Singularities of the Darboux ruled surface of a space curve in the pseudo-Galilean space

Tevfik Şahin

Amasya University, Faculty of Arts and Sciences, Department of Mathematics, Amasya-Turkey
tevfik.sahin@amasya.edu.tr

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Abstract

In this article, we establish the singularity theory in a pseudo-Galilean space G_1^3, a special case of Cayley-Klein spaces. We consider the cases where the Darboux ruled surface in G_1^3 is diffeomorphic to some surfaces in the neighbourhood of a singular point. In addition, we investigate the relationship between singularities of discriminant, bifurcation sets of the function, and geometric invariants of curves in G_1^3.
Keywords — Height function, singularities, Darboux ruled surface, pseudo-Galilean space.

1 Introduction

The singularity theory of smooth mappings is strongly related to both the Morse theory and the theory of immersions and embeddings of manifolds. Indeed, these two theories are originated from the theory of smooth functions in one-variable. We recall that the roots of derivative of a function are called critical or singular points. The graph of function behaves differently in the vicinity of a singular point. This effects the shape of graph of the function in the neighborhood of a singular point. As a result, one can infer that singular points of a function reveals crucial information about the shape of graphs of the function.

A family of functions containing f is called an *unfolding* of f: the family unfolds to reveal all these functions which are f's close relations. Singularity theory is more concerned with two other properties of such families. First, 'almost all' families of functions are universal unfoldings (strictly, 'versal' unfoldings) of each function in the family. Versal unfolding has been a central tool in almost all applications of singularity theory inside and outside mathematics. Consequently one can expect these unfoldings to arise in virtually any situation when studying families of functions. Secondly, these unfoldings are in a certain sense unique, that is, they depend only on the functions that they are unfolding. Thus one can expect same models, describing the geometry of the unfolding, to arise in many (almost all) situations. In other words, the bifurcation set or discriminant of the family is diffeomorphic to the bifurcation or discriminant set of a "standard" versal deformation of a function having the same type of singuarity. For example, the standard $\Re_e-$versal deformation (i.e., deformation which is versal for $R-$equivalence) of an A_3 singularity of $f(x)=x^4$ is $G(x,a,b,c)=x^4+ax^2+bx+c$ and therefore any $\Re_e-$versal deformation F of a function having an A_3 singularity has discriminant diffeomorphic to the discriminant SW of G. This is a well-known fact for swallowtail surface SW. For more details we refer the reader to [1].

The aim of this article is to show that certain germs of geometrically defined subsets of the pseudo Galilean space G_1^3 are diffeomorphic to cusp, cuspidal edge and swallowtail singularities by using some standard arguments from singularity theory. Initially, we want to construct a germ of family of functions $F: X\times G_1^3, (x_0,w_0)\rightarrow K$ $(K=R \; or \; K=C)$

on some space X (it doesn't matter what X is), with parameter space G_1^3, such that the germ at w_0 of the subset in question is the bifurcation set or discriminant of the family. We then strive to show that the family of functions is a versal deformation (with respect to some notion of parametrised equivalence of unfoldings) of the germ at x_0 of the function f_{w_0} defined by $f_{w_0}(x) = F(x, w_0)$.

General theory of differential geometry of curves and surfaces in Cayley-Klein spaces can be found in [2].

Study of singularities of curves and surfaces in Euclidean and non-Euclidean ambient spaces does not have a long history. There are some applications of singularity theory in the Euclidean and non-Euclidean geometry. Several references on these applications in Euclidean space, Minkowski space, and Galilean space can be found in [1, 3-8].

In this article, we apply elementary singularity theory techniques, along the lines developed in the basic book [1], to the study of geometrical invariants of curves in G_1^3. To this purpose, we introduce the notion of height function on space curves in G_1^3. The height function is quite useful for the study of singularities of the spherical Darboux ruled (we abbreviate as s.D.r.) surface of space curves in G_1^3. We also introduce the notion of the line of striction of the s.D.r. surface and the spherical Darboux images of space curves in G_1^3 As a result, we establish several relationships between the singularities of the above two subjects and geometric invariants of a curve under the action of G_1^3 group as applications of ordinary techniques of singularity theory for the above function. Therefore, the singularities of the spherical Darboux image describe how the shape of a curve is similar to helix.

