Finding the Lie Symmetries of Some First-Order Odes via Induced Characteristic

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Abstract

In this paper, the first-order ODEs which have no systematic way to find their Lie point symmetries - unlike higher order ODEs which have systematic ways- are reconsidered. As a first step, we considered first order PDEs which correspond to these equations by introducing reduced characteristic Q that used in the Lie's theory. Following this step, we tried to obtain solutions of the PDEs using their Lie point symmetries. But in this process, we met some difficulties, so by taking into account some assumptions we obtained the symmetries of ODEs which are in the special form, and also their solutions.

Keywords — Lie point symmetry, Ordinary differential equations, Ricatti differential equations, scale symmetry, prolonged infinitesimals.

1 Introduction

Lie point symmetries of the classic form first-order ODEs when they are linear, homogenous and exact can be easily found, that's the reason why one can generally say that this method is more attractive then others[2]. However, for first-order ODEs, Lie's method is not as useful as for the high order case because of the fact that determining equations whose solutions give Lie symmetries have the original differential equations in its characteristic strip. That's why some researchers, E.S. Cheb-Terrab, A.D. Roche (1998)[4], T. Kolokolnikov and his colleagues (2003)[3], used an alternative approach in which they tried to establish a catalog of ODEs which has special forms by introducing some special Lie point symmetries like $[\xi = 0, \eta = F(x)G(x)], [\xi = F(x), \eta = P(x)y + Q(x)]$ and $[\xi = 0, \eta = F(x)(y - H(x))]$ for some arbitrary functions. Last tangent vector leads to Riccati differential equations. But unfortunately these ODEs can not be solved because of including same differential equations in H. In addition there exists another way to solve ODEs providing that there can be found all symmetries of corresponding second-order ODEs and so the authors have solutions of Riccati differential equations considered by using Lie's symmetries obtained (2013)[1]. Sometimes inspite of the presence of upper methods, it may be really difficult to obtain symmetries of first-order ODEs because of having powerful nonlinearity part. Our study introduces how to find such symmetries which overcome this issue.

2 Finding Lie Symmetries of Some First-Order ODEs

Consider first-order ODE

$$y' = \omega(x, y) \tag{1}$$

and following differential operator

$$X^{(1)} = \xi \partial_x + \eta \partial_y + \eta^{(1)} \partial_{y'}$$

which is called prolonged infinitesimals of Lie group where $(\xi(x,y),\eta(x,y))$ is infinitesimals and $\eta^{(1)} = D_x\eta - y'D_x\xi$ for total derivative operator D_x. Taking into account (1), the linearized condition for the differential equation is

$$X^{(1)}\big(y' - \omega(x,y)\big) = 0$$

and this leads to

$$\eta_x + (\eta_y - \xi_x)\omega - \xi_y\omega^2 - \xi\omega_x - \eta\omega_y = 0. \tag{2}$$

When this differential equation is solved, Lie symmetry group of (1) could be found, which is also a partial differential equation including two dependent variables and two independent variables. Sometimes this equation can be hard to solve, that's why we are using another approach here to obtain the Lie's symmetries of (1).

3 Our Approach

Lets begin with following function

$$Q(x,y) = \eta - \omega\xi$$

so-called induced characteristic and remember (1). Using this we have

$$Q_x + \omega Q_y = \omega_y Q \tag{3}$$

Then tangent vector of (1) is $(\xi, Q + \omega\xi)$ where $\xi = \xi(x,y)$ is an arbitrary function. In order to obtain Q, we use Lie's point symmetries of (3). Then we have the determining equations as follows:

$$\begin{aligned}\xi\omega_x + \tau\omega_y + Q\omega\omega_y\xi_Q + \omega\xi_x + \omega^2\xi_y - Q\omega_y\tau_Q - \tau_x \\ - \omega\tau_y = 0 \\ -Q\xi\omega_{xy} - \eta\omega_y - Q\tau\omega_{yy} + Q\eta_Q\omega_y + \eta_x + \omega\eta_y \\ -Q^2\omega_y^2\xi_Q - Q\omega_y\xi_x - Q\omega\omega_y\xi_y = 0\end{aligned} \tag{4}$$

where (ξ,τ,η) is tangent vector for (3)[2]. This is more complicated and difficult than (2). Thus ansatz can be used, for example, scale symmetries

