Independence Saturation In Complementary Product Types of Graphs

Zeynep Nihan Berberler[1*], Murat Erşen Berberler[2]

[1,2] Dokuz Eylul University, Faculty of Science, Department of Computer Science,
35160, Izmir, TURKEY, + 90 232 3019514,
zeynep.berberler@deu.edu.tr
*Corresponding author

Recieved: 19th November 2016
Accepted: 29th April 2017
DOI: 10.18466/cbayarfbe.319783

Abstract

The independence saturation number $IS(G)$ of a graph $G=(V,E)$ is defined as $\min\{IS(u): u \in V\}$, where $IS(u)$ is the maximum cardinality of an independent set that contains vertex u. Let $\bar{G}$ be the complement graph of G. Complementary prisms are the subset of complementary product graphs. The complementary prism $G\bar{G}$ of G is the graph formed from the disjoint union of G and $\bar{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\bar{G}$. In this paper, the independence saturation in complementary prisms are considered, then the complementary prisms with small independence saturation numbers are characterized.

Keywords — Complementary prisms, graph theory, independence, independence saturation, network topology.

1 Introduction

A subset of pairwise nonadjacent vertices in a graph G is called independent (or stable/ internally stable). The cardinality of a maximum size independent set in G is called the independence number (or stability number) (or coefficient of internal stability [1]) of G and is denoted by $\beta(G)$. The independence number of a graph is one of the basic numerical characteristics of a graph and most fundamental and well-studied graph parameters. We refer to [2] for a review concerning algorithms, applications, and complexity issues of this problem. Among the independence-type parameters that have been studied, the independence saturation number is one of the fundamental ones introduced by Subramanian [3]. For a vertex v of a graph G, $IS(v)$ denotes the maximum cardinality of an independent set in G which contains v. The independence saturation number of G, denoted by $IS(G)$, is the value $\min\{IS(v): v \in V\}$. Hence $IS(G)$ is the largest positive integer k such that every vertex of G is included in an independent set with cardinality k. If for a vertex $v \in V$, $IS(v) = IS(G)$, then any independent set with cardinality $IS(G)$ containing v is an IS-set. For any graph G, the problem of determining whether $IS(G) \geq k$ is NP-complete. We refer to [15] and [16] for computation of independence saturation numbers of some classes of graphs.

In this paper, we consider finite undirected graphs without loops and multiple edges. Let $G=(V(G),E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The order of G is the number of vertices in G. The open neighborhood of v is $N(v)=\{u \in V(G) \mid uv \in E(G)\}$ and the closed neighborhood of v is $N[v]=\{v\} \cup N(v)$. For a set $S \subseteq V$, $N(S)=\bigcup_{v \in S} N(v)$ and $N[S]=N(S) \cup S$. The degree of a vertex v is $\deg_G(v)=|N(v)|$. A graph is regular if its vertices all have the same degree. A r-regular graph is the graph in which the degree of each vertex is r. A vertex of degree zero is an isolated vertex or an isolate. A leaf or an endvertex or a pendant vertex is a vertex of degree one and its neighbor is called a support vertex. The maximum degree of G is $\Delta(G)=\max\{\deg_G(u)|u\in V(G)\}$ whereas the minimum degree of G is $\delta(G)=\min\{\deg_G(u)|u\in V(G)\}$. For $S\subseteq V(G)$, the subgraph of G induced by X is denoted by $G[X]$. The *distance* between two vertices u and v in G is the length of a shortest path between u and v, and denoted by $d(u,v)$. If $u=v$, $d(u,v)=0$ and if u and v are not connected, then $d(u,v)=\infty$. The *eccentricity* of a vertex v in G is the maximum distance from v to a vertex in G. The *diameter* of G is the maximum distance between two vertices of G, and denoted by $diam(G)$ [4,5].

Complementary prisms were first introduced by Haynes, Henning, Slater and Van der Merwe in [6]. For a graph G, its complementary prism, denoted by $G\bar{G}$, is formed from a copy of G and a copy of $\bar{G}$ by adding a perfect matching between corresponding vertices. For each $v\in V(G)$, if $\bar{v}$ denote the vertex v in the copy of $\bar{G}$, then $G\bar{G}$ is formed from $G\cup\bar{G}$ by adding the edge $v\bar{v}$ for $\forall v\in V(G)$. We refer to [8,9,10,11,12,13,14,17,18] for recent studies on complementary prims.