The main result in this paper is Theorem 3.1. The theorem is about the singularities of the spherical Darboux ruled surface. We describe the geometric interpretation of Theorem 3.1 in section 3.1 and 3.2. Our basic techniques here follow those of Bruce and Giblin [1]. For this paper, we are inspired by [5-6].

2 Preliminaries on pseudo-Galilean Geometry

The pseudo-Galilean space G_1^3 is one of the Cayley-Klein spaces equipped with the projective metric of signature $(0,0,+,-)$ [9]. Note that G_1^3 is called the Galilean space of index 1. The absolute figure of the pseudo-Galilean space is the ordered triple $\{w, f, I\}$, where w is an ideal (absolute) plane, in the real three-dimensional projective space $P^3(R)$, f is a line (absolute line) in w, and I is a fixed hyperbolic involution of points of f.

In non-homogeneous coordinates the group of motion of G_1^3 (i.e. the group of isometries of G_1^3) has the form define:

$$\begin{aligned} \bar{x} &= a_1 + x, \\ \bar{y} &= a_2 + a_3 x + y\cosh\varphi + z\sinh\varphi, \\ \bar{z} &= a_4 + a_5 x + y\sinh\varphi + z\cosh\varphi, \end{aligned} \tag{2.1}$$

where a_1, a_2, a_3, a_4, a_5 and φ are real numbers [10]. If the first component of a vector is not zero, then the vector is called as non-isotropic, otherwise it is called isotropic vector [10].

The scalar product of two vectors $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ in G_1^3 is defined by

$$\mathbf{v} \cdot_{pG} \mathbf{w} = \begin{cases} v_1 w_1, & \text{if } v_1 \neq 0 \text{ or } w_1 \neq 0 \\ v_2 w_2 - v_3 w_3, & \text{if } v_1 = 0 \text{ and } w_1 = 0. \end{cases}$$

If $\mathbf{v} \cdot_{pG} \mathbf{w} = 0$, then $\mathbf{v}$ and $\mathbf{w}$ are perpendicular. In particular, every isotropic vector is perpendicular to every non-isotropic vector. The norm of $\mathbf{v}$ is defined by $\|\mathbf{v}\|_{pG} = \sqrt{|\mathbf{v} \cdot_{pG} \mathbf{v}|}$.

Let $I \subset R$ and let $\alpha : I \to G_1^3$ be a curve parametrized by arc length (we abbreviate as p.b.a.l) with curvature $\kappa > 0$ and torsion τ. If α is a curve p.b.a.l. that is,

$$\alpha(x) = (x, y(x), z(x)),$$

then the Frenet frame fields are given by

$$\begin{aligned} T(x) &= \alpha'(x), \\ N(x) &= \frac{\alpha''(x)}{\|\alpha''(x)\|_{pG}}, \\ B(x) &= \frac{1}{\kappa(x)}\left(0, \varepsilon z''(x), \varepsilon y''(x)\right), \end{aligned} \tag{2.2}$$

where $\kappa(x)$ and $\tau(x)$ are defined by

$$\begin{aligned} \kappa(x) &= \|\alpha''(x)\|_{pG}, \\ \tau(x) &= \frac{\det\left(\alpha'(x), \alpha''(x), \alpha'''(x)\right)}{\kappa^2(x)}. \end{aligned} \tag{2.3}$$

Also, where $\varepsilon = \pm 1$ determined by the criterion $\det(T, N, B) = \pm 1$. The vectors T, N and B are called the vectors of the tangent, the principal normal and the binormal vector field, respectively [10]. Therefore, the Frenet-Serret formulas can be written as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{2.4}$$

If D is the vector field $-\tau(x)T(x) + \kappa(x)B(x)$ on a unit speed curve α, then we can show that Frenet formulas become

$$\begin{aligned} T' &= D \times_{pG} T, \\ N' &= D \times_{pG} N, \\ B' &= D \times_{pG} B \end{aligned} \tag{2.5}$$

The vector field D is called a *Darboux vector* of α [5].

