Inequalities for log −convex functions via three times differentiability

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Abstract

In this paper, some new integral inequalities like Hermite-Hadamard type for functions whose third derivatives absolute value are log −convex are established. Some applications to quadrature formula for midpoint error estimate are given.

Keywords- Convexity, log −convex functions, Hermite-Hadamard inequality, Hölder integral inequality, Power-mean integral inequality

1 Introduction

We shall recall the definitions of convex functions and log −convex functions:

Let I be an interval in $\mathbb{R}$. Then $f: I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and all $\alpha \epsilon [0,1]$,

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) \quad (1.1)$$

holds. If (1.1) is strict for all $x \neq y$ and $\alpha \in (0,1)$, then f is said to be strictly convex. If the inequality in (1.1) is reversed, then f is said to be concave. If it is strict for all $x \neq y$ and $\alpha \in (0,1)$, then f is said to be strictly concave.

A function is called log −convex or multiplicatively convex on a real interval $I = [a,b]$, if $log f$ is convex , or, equivalently if for all $x, y \in I$ and all $\alpha \in [0,1]$,

$$f(\alpha x + (1-\alpha)y) \leq f(x)^{\alpha} . f(y)^{(1-\alpha)}. \quad (1.2)$$

It is said to be log-concave if the inequality in (1.2) is reversed. For some results for log − convex functions see [1,2,3,4,5,6,7].

The following inequality is called Hermite-Hadamard inequality for convex functions: Let $f: I \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

holds.

The main purpose of this paper is to obtain some new integral inequalities like Hermite-Hadamard type for functions whose third derivatives absolute value are log −convex.

In order to prove our main results for log −convex functios we need the following Lemma from [8]:

Lemma 1.1. Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a three times differentiable mapping on I° (the interior of I) and $a, b \in I^{\circ}$ with $a < b$. If $f^{(3)} \in L_1[a,b]$, then

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right)$$

$$= \frac{(b-a)^3}{96}\left[\int_0^1 t^3 f^{(3)}\left(\frac{t}{2}a + \frac{2-t}{2}b\right)dt - \int_0^1 t^3 f^{(3)}\left(\frac{2-t}{2}a + \frac{t}{2}b\right)dt\right].$$

In the sequel of paper, we deduce

$$L_p[a,b] = \left\{f: \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} < \infty, 1 \leq p < \infty\right\}$$

where $[a,b]$ is a closed interval.

2 Inequalities for log-convex functions

We shall start the following result:

Theorem 2.1. Let $f: I \to [0,\infty)$, be a three times differentiable mapping on I° such that $f''' \in L_1[a,b]$ where $a,b \in I^\circ$ with $a<b$. If $\left|f'''\right|$ is log −convex on $[a,b]$, then the following inequality holds:

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^3}{96}\left\{\left|f'''(b)\right|\mu_K + \left|f'''(a)\right|\mu_M\right\}$$

where

$$\mu_K = \frac{2K^{\frac{1}{2}}(\ln K - 6)}{(\ln K)^2} + \frac{48K^{\frac{1}{2}}(\ln K - 2)}{(\ln K)^4} + \frac{96}{(\ln K)^4},$$

$$\mu_M = \frac{2M^{\frac{1}{2}}(\ln M - 6)}{(\ln M)^2} + \frac{48M^{\frac{1}{2}}(\ln M - 2)}{(\ln M)^4} + \frac{96}{(\ln M)^4}$$

and

$$K = \frac{\left|f'''(a)\right|}{\left|f'''(b)\right|}, M = \frac{\left|f'''(b)\right|}{\left|f'''(a)\right|}.$$

In the sequel of the paper, we set $K, M \neq 1$.

Proof. From Lemma 1.1, property of the modulus and log −convexity of $\left|f'''\right|$ we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^3}{96}\left\{\int_0^1 t^3\left|f'''\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\right|dt\right.$$

$$\left. + \int_0^1 t^3\left|f'''\left(\frac{t}{2}b + \frac{2-t}{2}a\right)\right|dt\right\}$$

$$\leq \frac{(b-a)^3}{96}\left\{\int_0^1 t^3\left|f'''(a)\right|^{\frac{t}{2}}\left|f'''(b)\right|^{1-\frac{t}{2}}dt\right.$$

$$\left. + \int_0^1 t^3\left|f'''(b)\right|^{\frac{t}{2}}\left|f'''(a)\right|^{1-\frac{t}{2}}dt\right\}$$

$$= \frac{(b-a)^3}{96}\left\{\left|f'''(b)\right|\int_0^1 t^3\left[\frac{\left|f'''(a)\right|}{\left|f'''(b)\right|}\right]^{\frac{t}{2}}dt\right.$$

$$\left. \left|f'''(a)\right|\int_0^1 t^3\left[\frac{\left|f'''(b)\right|}{\left|f'''(a)\right|}\right]^{\frac{t}{2}}dt\right\}.$$

The proof is completed by making use of the neccessary computation.