The corona graph $G\circ K_1$ is a graph constructed from a copy of the graph G where each vertex of $V(G)$ is adjacent to exactly one vertex of degree one. We use $\lfloor a\rfloor$ to denote the greatest integer not greater than a, and $\lceil a\rceil$ to denote the least integer not less than a.

The paper proceeds as follows. In section 2, existing literature on independence saturation number is reviewed. The independence saturation numbers for complementary prism $G\bar{G}$ when G is a specific type of graphs are computed and exact formulae are derived. The graphs (resp. vertices) G for which $IS(G\bar{G})$ (resp. $IS(v)$) is small are characterized.

2 Independence Saturation

2.1 Known Results

Theorem 2.1.1. [7] *If G is an r-regular graph on n vertices with $r>0$, then $IS(G)\le n/2$. Further equality holds if and only if G is bipartite.*

Theorem 2.1.2. [7] *Let G be any graph on n vertices. Then, $IS\le n-\Delta$. For a tree T, $IS=n-\Delta$ iff $V-N(v)$ is an independent set for every vertex v of degree Δ and $p_u\le p_v$ for $\forall u\in N(v)$, where p_x is the number of pendant vertices adjacent to x.*

Theorem 2.1.3. [7] *For any graph G with at least three vertices,*

(a) $3\le IS+\overline{IS}\le n+1-(\Delta-\delta)$ *and* $2\le IS.\overline{IS}\le(p-\Delta)(\delta+1)$.

(b) *The following statements are equivalent:*
   - (a) $IS+\overline{IS}=3$.
   - (b) $IS.\overline{IS}=2$.
   - (c) *G or $\bar{G}$ has the property that it has a unique vertex of degree $p-1$ and has at least one pendant vertex.*

(c) $IS+\overline{IS}=p+1$ *iff G is either K_p or $\bar{K}_p$.*

Theorem 2.1.4. [7] *The independence saturation of*

(a) *the complete graph K_n is 1;*

(b) *the cycle C_n is $\lfloor n/2\rfloor$;*

(c) *the complete bipartite graph $K_{m,n}$ is $\min\{m,n\}$;*

(d) *the star $K_{1,n}$ is 1.*

Observation 2.1.1.

(a) $\beta(P_n)=\left\lceil\frac{n}{2}\right\rceil$.

(b) $\beta(C_n)=\left\lfloor\frac{n}{2}\right\rfloor$.

(c) For $n>3$ $\beta(\bar{C}_n)=2$.

Definition 2.1.1. [5] Let G and H be two disjoint graphs. The *union* graph of G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph $G^* = G \cup H$ with vertex set $V(G^*) = V(G) \cup V(H)$ and edge set $E(G^*) = E(G) \cup E(H)$.

Observation 2.1.2. *Let $G_1, G_2, \cdots, G_n$ be disjoint graphs. If $G = \cup_{i=1}^{n} G_i$, then $\beta(G) = \Sigma_{i=1}^{n} \beta(G_i)$.*

2.2 Complementary Prisms

We begin this subsection by determining the independence saturation number of the complementary prism $G\bar{G}$ in which G is a specific type of graphs.

Theorem 2.2.1

(a) If $G = K_n$, then $IS(G\bar{G}) = n$.

(b) If $G = tK_2$, then $IS(G\bar{G}) = t+1$.

(c) If $G = K_t \circ K_1$, then $IS(G\bar{G}) = t+1$.

(d) If $G = K_{1,n}$, then $IS(G\bar{G}) = 2$.

(e) If $G = K_{m,n}$ where $2 \le m \le n$, then $IS(G\bar{G}) = m+1$.

(f) If $G = C_n$, then

$$IS(G\bar{G}) = \begin{cases} n/2+1, & \text{if } n \text{ is even;} \\ \lfloor n/2 \rfloor + 2, & \text{if } n \text{ is odd.} \end{cases}$$

(g) If $G = W_n$ $(n > 3)$, then $IS(G\bar{G}) = 3$.