Also, where $\times_{pG}$ is the pseudo-Galilean cross product defined by

$$\mathbf{v} \times_{pG} \mathbf{w} = \begin{vmatrix} 0 & \mathbf{e}_2 & -\mathbf{e}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{2.6}$$

for $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ [11]. The pseudo-Galilean Sphere S^2_{pG} with center x_0 and radius r is defined by $S^2_{pG} = \left\{(x, y, z) \in G^3_1 \mid |x - x_0| = r\right\}$.

We refer to [10-12, 13] for detailed treatment of Galilean and pseudo-Galilean geometry.

3 Singularities of the Darboux ruled surfaces in G_1^3

Ruled surfaces are the classical subject in differential geometry. There are important classes of ruled surfaces defined by Frenet vectors of a given regular curve such as natural developable (tangent, focal and rectifying) and Darboux ruled surfaces. Recently there appeared several articles concerning on singularities of these ruled surfaces (ruled surfaces are also special surfaces in general singular surfaces) in Euclidean and Non-Euclidean geometry [4-8].

We define a spherical curve $d : I \to S^2_{pG}$ by $d(x) = \dfrac{D(x)}{\|D(x)\|_{pG}}$, where $D(x)$ is the Darboux vector, and surface

$$dR(\alpha) = \{d(x) + \lambda N(x) : \lambda \in R, \ x \in I\},$$

and curve

$$\gamma(x) = \left\{d(x) - \frac{1}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x) N(x) : x \in I\right\}.$$

We call the image of d as the pseudo-Galilean spherical Darboux image, the surface $dR(\alpha)$ as the pseudo-Galilean spherical Darboux ruled (s.D.r.) surface of γ and the curve $\gamma(x)$ as the line of striction of the Darboux ruled surface. The following is the main theorem of this paper.

Theorem 3.1 Let $\alpha : I \to G_1^3$ be curve p.b.a.l. with $\kappa(x) \neq 0$, and we assume $\tau(x) \neq 0$. Then we have:
1) The line of striction of the pseudo-Galilean s.D.r. surface image is locally diffeomorphic to the ordinary cusp C at $\alpha(x_0)$ iff

$$\left(\frac{\kappa}{\tau}\right)''(x_0) = \left(\frac{\tau'}{\tau}\left(\frac{\kappa}{\tau}\right)'\right)(x_0) \text{ and}$$

$$\left(\frac{\kappa}{\tau}\right)'''(x_0) \neq \left(\frac{\tau''}{\tau}\left(\frac{\kappa}{\tau}\right)'\right)(x_0).$$

2-a) The pseudo-Galilean s.D.r. surface is locally diffeomorphic to the cuspidal edge $\boldsymbol{C}\times\boldsymbol{R}$ at

$d(x_0)+\lambda_0 N(x_0)$ iff $\lambda_0 = -\left(\frac{1}{\tau}\left(\frac{\kappa}{\tau}\right)'\right)(x_0)$ and

$$\left(\frac{\kappa}{\tau}\right)''(x_0) \neq \left(\frac{\tau'}{\tau}\left(\frac{\kappa}{\tau}\right)'\right)(x_0).$$

2-b) The pseudo-Galilean s.D.r. surface is locally diffeomorphic to the swallowtail SW at $d(x_0)+\lambda_0 N(x_0)$ if and only if

$$\lambda_0 = -\left(\frac{1}{\tau}\left(\frac{\kappa}{\tau}\right)'\right)(x_0),$$

$$\left(\frac{\kappa}{\tau}\right)''(x_0) = \left(\frac{\tau'}{\tau}\left(\frac{\kappa}{\tau}\right)'\right)(x_0) \text{ and}$$

$$\left(\frac{\kappa}{\tau}\right)'''(x_0) \neq \left(\frac{\tau''}{\tau}\left(\frac{\kappa}{\tau}\right)'\right)(x_0).$$

Here, $C=\{(x,y): x^2 = y^3\}$ is ordinary cusp and

$SW=\{(x,y,z): x=3u^4+u^2v, y=4u^3+2uv, z=v\}$ is the swallowtail.

