGebra toward further behaviors of the proving process. It was observed that a simple proving process such as heuristic verification occurred only in the P&P environment. It is noteworthy that the process requiring inductive explanation occurred in both environments but that there was significant differentiation in favor of GeoGebra. In the process that required high-level abstract proving, GeoGebra was seen to have played an important role in revealing the behavior.

CONCLUSION AND DISCUSSION

As a result of this study, the work undertaken by the students in the specializing step were seen to be similar in both the P&P and GeoGebra environments. However, studies show DGS as useful in helping discovering geometric relationships (Köse et al., 2012; Sarracco, 2005; Stylianides & Stylianides, 2005; Yavuzsoy-Köse et al., 2012). In the present study, on the other hand, there was no difference between P&P and GeoGebra environments in terms of exploring the relationships. Specializing includes behaviors that can be considered easy, such as examining examples (Stacey, 2006) and making generalizations by gathering evidence (Stacey et al., 1985). Specializing is not only a simple process but also an introductory behavior. In the present study, the students were able to perform specializations related to the given problem situations in the P&P environment. Gifted students are already successful at this step. They do not need a new environment to do this. Therefore, GeoGebra did not provide them with any additional experience.

In the present study, generalizing was handled in two different steps. The first was G1, which included testing different special cases, making correct attempts to identify the relationship and pattern, and trial and error work. The gifted students were already able to perform such work easily in the P&P environment. Therefore, they did not need a different environment. In this context, no differentiation was observed between the P&P and GeoGebra environments in terms of testing different special cases, making correct attempts to find the relationships and patterns, and trial and error work. The second step was that where concrete generalizations needed to be made, which included determining the given situation-specific pattern and relationship, called G2. Although the gifted students perfectly performed the trial and error work specific to the situation given in the P&P environment, they could not always clearly reveal the pattern or relationship. However, when they worked in GeoGebra environments, they were able to reach correct and complete relationships and patterns by concluding their trial and error work and their search for relationship. G2 required higher-level work than G1. In this context, it can be said that GeoGebra had the potential to reveal high-level work at the generalizing step. In the literature, it was emphasized that the gifted students were successful in understanding patterns and relationships, and generalizing (Jensen Sheffield, 1994). There are studies in the literature, emphasize that DGS allowed generalizations (Kondratieva, 2013). In the present study, the functions of the different environments in the work for the sub-steps of generalizing were determined. Thus, it can be said that GeoGebra was functional in revealing high-level skills for the generalizing step of mathematical thinking.

In the present study, the conjecturing step was taken as comprising three sub-steps. Two students were very successful in the P&P environment in three steps. Therefore, they were able to make assumptions without the need for a different environment. However, the same could not be said for GS2. GS2 was not as successful in the three sub-steps of conjecturing in the P&P environment. However, when they were working in the GeoGebra environment, they were able to perform in accordance with the three sub-steps of conjecturing in all activities. We see that one of the three students diagnosed as gifted had a different level of mathematical thinking process than the other students did. This was applied to the P&P environment. When we provided a different environment (GeoGebra) to the student, they were able to exhibit high-level conjecturing behavior for each activity. On the other hand, as we moved from verbal assumptions to abstract assumptions (from C1 to C3), the potential of GeoGebra became clearer. Although GS2 seemed to be at a disadvantage when compared with the other two students in the P&P environment, it was seen that they had similar mathematical thinking skills when they were given appropriate opportunities. Different environments seemed to be important in revealing the ability to make assumptions. Similarly, studies emphasize that DGS provided a positive effect on making mathematical predictions (Baltacı et al., 2015). In the present study, the potential of the GeoGebra environments in revealing the conjecturing behavior in a high-level behavior such as mathematical thinking was identified.

The proving step was discussed in three sub-steps. The first step, called P1, involved demonstrating the accuracy of the claim using heuristic strategies. While the students were working in only one activity for this step in the P&P environment, they never worked in the GeoGebra environment. This is because at the above-mentioned activities, GeoGebra provided them with formal proofs. In the GeoGebra environment, students could work directly on formal proofs without the need for heuristic verification. In general, it was seen that gifted students did not tend to validate heuristically in either environment. This shows that gifted students did not need heuristic verification but tended toward high-level proof such as formal proofs. Thus, it can be said that the GeoGebra environment was not suitable for enabling heuristic verification. P2 required demonstrating the accuracy of the assumptions with the help of special cases. It was seen that GeoGebra provided opportunities for working toward this step. When the students' work for this step was compared, it was seen that GS2 and GS3 were disadvantaged in the P&P environment. However, when they were offered the opportunity to work in GeoGebra, they were able to exhibit work on this step in more activities than GS1 was. This shows that the provision of suitable environments will make a difference in determining superior ability for mathematical thinking. Work on the step involving formal proofs, such as transformational abstraction, called P3, and making generalizations reveals this situation more clearly. No student could work on this step in the P&P environment, whereas three students were able to work on this step in the GeoGebra environment, albeit in a small number of activities. This shows that higher-order thinking skills for proof can be revealed with GeoGebra. Similarly, it is known that DGS provides the opportunity to demonstrate the proving process and to reveal different thinking structures (Dede & Karakuş, 2014) and verify the discovered relationships (Köse et al., 2012; Stylianides & Stylianides, 2005; Yavuzsoy-Köse et al., 2012). In the present study, the difference in the sub-steps of proving was revealed.

Studies reveal DGS's effectiveness in establishing relationships between mathematical concepts, realizing different solution strategies, and revealing the skills of students by exerting a positive effect on almost every step of the solution process (Hıdıroğlu & Bukova-Güzel, 2014). Yıldız (2016) and Baltacı et al. (2016) demonstrated that gifted students were able to come up with different solutions in their P&P and GeoGebra work, and GeoGebra provided opportunities to reveal the creativity of gifted people. Edwards and Jones (2006) stated that dynamic software helped not only improve shape-forming skills but also exhibit mathematical thinking skills. In the present study, it was seen that GeoGebra was effective in revealing higher-order thinking structures in some steps of mathematical thinking. Unlike the literature, it was observed that different environments (P&P and GeoGebra) were not discriminating in determining giftedness in some simple steps for gifted students. In the future researches, it can be investigated whether the results change by making the customization step of the activities more difficult. Considering the potential of GeoGebra to reveal higher-order thinking skills, it can be said that using it in determining field-specific giftedness can change the results in diagnosing giftedness. In this context, GeoGebra can be used as a tool to determine giftedness for mathematical thinking. In addition, GeoGebra can be employed in learning environments because it can contribute to the diagnosis of giftedness at an earlier stage. It is also emphasized that the mathematical thinking processes of gifted students can be positively affected when appropriate differentiated instructions are provided (Mohd-Hasrul et al., 2022). Considering the potential of GeoGebra revealed in this study, this effect can be investigated in environments enriched with GeoGebra. Jablonski and Ludwig (2022) infer that the reasoning skills of mathematically gifted children may change over time. It can be investigated whether mathematical thinking skills determined in GeoGebra-supported environments change over time.

Although the potential of GeoGebra to reveal superior ability for mathematical thinking is demonstrated in the present study, there are some limitations, the most important of which is that the study is limited to three students. It was observed that different environments provided some students with more opportunities than they did to others. In this context, it can be examined whether the results would change by increasing the number of students. In addition, in the present study, field-specific giftedness was limited to mathematical thinking. The role of GeoGebra in determining giftedness for other field-specific skills can be examined.

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