(h) If $G = P_n$ $(n \ge 2)$, then $IS(G\bar{G}) = \lceil (n+2)/2 \rceil$.

Proof. To prove (a), for $G = K_n$, the complementary prism $G\bar{G}$ is the corona $K_n \circ K_1$. Let v be a vertex of $G\bar{G}$. If v is a support vertex, then the $IS(v)$-set must contain each leaf of the other support vertices except v. If v is a leaf, then an $IS(v)$-set must contain either each leaf or a support vertex which is not adjacent to v and each leaf of the other support vertices. Hence, $IS(v) = n$. Then, for any $v \in V(G\bar{G})$, $IS(v) = n$ and thus $IS(G\bar{G}) = n$. ■

To prove (b), label $2t$ vertices of $V(G)$ as u_i, v_i where $1 \le i \le t$ such that $u_i v_i \in E(G)$. Then, there exist four cases depending on the different type of vertices of $G\bar{G}$:

Case 1. The set $\{\bar{u}_i\} \cup \{v_1, \cdots, v_t\}$ is an $IS(v)$-set of $G\bar{G}$ for $v = \bar{u}_i$ with cardinality $t+1$.

Case 2. The set $\{\bar{v}_i\} \cup \{u_1, \cdots, u_t\}$ is an $IS(v)$-set of $G\bar{G}$ for $v = \bar{v}_i$ with cardinality $t+1$.

Case 3. The set $\{u_1, \cdots, u_t\} \cup \{\bar{v}\}$ is an $IS(v)$-set of $G\bar{G}$ for $v = u_i$ with cardinality $t+1$.

Case 4. The set $\{v_1, \cdots, v_t\} \cup \{\bar{u}\}$ is an $IS(v)$-set of $G\bar{G}$ for $v = v_i$ with cardinality $t+1$.

By *Cases 1, 2, 3,* and *4,* for any $v \in V(G\bar{G})$, $IS(v) = t+1$, and we have $IS(G\bar{G}) = t+1$. ■

To prove (c), let $G = K_t \circ K_1$. If $t = 1$, then $G = K_2$ and from (a), $IS(G\bar{G}) = 2$. Assume that $t \ge 2$, and label the vertices of G as follows: let $A = \{a_i | 1 \le i \le t\}$ be the set of t vertices that induce the subgraph K_t of G, and let $B = \{b_i | 1 \le i \le t\}$ be the end-vertices of G adjacent to vertices in A such that $a_i b_i \in E(G)$. The set $\bar{A} \cup B$ is an $IS(v)$-set for $v = \bar{a}_i$ or b_i with cardinality $2t$. If $v = a_i$, then an $IS(v)$-set is composed of $a_i \cup B \setminus \{b_i\} \cup \bar{A} \setminus \{\bar{a}_i\}$ with cardinality $1 + (t-1) + (t-1) = 2t-1$ and if $v = \bar{b}_i$, then $IS(v)$-set is composed of $\bar{b}_i \cup \bar{a}_i \cup B \setminus \{b_i\}$ with cardinality $t+1$. Thus, $IS(G\bar{G}) = \min\{2t, 2t-1, t+1\} = t+1$. ■

To prove (d), since G is a star, the support vertex t in G is an isolated vertex $\bar{t}$ in $\bar{G}$ and a leaf in $G\bar{G}$. Denote the n leaves of G by $\{u_1, u_2, \cdots, u_n\}$. The leaves in G will form a complete graph with n vertices in $\bar{G}$. Let I be an $IS(v)$-set of $G\bar{G}$. Then, there are four cases according to the types of vertices in $G\bar{G}$.

Case 1. If $v = t$, then $I = t \cup \bar{u}_i$, and so $IS(v) = 2$.

Case 2. If $v = \bar{t}$, then $I = \bar{t} \cup \{u_i : 1 \le i \le n\}$ or $I = \bar{t} \cup \bar{u}_i \cup \{u_j : \forall j \ne i, 1 \le j \le n\}$, and so $IS(v) = n+1$.