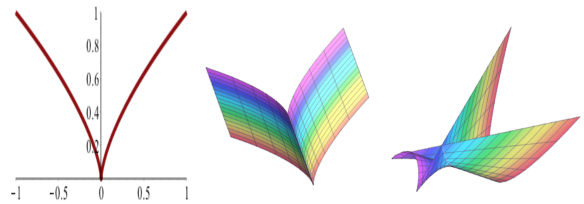

Figure 1: The cusp curve, cuspidal edge and swallowtail surface

The main goal of this paper is to give a proof for the Theorem 3.1. To this purpose, we shall study the singularities of the height function in G_1^3 in section 3.1. Since we need unfoldings of functions in G_1^3, we describe them in detail in section 3.2.

3.1 Families of smooth functions on a space curve in G_1^3

From now on, unless we explicitly state otherwise, we will only consider curves parametrized by arc length (p.b.a.l.) with $\kappa(x) \neq 0$, and we assume $\tau(x) \neq 0$.
In this part, we now introduce some families of functions that useful for the study of singularities of a space curve. Such functions are.

3.1.1 Height function in G_1^3

Consider the folowing two-parameter family of smooth functions on I:

$H: I\times S_{pG}^2 \to R$

with $H(x,w) = \det(T(x), B(x), w)$. We call H as the height function on α. We use the notation $h_{\mathbf{w}}(x) = H(x,\mathbf{w})$ for any $\mathbf{w} \in S_{pG}^2$. Then, we obtain the following proposition.

Proposition 3.2 Let $\alpha : I \to G_1^3$ be a curve. Then,

1) $h_{\mathbf{w}}'(x) = 0$ iff there exists a real number $u \in R$ such that $w = \pm T(x) + uN(x) \mp \left(\frac{\kappa}{\tau}\right)(x)B(x)$

2) $h_{\mathbf{w}}'(x) = h_{\mathbf{w}}''(x) = 0$ iff

$$\mathbf{w} = \pm\left(T(x) + \frac{1}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x)N(x) - \left(\frac{\kappa}{\tau}\right)(x)B(x)\right).$$

3) $h_{\mathbf{w}}'(x) = h_{\mathbf{w}}''(x) = h_{\mathbf{w}}'''(x) = 0$ iff

$$\mathbf{w} = \pm\left(T(x) + \frac{1}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x)N(x) - \left(\frac{\kappa}{\tau}\right)(x)B(x)\right),$$

$$\left(\frac{\kappa}{\tau}\right)''(x) = \frac{\tau'(x)}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x).n$$

4) $h_{\mathbf{w}}'(x) = h_{\mathbf{w}}''(x) = h_{\mathbf{w}}'''(x) = h_{\mathbf{w}}^{(4)}(x) = 0$ iff

$$\mathbf{w} = \pm\left(T(x) + \frac{1}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x)N(x) - \left(\frac{\kappa}{\tau}\right)(x)B(x)\right),$$

$$\left(\frac{\kappa}{\tau}\right)''(x) = \frac{\tau'(x)}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x),$$

$$\left(\frac{\kappa}{\tau}\right)'''(x) = \frac{\tau''(x)}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x)$$

Proof. From the Frenet-Serret formula we have:

i.

$$h_{\mathbf{w}}'(x) = \kappa(x)\left|N(x)\ B(x)\ \mathbf{w}\right| + \tau(x)\left|T(x)\ N(x)\ \mathbf{w}\right|$$

ii.

$$\begin{aligned} h_{\mathbf{w}}''(x) = \ & \kappa'(x)\left|N(x)\ B(x)\ \mathbf{w}\right| \\ & + \tau'(x)\left|T(x) N(x)\ \mathbf{w}\right| \\ & + \tau^2(x)\left|T(x)\ B(x)\ \mathbf{w}\right| \end{aligned}$$

iii.

$$\begin{aligned} h_{\mathbf{w}}'''(x) = \ & \left(\kappa''(x) + \kappa(x)\tau^2(x)\right)\left|N(x)\ B(x)\ \mathbf{w}\right| \\ & + \left(\tau^3(x) + \tau''(x)\right)\left|T(x)\ N(x)\ \mathbf{w}\right| \\ & + 3\tau(x)\tau'(x)\left|T(x)\ B(x)\ \mathbf{w}\right|. \end{aligned}$$

iv.