Corollary 2.1. Let μ_K, μ_M, K and M be defined as in Theorem 2.1. If we choose $f''\left(\frac{a+b}{2}\right) = 0$ in Theorem 2.1, we obtain the following inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^3}{96}\left\{\left|f'''(b)\right|\mu_K + \left|f'''(a)\right|\mu_M\right\}.$$

Theorem 2.2. Let $f: I \to [0,\infty)$, be a three times differentiable mapping on I° such that $f''' \in L_1[a,b]$ where $a,b \in I^\circ$ with $a<b$. If $\left|f'''\right|$ is log −convex on $[a,b]$, then the following inequality holds for some fixed $q > 1$

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^3}{96}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}\left\{\left|f'''(b)\right|\left(\frac{2}{q\ln K}\left[K^{\frac{q}{2}} - 1\right]\right)^{\frac{1}{q}}\right.$$

$$\left. + \left|f'''(a)\right|\left(\frac{2}{q\ln M}\left[M^{\frac{q}{2}} - 1\right]\right)^{\frac{1}{q}}\right\}$$

where K and M are as in Theorem 2.1. and

$\frac{1}{p}+\frac{1}{q}=1.$

Proof. From Lemma 1.1 and using the Hölder integral inequality, we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^3}{96}\left\{\left(\int_0^1 t^{3p}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 t^{3p}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'''\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$

Since $\left|f'''\right|$ is log −convex on $[a,b]$ we can say $\left|f'''\right|^q$ is also log −convex on $[a,b]$. If we use the log-convexity of $\left|f'''\right|^q$ above, we can write

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^3}{96}\left\{\left(\int_0^1 t^{3p}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'''(a)\right|^{\frac{qt}{2}}\left|f'''(b)\right|^{q-\frac{qt}{2}} dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 t^{3p}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'''(b)\right|^{\frac{qt}{2}}\left|f'''(a)\right|^{q-\frac{qt}{2}} dt\right)^{\frac{1}{q}}\right\}$$

$$= \frac{(b-a)^3}{96}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}\left\{\left|f'''(b)\right|\left(\frac{2}{q\ln K}\left[K^{\frac{q}{2}}-1\right]\right)^{\frac{1}{q}}\right.$$

$$\left.+\left|f'''(a)\right|\left(\frac{2}{q\ln M}\left[M^{\frac{q}{2}}-1\right]\right)^{\frac{1}{q}}\right\}.$$

The proof is completed.

Corollary 2.2. Let K and M be defined as in Theorem 2.2. If we choose $f''\left(\frac{a+b}{2}\right)=0$ in Theorem 2.2, we obtain the following inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^3}{96}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}\left\{\left|f'''(b)\right|\left(\frac{2}{q\ln K}\left[K^{\frac{q}{2}}-1\right]\right)^{\frac{1}{q}}\right.$$

$$\left.+\left|f'''(a)\right|\left(\frac{2}{q\ln M}\left[M^{\frac{q}{2}}-1\right]\right)^{\frac{1}{q}}\right\}$$

where $q>1,\ \frac{1}{p}+\frac{1}{q}=1.$

Theorem 2.3. Let $f: I\to[0,\infty)$, be a three times differentiable mapping on I° such that $f'''\in L_1[a,b]$ where $a,b\in I^\circ$ with $a<b$. If $\left|f'''\right|$ is log − convex on $[a,b]$. Then the following inequality holds for some fixed $q\geq 1$:

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^3}{96}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left\{\left|f'''(b)\right|\left(\mu_{K,q}\right)^{\frac{1}{q}}+\left|f'''(a)\right|\left(\mu_{M,q}\right)^{\frac{1}{q}}\right\}$$

where

$$\mu_{K,q}=\frac{2K^{\frac{q}{2}}(q\ln K-6)}{(q\ln K)^2}+\frac{48K^{\frac{q}{2}}(q\ln K-2)}{(q\ln K)^4}+\frac{96}{(q\ln K)^4},$$

$$\mu_{M,q}=\frac{2M^{\frac{q}{2}}(q\ln M-6)}{(q\ln M)^2}+\frac{48M^{\frac{q}{2}}(q\ln M-2)}{(q\ln M)^4}+\frac{96}{(q\ln M)^4}$$

and K, M are as in Theorem 2.1.