$$\big(\hat{x},\hat{y},\hat{Q}\big) = (xe^{a\varepsilon}, ye^{b\varepsilon}, Qe^{c\varepsilon})$$

where a, b and c are constants. Then substituting these new variables into the equation (3) we have

$$\begin{aligned}e^{(a-b)\varepsilon}\hat{Q}_{\hat{x}} + \omega(e^{-a\varepsilon}\hat{x}, e^{-b\varepsilon}\hat{y})\hat{Q}_{\hat{y}} \\ = \omega_{\hat{y}e^{-b\varepsilon}}(\hat{x}e^{-a\varepsilon}, \hat{y}e^{-b\varepsilon})\hat{Q}e^{-b\varepsilon}\end{aligned}$$

So the variables become

$$(\hat{x},\hat{y},\hat{Q}) = (x, y, Qe^{c\varepsilon})$$

But this symmetry is not useful because from invariant surface condition it is found

$$\eta - y'\xi = \omega\xi - y'\xi = \xi(\omega - y') = 0$$

for (1), therefore this is the trivial symmetry for (1). As it is seen from above it is convenient to seek point symmetries which have the form

$$\big(\hat{x},\hat{y},\hat{Q}\big) = \Big(x, y, \hat{Q}(x,y,Q)\Big)$$

because of the fact that ω is arbitrary and depends on just x and y. Then the determining equations (4) become

$$\eta_x + \omega\eta_y + Q\omega_y\eta_Q - \omega_y\eta = 0. \tag{5}$$

If we find η which has the form $\eta = Q - f(x,y)$, then we can obtain solutions of (1) using canonical coordinate

$$r = x \qquad s = \int \frac{dy}{f(r,y)}\bigg|_{r=x}$$

In fact it can be found for $f = f(x)$ and $f = f(y)$ easily but these are valid for just linear and separable ordinary differential equations, respectively. Furthermore for other forms that can be chosen arbitrarily, this choice leads to $f_x + \omega f_y = a_y f$ equation that seems (1).

Therefore we use another approach which includes $Q = \varphi(x)\psi(y)$ and the method of separation of variables. Then we obtain

$$(\xi,\eta) = (0, \exp\,(\lambda x + \textstyle\int f\,(y)dy)) = (0, \psi(y)\exp\,(\lambda x))$$

for

$$y' = \lambda\psi(y)[\int^{\tau=y} \frac{d\tau}{\psi(\tau)} + g(x)]$$

and

$$(\xi,\eta) = (0, \exp\,(\lambda y + \int h\,(x)dx))$$

for

$$y' = \exp\,(\lambda y)G(x) - \frac{h(x)}{\lambda}$$

where f, g, h, G, ψ are arbitrary functions providing that satisfy suitable regularization conditions and λ

is an arbitrary constant. We find solution to these differential equations by using their Lie's point symmetries respectively as follows:

$$\int^{v=y} \exp\left(-\int^{u=v} f(u)du\right) dv = e^{\lambda x}\left[\int^{r=x} e^{-\lambda r} g(r)dr + C\right]$$

and

$$y(x) = -\frac{1}{\lambda}\ln\left| -\lambda \exp\left(\int^{u=x} f(u)du\right) \left[\int^{r=x}\left(g(r)\exp\left(-\int^{u=r} f(u)du\right)\right)dr + c\right]\right|.$$

But we see that they are linear and Bernoulli type differential equations by using transformations

$$u = \int^{\tau=y} \frac{d\tau}{\psi(\tau)} \quad , \quad v = \exp(\lambda y)$$

respectively.

However we can use induced characteristic in order to determine symmetries of ODEs which have special forms or vice versa as well as canonical coordinates: By using (3) we obtain first order ODEs

$$y' = Q(x,y)\left\{\int^{a=y} \frac{Q_x(x,a)}{Q^2(x,a)} da + H(x)\right\} \tag{6}$$

that admit Lie's symmetries

$$(\xi, \eta) = (0, Q(x,y)) \tag{7}$$

for an arbitrary function Q in x, y which satisfies suitable regularization conditions where $H(x)$ is an arbitrary continuous function in x. For these differential equations we can also use symmetry generators $(\xi, \omega\xi + Q(x,y))$ where ξ is an arbitrary function. Therefore without using canonical coordinates, it can be obtained ordinary differential equations that admit some special form of symmetries by Eqs.(6) and (7), so this way is much more easy.