Case 3. If $v = u_i$ where $1 \le i \le n$, then $I = \bar{t} \cup \{u_i : 1 \le i \le n\}$ or being $j \ne i$ and $1 \le j \le n$, $I = u_i \cup \bar{t} \cup \bar{u}_j \cup \{u_t : \forall t \ne j \wedge t \ne i, 1 \le t \le n\}$, and so $IS(v) = n+1$.

Case 4. If $v = \bar{u}_i$ where $1 \le i \le n$, then $I = \bar{t} \cup \bar{u}_i \cup \{u_j : \forall j \ne i, 1 \le j \le n\}$, and so $IS(v) = n+1$.

Consequently, by *Cases 1, 2, 3,* and *4,* $IS(G\bar{G}) = \min\{2, n+1\} = 2$. ■

To prove (e), let $G = K_{m,n}$ $(2 \le m \le n)$, where R and S are the partite sets of G with cardinality m and n, respectively. Let $R = \{r_1, r_2, \cdots, r_m\}$ and $S = \{s_1, s_2, \cdots, s_n\}$. The vertices of R and S will form complete graphs K_m and K_n with m and n vertices, respectively, in $\bar{G}$. Then, there exist four cases depending on the types of vertices of $G\bar{G}$:

Case 1. If $r_i \in R$ where $1 \le i \le m$, then $N_{G\bar{G}}(r_i) = S \cup \{\bar{r}_i\}$. Let I be an $IS(r_i)$-set of $G\bar{G}$. Since $G\bar{G}[\bar{S}] = K_n$, $I = \{\bar{s}\} \cup R$, and so $IS(r_i) = m+1$ for $1 \le i \le m$.

Case 2. If $s_i \in S$ where $1 \le i \le n$, then $N_{G\bar{G}}(s_i) = R \cup \{\bar{s}_i\}$. If I is an $IS(s_i)$-set of $G\bar{G}$, then since $G\bar{G}[\bar{R}] = K_m$, $I = \{\bar{r}\} \cup S$. Therefore, $IS(s_i) = n+1$ for $1 \le i \le n$.

Case 3. If $\bar{r}_i \in \bar{R}$ where $1 \le i \le m$, then $N_{G\bar{G}}(\bar{r}_i) = \{r_i\} \cup \bar{R} \setminus \{\bar{r}_i\}$. Let I be an $IS(\bar{r}_i)$-set of $G\bar{G}$. We have $I = \bar{r}_i \cup S$ with cardinality $n+1$. Therefore, for $1 \le i \le m$, $IS(\bar{r}_i) = n+1$.

Case 4. If $\bar{s}_i \in \bar{S}$ where $1 \le i \le n$, then $N_{G\bar{G}}(\bar{s}_i) = \{s_i\} \cup S \setminus \{\bar{s}_i\}$. If I is an $IS(\bar{s}_i)$-set of $G\bar{G}$, then we have $I = \{\bar{s}_i\} \cup \{\bar{r}\} \cup S \setminus \{s_i\}$. Thus, for $1 \le i \le n$, $IS(\bar{s}_i) = n+1$.

As a consequence, by *Cases 1, 2, 3,* and *4,* $IS(G\bar{G}) = \min\{m+1, n+1\} = m+1$. ■

To prove (f), let the vertices of $G = C_n$ be labeled sequentially as $u_0, u_1, \cdots, u_{n-1}$. There exist two cases depending on the number of vertices of C_n:

Case 1. If n is even;

If I is an $IS(u_j)$-set of $G\bar{G}$ where $0 \le j \le n-1$, then I contains the independent set of G with maximum cardinality including u_j and a vertex $\bar{u}_k$ of $\bar{G}$, where k is even, if j is odd; k is odd, if j is even. Hence, $IS(u_j) = \beta(G)+1$ and since $\beta(C_n) = n/2$, $IS(u_j) = n/2+1$, for $0 \le j \le n-1$.

If I is an $IS(\bar{u}_j)$-set of $G\bar{G}$ where $0 \le j \le n-1$, then let $S = V(G\bar{G}) \setminus N_{G\bar{G}}[\bar{u}_j]$. Since $G\bar{G}[S] = C_{n+1}$, $IS(\bar{u}_j) = 1 + \beta(C_{n+1}) = 1 + \lfloor (n+1)/2 \rfloor = 1 + n/2$.