$$\begin{aligned} h_{\mathbf{w}}^{(4)}(x) = \ & \left(\kappa''' + \kappa'\tau^2 + 5\kappa\tau\tau'\right)(x)\left|N(x)\ B(x)\ \mathbf{w}\right| \\ & + \left(\tau'''(x) + 6\tau^2(x)\tau'(x)\right)\left|T(x)\ N(x)\ \mathbf{w}\right| \\ & + \left(3\tau'^2 + \tau^4 + 4\tau\tau''\right)(x)\left|T(x)\ B(x)\ \mathbf{w}\right|. \end{aligned}$$

Now we prove each part of the theorem:

1) The assertion is trivial by the formula (i). By the assumption $\mathbf{w} \in S_{pG}^2$, we have $\mathbf{w} = \pm T(x) + \mu N(x) + \lambda B(x)$. It follows from (i) that $h_{\mathbf{w}}'(x) = \pm\kappa(x) + \lambda\tau(x)$. Therefore we have $W = \pm T(x) + \mu N(x) \mp \left(\frac{\kappa}{\tau}\right)(x)B(x)$

2) By (1) in (ii), we get $\mu = \pm\frac{1}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x)$.

Therefore, we have

$$\mathbf{w} = \pm\left(T(x) + \frac{1}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x)N(x) - \left(\frac{\kappa}{\tau}\right)(x)B(x)\right).$$

3) By writing

$$\mathbf{w} = \pm\left(T(x) + \frac{1}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x)N(x) - \left(\frac{\kappa}{\tau}\right)(x)B(x)\right)$$

in (iii), we get $\left(\frac{\kappa}{\tau}\right)''(x) = \frac{\tau'(x)}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x)$.

4) By using (3) in (iv) then we get

$$\left(\frac{\kappa}{\tau}\right)'''(x)=\frac{\tau''(x)}{\tau(x)}\left(\frac{\kappa}{\tau}\right)'(x).$$

3.2 Unfoldings of functions by one-variable

We start with covering some fundamental results on singularity theory. For more details, we refer the reader to [1].

Tanım 3.1 [1]. Let $F: R\times R^r,(x_0,w_0)\to R$ be a smooth function. We write $f_{w_0}: R, x_0 \to R$ for the function $f_{w_0}(x)=F(x,w_0)$. F as above is called an $r-$*parameter unfolding of* f_{w_0}. From now on f will stand for f_{w_0} unless otherwise stated. Let $F: I\times R^r,(x_0,w_0)\to R$ be a function germ. We say that f has A_k-singularity at x_0 if $f'(x_0)=f''(x_0)=\ldots=f^{(k)}(x_0)=0$ and $f^{(k+1)}(x_0)\neq 0$. Let F be an unfolding of f and let $f(x)$ have A_k-singularity $(k\geq 1)$ at x_0. Let us denote the $(k-1)-$jet of the partial derivative $\frac{\partial F}{\partial w_i}$ at x_0 by

$$J^{k-1}\left(\frac{\partial F}{\partial w_i}(x,w_0)\right)(x_0)=\sum_{j=1}^{k-1}\alpha_{ij}x^j, \qquad i=1,...,r.$$

Tanım 3.2 [1]. The unfolding $G: R\times R^{r-1}\to R$, given by $G(t,x)=\pm t^{k+1}+x_1t+x_2t^2+\cdots+x_{k-1}t^{k-1}$ is a $(p)-$versal unfolding of $g(t)=\pm t^{k+1}$ at $t_0=0$.

The unfolding $G: R\times R^k\to R$, given by $G(t,x)=\pm t^{k+1}+x_1+x_2t+\cdots+x_kt^{k-1}$ is a versal unfolding of $g(t)=\pm t^{k+1}$ at $t_0=0$ [1].

Next, we will give matrix criterion for versality and $(p)-$ versality.