4 Applications

Example 1. Consider $Q_1(x,y) = F(x) + G(y)$ and $Q_2(x,y) = f(x)g(y)$. Then it is seen from Eqs.(6)-(7) that $(0, Q_1(x,y))$, $(0, Q_2(x,y))$ are admitted by differential equations

$$y' = [F(x) + G(y)]\left\{F'(x)\int^{a=y} \frac{da}{[F(x)+G(y)]^2} + H(x)\right\}$$

and

$$y' = g(y)\left\{\frac{f'(x)}{f(x)}\int^{a=y} \frac{da}{g(y)} + h(x)\right\}$$

respectively. This results can be found also in [4]. Even if one can use more general choices, for example $Q = F(x)G(y) + f(x) + g(y)$, the differential equations which admit these symmetries can be obtained easily from (6)-(7).

Example 2. Let's consider the differential equation

$$y' = e^{-x}y^2 + y + e^x.$$

By using the method in [1], the generator of Lie's symmetries is found as $X = \xi\partial_x + \eta\partial_y = \partial_x + y\partial_y$. So we have $Q(x,y) = -e^x - y^2e^{-x}$. Using (6) with this $Q(x,y)$ more general differential equations can be obtained

$$y' = -e^{-x}(e^{2x} + y^2)\left\{\int^{a=y} \frac{a^2e^{-x} - e^x}{(e^x + a^2e^{-x})^2} da - H(x)\right\}$$

$$= H(x)y^2 + y + H(x)e^{2x}.$$

The differential equation above implies the Riccati differential equation

$$y' = f_2(x)y^2 + f_1(x)y + f_0(x)$$

where $f_2 = H(x), f_1(x) = 1$ and $f_0 = H(x)e^{2x}$. Generally, it is very hard or impossible to compute Lie's symmetries of Riccati differential equation even if it is in above form. That's why this approach introduce us to quite an important result. To obtain solution, we use canonical coordinates

$$r = x \quad , \quad s = -\arctan\left(\frac{y}{e^x}\right).$$

According to these coordinates, the solution becomes

$$s = C - \int^{r=x} H(r)e^r dr$$

or in terms of original variables

$$y = e^x \tan\left[\int^{r=x} H(r)e^r dr - C\right]$$

Now let's take more general $Q(x,y)$ as

$$Q(x,y) = f(x) + h(x)y^2.$$

Then we obtain a nonlinear differential equation which includes also nonlinearity with inverse trigonometric function. If it is desired to obtain a more general Riccati differential equation than above correspondingly, it is concluded that functions f and h must satisfy condition $(fg)' = 0$ that is $h = \frac{\gamma}{f}$ where γ is an arbitrary constant and $f \neq 0$. So we obtain the differential equation

$$y' = \frac{\gamma H(x)}{f(x)} y^2 + \frac{f'(x)}{f(x)} y + H(x)f(x).$$

Because of the fact that H is an arbitrary function in x, if we take $f(x) = -e^x$ and $H(x) = -\widetilde{H}(x)e^x$ then we have the result above again.

Example 3. Consider differential equation

$$y' = \frac{3y}{x} + \frac{x^5}{2y+x^3}.$$

From [1] we know that the differential equation has symmetries generator $X = x\partial_x + 3y\partial_y$. By using again the above process, it is seen that the differential equation

$$y' = \frac{3y}{x} + \frac{H(x)}{x(2y+x^3)}$$

admits symmetries $(\xi, \eta) = (0, -\frac{x^6}{2y+x^3})$.

5 Conclusions

In this work, we firstly tried to find Lie's point symmetry of general first order ordinary differential equation $y' = \omega(x, y)$. Then we were led to consider some special Lie's symmetries of differential equation correspondingly by using the method of separation of variables in order to find exact solution for Linearized symmetry condition via induced characteristic. If one uses different methods in order to find solution to the linearized symmetry condition, it can be obtained symmetries that have different forms for some special first order ODEs or perhaps symmetries that admitted by $y' = \omega(x, y)$ for an arbitrary ω. Also we considered a formulation derived from (3) to construct some differential equations that admit chosen symmetries and we gave applications to see both compatibility and effectiveness of formulation in which we firstly considered some differential equations from [5] and found symmetries of them. Moreover with their more general form, we obtain more general differential equations whose symmetries can not be obtain easily, like Riccati differential equations.

6 References

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