As a consequence, for any $v \in V(G\bar{G})$, $IS(v) = n/2+1$ and so $IS(G\bar{G}) = n/2+1$ for $G = C_n$ and n is even.

Case 2. If n is odd;

If I is an $IS(u_j)$-set of $G\bar{G}$ where $0 \le j \le n-1$, then I contains the independent set of G with maximum cardinality including u_j and vertices $\bar{u}_{j+1}$ and $\bar{u}_{j+2}$ of $\bar{G}$ where $j+1$ and $j+2$ are taken modulo n. Hence, $IS(u_j) = \beta(G)+2$ and

$IS\left(u_j\right) = \lfloor n/2 \rfloor + 2$.

If I is an $IS\left(\bar{u}_j\right)$-set of $G\bar{G}$ where $0 \le j \le n-1$, then let $S = V\left(G\bar{G}\right) \setminus N_{G\bar{G}}\left[\bar{u}_j\right]$. Since $G\bar{G}[S] = C_{n+1}$, $IS\left(\bar{u}_j\right) = 1 + (n+1)/2 = (n+3)/2$.

As a consequence, for any $v \in V\left(G\bar{G}\right)$, $IS(v) = \lfloor n/2 \rfloor + 2$ and so $IS\left(G\bar{G}\right) = \lfloor n/2 \rfloor + 2$ for $G = C_n$ and n is odd.

Thus, the proof is complete. ■

To prove (g), let G be a wheel of order $n+1$ and consider $G\bar{G}$. Since the center vertex c of G is adjacent to all other vertices of G, it is an isolate in $\bar{G}$ and a leaf in $G\bar{G}$. Therefore, if I is an $IS(c)$-set of $G\bar{G}$, then first $c \in I$. Since $W_n = C_n + K_1$, if the vertices of C_n are labeled sequentially as $u_0, u_1, \cdots, u_{n-1}$ in $G\bar{G}$, then $N_{G\bar{G}}(c) = \{\bar{c}\} \cup \{u_i \,|\, 0 \le i \le n-1\}$. Let $S = V\left(G\bar{G}\right) \setminus N_{G\bar{G}}[c]$. Eventually, $G\bar{G}[S] = \bar{C}_n$. Therefore, an independent set of maximum cardinality of $\bar{C}_n$ should be included in $IS(c)$-set. Hence, by Observation 2.1.1(c), we have that $IS(c) = 1 + \beta\left(\bar{C}_n\right) = 3$.

Now, consider the leaf $\bar{c}$ in $G\bar{G}$. Let $S = V\left(G\bar{G}\right) \setminus N_{G\bar{G}}[\bar{c}]$. Then $G\bar{G}[S] = C_n\bar{C}_n$. From (f), we know that

$$IS\left(C_n\bar{C}_n\right) = \begin{cases} \beta\left(C_n\right) + 1, & if\ n\ is\ even; \\ \beta\left(C_n\right) + 2, & if\ n\ is\ odd. \end{cases}$$

Thus, $IS(\bar{c}) = \begin{cases} \beta\left(C_n\right) + 2, & if\ n\ is\ even; \\ \beta\left(C_n\right) + 3, & if\ n\ is\ odd. \end{cases}$,

that is, $IS(\bar{c}) = \begin{cases} (n+4)/2, & if\ n\ is\ even; \\ (n+5)/2, & if\ n\ is\ odd. \end{cases}$

Consider the vertices of C_n which are labeled sequentially as $u_0, u_1, \cdots, u_{n-1}$ in $G\bar{G}$. Then, the proof is very similar to the proof of (f). Since $c \in N\left(u_j\right)$ where $0 \le j \le n-1$, the leaf $\bar{c}$ should also be included in the independent set of maximum cardinality including u_j. Therefore,

$$IS\left(u_j\right) = \begin{cases} 1 + \beta\left(C_n\right) + 1, & if\ n\ is\ even; \\ 1 + \beta\left(C_n\right) + 2, & if\ n\ is\ odd. \end{cases},$$

that is, $IS\left(u_j\right) = \begin{cases} (n+4)/2, & if\ n\ is\ even; \\ (n+5)/2, & if\ n\ is\ odd. \end{cases}$