Then F is called {a $(p)-$versal unfolding }if the $(k-1)\times r$ matrix of coefficients (α_{ij}) has rank $k-1$ $(k-1\leq r)$. Under the same conditions as the above, then F is called {a versal unfolding} if the $k\times r$ matrix of coefficients $(\alpha_{0i},\alpha_{ij})$ is of rank k $(k\leq r)$, where $\alpha_{0i}=\frac{\partial F}{\partial w_i}(x_0,w_0)$. In the following, we describe a set of results related to foregoing notions. The *discriminant set* of $\boldsymbol{F}$ is the set

$$D_F=\left\{w\in R^r : F=\frac{\partial F}{\partial x}=0 \ at \ (x,w)\ for\ some\ x\right\},$$

and the *bifurcation set* B_F of F is the set

$$B_F=\left\{w\in R^r : F=\frac{\partial F}{\partial x}=\frac{\partial^2 F}{\partial x^2}=0 \ at \ (x,w)\ at\ x\right\}.$$

What follows is an interpretation of uniqueness theorems for bifurcation and discriminant sets given in [1]. Now let F and G be any two $(p)-$ versal $r-$ parameter unfoldings of f (at t_0) and g (at t_1) respectively, both of type A_k $(k\geq 1)$.

Thus the discriminant sets D_F and D_G (bifurcation sets B_F and B_G) are locally diffeomorphic: the local picture is the same up to diffeomorphism for any $r-$ parameter versal ($(p)-$ versal) unfolding of any A_k singularity. That is, we can say that the bifurcation set (or discriminant) of the family is diffeomorphic to the bifurcation set (or discriminant) of a "standard" versal deformation of a function having the same type of singularity. For example, the standard $\mathfrak{R}_e-$ versal deformation (i.e. deformation which is versal for $\mathfrak{R}-$ equivalence) of an A_3 singularity $f(x)=x^4$ is $F(x,a,b,c)=x^4+ax^2+bx+c$ and thus any $\mathfrak{R}_e-$ versal deformation G of a function having an A_3 singularity has discriminant diffeomorphic to the discriminant SW of F, which is the well-known swallowtail surface. Then we have the following well-known result [1].

Theorem 3.3 [1]. Let F be an $r-$parameter unfolding of $f(x)$ having an A_k- singularity at x_0. Suppose that F is a (p)- versal unfolding. Then,
a) If $k=1$ ($k=2$), then D_F (B_F) is locally diffeomorphic to $\{0\}\times R^{r-1}$.
b) If $k=2$ ($k=3$), then D_F (B_F) is locally diffeomorphic to $C\times R^{r-2}$.
c) If $k=3$ ($k=4$), then D_F (B_F) is locally diffeomorphic to $SW\times R^{r-3}$.
Here, $C=\{(x,y): x^2=y^3\}$ is the ordinary cusp and $SW=\{(x,y,z): x=3u^4+u^2v,\ y=4u^3+2uv,\ z=v\}$
is the swallowtail as shown in Figure 1.

In order to apply the results of singularity theory, we need to decide whether H is a versal unfolding of $h_{\mathbf{v}_0}$ at x_0 by finding $\frac{\partial H}{\partial w_i}$ and using matrix criterion given above. If the criterion is satisfied then locally (near x_0) the bifurcation set or discriminant set is diffeomorphic to the standard model applicape to the values of r and k in question. For the proof of Theorem 3.1, we have the following key propositions.

Proposition 3.4 Let $\alpha(x)$ be curve and assume that $H: I\times S^2_{pG}\rightarrow R$ is the height function on $\alpha(x)$. If $h_{\mathbf{w}_0}$ has A_k- singularity $(k=2,3)$ at the point x_0, then H is a $(p)-$ versal unfolding of $h_{\mathbf{w}_0}$.

Proof: Let $\alpha(x)=(x,y(x),z(x))$
and $\mathbf{w}=(1,w_2,w_3)$. By definition,

$$H(x,\mathbf{w})=\left|T(x)\ B(x)\ \mathbf{w}\right|$$
$$=\frac{1}{\kappa(x)}[y'(x)y''(x)-z'(x)z''(x)+z''(x)w_3-y''(x)w_2].$$