Proof. From Lemma 1.1, using the well-known power-mean integral inequality and log −convexity of $\left|f'''\right|^q$ we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^3}{96}\left\{\left(\int_0^1 t^3 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^3\left|f'''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 t^3 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^3\left|f'''\left(\frac{t}{2}b+\frac{2-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right\}$$

$$\leq \frac{(b-a)^3}{96}\left\{\left(\int_0^1 t^3 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^3\left|f'''(a)\right|^{\frac{qt}{2}}\left|f'''(b)\right|^{q-\frac{qt}{2}} dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 t^3 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^3\left|f'''(b)\right|^{\frac{qt}{2}}\left|f'''(a)\right|^{q-\frac{qt}{2}} dt\right)^{\frac{1}{q}}\right\}.$$

The proof is completed by making use of the neccessary computation.

Corollary 2.3. Let $\mu_{K,q}, \mu_{M,q}$ be defined as in Theorem 2.3 and *K, M* be defined as in Theorem 2.1. If we choose $f''\left(\frac{a+b}{2}\right)=0$ in Theorem 2.3 , we obtain the following inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{(b-a)^3}{96}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left\{\left|f'''(b)\right|\left(\mu_{K,q}\right)^{\frac{1}{q}}+\left|f'''(a)\right|\left(\mu_{M,q}\right)^{\frac{1}{q}}\right\}.$$

Corollary 2.4. From Corollaries 2.1-2.3, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \min\left\{\chi_1,\chi_2,\chi_3\right\}$$

where

$$\chi_1 = \frac{(b-a)^3}{96}\left\{\left|f'''(b)\right|\frac{2K^{\frac{1}{2}}(\ln K-6)}{(\ln K)^2}+\frac{48K^{\frac{1}{2}}(\ln K-2)}{(\ln K)^4}\right.$$

$$+\frac{96}{(\ln K)^4}+\left|f'''(a)\right|\frac{2M^{\frac{1}{2}}(\ln M-6)}{(\ln M)^2}$$

$$\left.+\frac{48M^{\frac{1}{2}}(\ln M-2)}{(\ln M)^4}+\frac{96}{(\ln M)^4}\right\},$$

$$\chi_2 = \frac{(b-a)^3}{96}\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}$$

$$\times\left\{\left|f'''(b)\right|\left(\frac{2}{q\ln K}\left[K^{\frac{q}{2}}-1\right]\right)^{\frac{1}{q}}\right.$$

$$\left.+\left|f'''(a)\right|\left(\frac{2}{q\ln M}\left[M^{\frac{q}{2}}-1\right]\right)^{\frac{1}{q}}\right\},$$

$$\chi_3 = \frac{(b-a)^3}{96}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left\{\left|f'''(b)\right|\left(\frac{2K^{\frac{q}{2}}(q\ln K-6)}{(q\ln K)^2}+\right.\right.$$

$$\left.+\frac{48K^{\frac{q}{2}}(q\ln K-2)}{(q\ln K)^4}+\frac{96}{(q\ln K)^4}\right)^{\frac{1}{q}}+\left|f'''(a)\right|$$

$$\times\left(\frac{2M^{\frac{1}{2}}(\ln M-6)}{(\ln M)^2}+\frac{48M^{\frac{1}{2}}(\ln M-2)}{(\ln M)^4}\right.$$

$$\left.\left.+\frac{96}{(\ln M)^4}\right)^{\frac{1}{q}}\right\}$$

and *K, M* are as in *Theorem 2.1.*

Remark 2.1. In Theorem 2.3 and Corollary 2.3, if we choose $q=1$, we obtain Theorem 2.1 and Corollary 2.1 respectively.

3 Applications to midpoint formula

We give some error estimates to midpoint formula by using the results of Section 2. Let d be a division $a=x_0<x_1<...<x_{n-1}<x_n=b$ of the interval $[a,b]$ and consider the formula $\int_a^b f(x)dx = M(f,d)+E(f,d)$

where $M(f,d)=\sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)$ for the midpoint version and $E(f,d)$ denotes the associated approximation error.