Consequently, consider the vertices of $\bar{C}_n$ in $G\bar{G}$. For a vertex $\bar{u}_j$ $(0 \le j \le n-1)$ of $G\bar{G}$, similar to the proof of (f), if I is an $IS\left(\bar{u}_j\right)$-set of $G\bar{G}$, then I contains the leaf $\bar{c}$ in addition. Thus,

$$IS\left(\bar{u}_j\right) = \begin{cases} \beta\left(C_n\right) + 2, & if\ n\ is\ even; \\ \beta\left(C_n\right) + 3, & if\ n\ is\ odd. \end{cases},$$

that is, $IS\left(\bar{u}_j\right) = \begin{cases} (n+4)/2, & if\ n\ is\ even; \\ (n+5)/2, & if\ n\ is\ odd. \end{cases}$

As a result,

$$IS\left(G\bar{G}\right) = \begin{cases} \min\{3, (n+4)/2\}, & if\ n\ is\ even; \\ \min\{3, (n+5)/2\}, & if\ n\ is\ odd. \end{cases}$$

Since, $n \ge 3$, $IS\left(G\bar{G}\right) = 3$ for $G = W_n$. This completes the proof. ■

To prove (h), consider the vertices of $\bar{P}_n$ in $G\bar{G}$.

If a vertex $\bar{v}$ of $\bar{P}_n$ in $G\bar{G}$ is an endvertex of P_n, then first an $IS(\bar{v})$-set must include $\bar{v}$. Then, let $S = V\left(G\bar{G}\right) \setminus N[\bar{v}]$. Since $G\bar{G}[S] = P_n$, the independent set of $G\bar{G}[S]$ of maximum cardinality must be included in $IS(\bar{v})$-set. Therefore,

$$IS(\bar{v}) = 1 + \beta\left(P_n\right) = 1 + \lceil n/2 \rceil. \tag{1}$$

If a vertex $\bar{v}$ of $\bar{P}_n$ in $G\bar{G}$ is not an endvertex of P_n, then let $S = V\left(G\bar{G}\right) \setminus N[\bar{v}]$, that is, $G\bar{G}[S] = P_{n+1}$, implying that, the independent set of $G\bar{G}[S]$ of maximum cardinality should be included in $IS(\bar{v})$-set. Therefore, $IS(\bar{v}) = 1 + \beta\left(P_{n+1}\right) = 1 + \lceil (n+1)/2 \rceil$.

(2)

Now consider the vertices of P_n in $G\bar{G}$. There exist two cases as being n odd or even.

Case 1. n is odd:
For a vertex v_i $\left(1 \le i \le \lceil n/2 \rceil\right)$ of P_n in $G\bar{G}$, if the vertex $\bar{v}_{i+1}$ is included in $IS(v_i)$-set, then there remains the induced subgraph

$$G\bar{G}[S] = \begin{cases} P_{n-1}, & \text{if } i = 1; \\ P_{n-i} \cup P_{i-2}, & \text{otherwise}. \end{cases},$$

where $S = V(G\bar{G}) \setminus \left\{ N[v_i] \cup N[\bar{v}_{i+1}] \right\}$. Thus, by Observation 2.1.2,

$$IS(v_i) = \begin{cases} 2 + \beta(P_{n-1}), & \text{if } i = 1; \\ 2 + \beta(P_{n-i}) + \beta(P_{i-2}), & \text{otherwise}. \end{cases}$$

, that is, $IS(v_i) = (n+3)/2$. (3)

For a vertex v_i $\left(\lceil n/2 \rceil + 1 \le i \le n\right)$ of P_n in $G\bar{G}$, if the vertex $\bar{v}_{i-1}$ is included in $IS(v_i)$-set, then there remains the induced subgraph

$$G\bar{G}[S] = \begin{cases} P_{n-1}, & \text{if } i = n; \\ P_{i-1} \cup P_{n-i-1}, & \text{otherwise}. \end{cases},$$

where $S = V(G\bar{G}) \setminus \left\{ N[v_i] \cup N[\bar{v}_{i-1}] \right\}$. Thus, by Observation 2.1.2,

$$IS(v_i) = \begin{cases} 2 + \beta(P_{n-1}), & \text{if } i = n; \\ 2 + \beta(P_{i-1}) + \beta(P_{n-i-1}), & \text{otherwise}. \end{cases}$$

, that is, $IS(v_i) = (n+3)/2$. (4)

By (1), (2), (3), and (4), if n is odd, then

$$\mathrm{IS}(G\bar{G}) = \min\left\{1 + \left\lceil \frac{n}{2} \right\rceil, 1 + \left\lceil \frac{n+1}{2} \right\rceil, \frac{n+3}{2}\right\} = \frac{n+3}{2}. \quad (5)$$

Case 2. n is even:
For a vertex v_i (i is odd) of P_n in $G\bar{G}$, if the vertex $\bar{v}_{i+1}$ is included in $IS(v_i)$-set, then there remains the induced subgraph

$$G\bar{G}[S] = \begin{cases} P_{n-1}, & \text{if } i = 1; \\ P_{n-3}, & \text{if } i = n-1; \\ P_{n-i} \cup P_{i-2}, & \text{otherwise}. \end{cases}$$

, where $S = V(G\bar{G}) \setminus \left\{ N[v_i] \cup N[\bar{v}_{i+1}] \right\}$. Thus, by Observation 2.1.2,

$$IS(v_i) = \begin{cases} 2 + \beta(P_{n-1}), & \text{if } i = 1; \\ 2 + \beta(P_{n-3}), & \text{if } i = n-1; \\ 2 + \beta(P_{n-i}) + \beta(P_{i-2}), & \text{otherwise}. \end{cases}$$

,that is, $$IS(v_i) = \begin{cases} \dfrac{n+2}{2}, & \text{if } i = n-1; \\ \dfrac{n+4}{2}, & \text{otherwise}. \end{cases} \quad (6)$$

For a vertex v_i (i is even) of P_n in $G\bar{G}$, if the vertex $\bar{v}_{i-1}$ is included in $IS(v_i)$-set, then there remains the induced subgraph

$$G\bar{G}[S] = \begin{cases} P_{n-3}, & \text{if } i = 2; \\ P_{n-1}, & \text{if } i = n; \\ P_{n-i-1} \cup P_{i-1}, & \text{otherwise}. \end{cases}$$

, where $S = V(G\bar{G}) \setminus \left\{ N[v_i] \cup N[\bar{v}_{i-1}] \right\}$. Thus, by Observation 2.1.2,

$$IS(v_i) = \begin{cases} 2 + \beta(P_{n-3}), & \text{if } i = 2; \\ 2 + \beta(P_{n-1}), & \text{if } i = n; \\ 2 + \beta(P_{n-i-1}) + \beta(P_{i-1}), & \text{otherwise}. \end{cases}$$

, that is, $$IS(v_i) = \begin{cases} (n+2)/2, & \text{if } i = 2; \\ (n+4)/2, & \text{otherwise}. \end{cases}. \quad (7)$$

By (1), (2), (6), and (7), if n is even, then

$$IS(G\bar{G}) = \min\left\{\frac{n+2}{2}, \frac{n+4}{2}\right\} = \frac{n+2}{2}. \quad (8)$$

By (5) and (8), we conclude that, $IS(G\bar{G}) = \left\lceil (n+2)/2 \right\rceil$ for $G = P_n$ and $n > 1$. ■

2.3 Small Values

Theorem 2.3.1 *For a graph G with n vertices and its complementary prism $G\bar{G}$, $IS(G\bar{G}) = 1$ iff $n = 1$.*

Proof. The sufficiency is immediate since if $n = 1$, then $G\bar{G} = K_2$. Thus, by Theorem 2.1.4(a),

$IS(G\bar{G})=1$. Now, suppose that $IS(G\bar{G})=1$. This implies that there is at least one vertex in $G\bar{G}$ that has $IS(v)=1$. Therefore, $G\bar{G}$ is the corona $K_1 \circ K_1$, and so $|V(G)|=1$. This establishes the necessity. ■