Proposition 3.1. Let $f: I \to [0,\infty)$ be a three times differentiable mapping on I° with $a,b \in I^\circ$

such that $a<b$. If $\left|f'''\right|$ is log −convex function with $f'''\in L_1[a,b]$, then for every division d of $[a,b]$, the midpoint error estimate satisfies

$$\left|E(f,d)\right| \le \sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^4}{96}\left\{\left|f'''(x_{i+1})\right|\mu_1+\left|f'''(x_i)\right|\mu_2\right\}$$

where

$$\mu_1=\frac{2K_1^{\frac{1}{2}}(\ln K_1-6)}{(\ln K_1)^2}+\frac{48K_1^{\frac{1}{2}}(\ln K_1-2)}{(\ln K_1)^4}+\frac{96}{(\ln K_1)^4},$$

$$\mu_2=\frac{2M_1^{\frac{1}{2}}(\ln M_1-6)}{(\ln M_1)^2}+\frac{48M_1^{\frac{1}{2}}(\ln M_1-2)}{(\ln M_1)^4}+\frac{96}{(\ln M_1)^4}$$

and

$$K_1=\frac{\left|f'''(x_i)\right|}{\left|f'''(x_{i+1})\right|},\ M_1=\frac{\left|f'''(x_{i+1})\right|}{\left|f'''(x_i)\right|}.$$

Also $K_1,M_1\neq 1$.

Proof. By applying Corollary 2.1 on the subintervals $[x_i,x_{i+1}]$, $(i=0,1,\ldots,n-1)$ of the division d we have

$$\left|\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(x)dx-f\left(\frac{x_i+x_{i+1}}{2}\right)\right| \le \frac{(x_{i+1}-x_i)^3}{96}\left\{\left|f'''(x_{i+1})\right|\mu_1+\left|f'''(x_i)\right|\mu_2\right\}.$$

By summing over i from 0 to $n-1$, we can write

$$\left|\int_a^b f(x)dx-M(f,d)\right| \le \sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^4}{96}\left\{\left|f'''(x_{i+1})\right|\mu_1+\left|f'''(x_i)\right|\mu_2\right\}$$

which completes the proof.

Proposition 3.2. Let $f:I\to[0,\infty)$ be a three times differentiable mapping on I° with $a,b\in I^\circ$ such that $a<b$. If $\left|f'''\right|^q$ is log – convex function with $f'''\in L_1[a,b]$ for some fixed $q>1$, then for every division d of $[a,b]$, the midpoint error estimate satisfies

$$\left|E(f,d)\right|\le\left(\frac{1}{3p+1}\right)^{\frac{1}{p}}\frac{1}{96}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^4 \times\left\{\left|f'''(x_{i+1})\right|\left(\frac{2}{q\ln K_1}\left[K_1^{\frac{q}{2}}-1\right]\right)^{\frac{1}{q}}+\left|f'''(x_i)\right|\left(\frac{2}{q\ln M_1}\left[M_1^{\frac{q}{2}}-1\right]\right)^{\frac{1}{q}}\right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and K_1,M_1 are as defined in Proposition 3.1.

Proof. The proof can be maintained by using Corollary 2.2 like Proposition 3.1.

Proposition 3.3. Let $f:I\to[0,\infty)$ be a three times differentiable mapping on I° with $a,b\in I^\circ$ such that $a<b$. If $\left|f'''\right|^q$ is log – convex function with $f'''\in L_1[a,b]$ for some fixed $q\ge 1$, then for every division d of $[a,b]$, the midpoint error estimate satisfies

$$\left|E(f,d)\right|\le\frac{1}{96}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^4 \times\left\{\left|f'''(x_{i+1})\right|(\mu_{1,q})^{\frac{1}{q}}+\left|f'''(x_i)\right|(\mu_{2,q})^{\frac{1}{q}}\right\}$$

where

$$\mu_{1,q}=\frac{2K_1^{\frac{q}{2}}(q\ln K_1-6)}{(q\ln K_1)^2}+\frac{48K_1^{\frac{q}{2}}(q\ln K_1-2)}{(q\ln K_1)^4}+\frac{96}{(q\ln K_1)^4},$$

$$\mu_{2,q}=\frac{2M_1^{\frac{q}{2}}(q\ln M_1-6)}{(q\ln M_1)^2}+\frac{48M_1^{\frac{q}{2}}(q\ln M_1-2)}{(q\ln M_1)^4}+\frac{96}{(q\ln M_1)^4}$$

and K_1,M_1 are as defined in Proposition 3.1.

Proof. The proof can be maintained by using Corollary 2.3 like Proposition 3.1.

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