Theorem 2.3.2 *For a graph G with n vertices, if either G or $\bar{G}$ has diameter one, then $IS(G\bar{G})=n$.*

Proof. If either G or $\bar{G}$ has diameter one, then $G\bar{G}$ is the corona $K_n \circ K_1$. Therefore, by Theorem 2.2.1(a), the proof is immediate. ■

Corollary 2.3.1 *If a vertex u has eccentricity one in graph G, then $IS(u)=1$.*

Corollary 2.3.2 Let *G be a graph with n vertices. If G has a vertex with eccentricity one, then $IS(G)=1$.*

4 Conclusion

Independence saturation in complementary prisms of particular types of graphs are considered in this paper while giving an insight of how to evaluate the parameter and derive formulae on complementary product types of graphs.

5 References

[1] Korshunov, A.D. Coefficient of Internal Stability of Graphs. Cybernetics. 1974; 10, 19-33.

[2] Bomze, I.; Budinich, M.; Pardalos, P.; Pelillo, M. The Maximum Clique Problem. Handbook of Combinatorial Optimization, Supplement Volume A; Du, D., Pardalos, P., Eds.; Kluwer Academic Press: 1999.

[3] Subramanian, M. Studies in Graph Theory-Independence Saturation in Graphs, Ph.D thesis, Manonmaniam Sundaranar University, 2004.

[4] West, D.B. Introduction to Graph Theory; Prentice Hall, NJ, 2001.

[5] Buckley, F.; Harary, F. Distance in Graphs; Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, 1990.

[6] Haynes, T.W.; Henning, M.A.; Slater, P.J.; Merwe, V.D. The Complementary Product of Two Graphs. Bulletinof the Institute of Combinatorics and its Applications. 2007; 51, 21-30.

[7] Arumugam, S.; Subramanian, M. Independence Saturation and Extended Domination Chain in graphs. AKCE International Journal of Graphs and Combinatorics. 2007; 4, 59-69.

[8] Gongora, J.A.; Haynes, T.W.; Jum, E. Independent Domination in Complementary Prisms. Utilitas Mathematica. 2013; 91, 3-12.

[9] Aytaç, A.; Turacı, T. Strong Weak Domination in Complementary Prisms. Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications & Algorithms. 2015; 22(2b), 85-96.

[10] Gölpek, T.H.; Turacı, T.; Coskun, B. On The Average Lower Domination Number and Some Results of Complementary Prisms and Graph Join. Journal of Advanced Research in Applied Mathematics. 2015; 7(1), 52-61.

[11] Desormeaux, W.J.; Haynes, T.W.; Vaughan, L. Double Domination in Complementray Prisms. Utilitas Mathematica. 2013; 91, 131-142.

[12] Desormeaux, W.J.; Haynes, T.W. Restrained Domination in Complementray Prisms. Utilitas Mathematica. 2011; 86, 267-278.

[13] Kazemi, A.P. *k*-Tuple Total Restrained Domination in Complementary Prisms. ISRN Combinatorics. 2013; doi:10.1155/2013/984549.

[14] Chaluvaraju, B.; Chaitra, V. Roman domination in Complementary Prism Graphs. International Journal of Mathematical Combinatorics. 2012; 2, 24-31

[15] Muthulakshmi, T.; Subramanian, M. Independence saturation number of some classes of graphs. Far East Journal of Mathematical Sciences. 2014; 86(1), 11-21.

[16] Berberler, Z.N.; Berberler, M.E. Independently Saturated Graphs. TWMS Journal of Applied and Engineering Mathematics. Accepted. 2017.

[17] Haynes, T.W.; Henning, M.A.; Merwe, V.D. Domination and total domination in complementary prisms. Journal of Combinatorial Optimzation. 2009; 18, 23-37.

[18] Holmes, K.R.S.; Koessler, D.R.; Haynes, T.W. Locating-domination in complementary prisms. Journal of Combinatorial Mathematics and Combinatorial Computing. 2010; 72, 163-171.