Let $J^{k-1}\left(\frac{\partial H}{\partial w_i}(x,\mathbf{w}_0)\right)(x_0)$ be the $(k-1)-$jet of $\frac{\partial H}{\partial w_i}$ at x_0 $(i=2,3)$. Then,

$$J^3\left(\frac{\partial H}{\partial w_i}(x,\mathbf{w}_0)\right)(x_0)=(-1)^{i+1}\left[N_i'(x_0)x+\frac{1}{2}N_i''(x_0)x^2+\frac{1}{6}N_i'''(x_0)x^3\right]$$

with $i=2,3$. Here,

$$N(x)=(0,N_2(x),N_3(x))=\frac{1}{\kappa(x)}(0,y''(x),z''(x))$$

by equation (2.4). In the following, we investigate two important cases:

Case 1: Suppose that $h_{\mathbf{v}_0}$ has A_2-singularity at x_0. We let a $1\times 2-$matrix M_1 be

$$M_1=\begin{bmatrix}\left(-\frac{y''(x_0)}{\kappa(x_0)}\right)' & \left(\frac{z''(x_0)}{\kappa(x_0)}\right)'\end{bmatrix}.$$

By equation (2.4), we get $N'(x)=\tau(x)B(x)\neq 0$, and therefore, $rankM_1=1$

Case 2: Suppose that $h_{\mathbf{w}_0}$ has A_3-singularity at the point x_0. We let a $2\times 2-$ matrix be

$$M_2=\begin{bmatrix}\left(-\frac{y''(x_0)}{\kappa(x_0)}\right)' & \left(\frac{z''(x_0)}{\kappa(x_0)}\right)' \\ \left(-\frac{y''(x_0)}{\kappa(x_0)}\right)'' & \left(\frac{z''(x_0)}{\kappa(x_0)}\right)''\end{bmatrix}.$$

From (2.4), we obtain
$$\left|M_2\right|=\left|T(x_0)\ N''(x_0)\ N'(x_0)\right|.$$
By plugging in the necessary derivatives above, we obtain $\left|M_2\right|=\tau^3(x_0)$. Since $\tau(x)\neq 0$, we conclude that $rankM_2=2$.

Let $\tilde{H}: I\times S^2_{pG}\times\mathbb{R}\rightarrow\mathbb{R}$, be a function such that

$\tilde{H}(x,\mathbf{w},v) = H(x,\mathbf{w}) - v$ and write $h_{\mathbf{w},v}(x) = \tilde{H}(x,\mathbf{w},v)$.

Proposition 3.5 If $h_{\mathbf{w}_0,v_0}$ has A_k-singularity $(k=1,2,3)$ at x_0, then H is a versal unfolding of $h_{\mathbf{w}_0,v_0}$.

Proof: We follow the similar notations used in proposition 3.4,

$$\tilde{H}(x,\mathbf{w},w_1) = \frac{1}{\kappa(x)}[y'(x)y''(x) - z'(x)z''(x) + z''(x)w_3 - y''(x)w_2] - w_1.$$

Let $J^{k-1}\left(\frac{\partial\tilde{H}}{\partial w_i}(x,\mathbf{w}_0)\right)(x_0)$ be the $(k-1)-$jet of $\frac{\partial\tilde{H}}{\partial w_i}$ at x_0 $(i=1,2,3)$. Then,

$$\frac{\partial\tilde{H}}{\partial w_1}(x_0,\mathbf{w}_0) + J^2\left(\frac{\partial\tilde{H}}{\partial w_1}(x,\mathbf{w}_0)\right)(x_0) = -1$$ and,

$$\frac{\partial\tilde{H}}{\partial w_i}(x_0,\mathbf{w}_0) + J^2\left(\frac{\partial\tilde{H}}{\partial w_i}(x,\mathbf{w}_0)\right)(x_0) =$$
$$(-1)^{i+1}\left[N_i(x_0) + N_i'(x_0)x + \frac{1}{2}N_i''(x_0)x^2\right]$$

with $i=2,3$. We now consider the following cases:

Case 1: Suppose that $h_{\mathbf{w}_0,v_0}$ has A_1-singularity at x_0.

If M_3 is defined as

$$M_3 = \begin{bmatrix} -1 & \left(-\frac{y''(x_0)}{\kappa(x_0)}\right) & \left(\frac{z''(x_0)}{\kappa(x_0)}\right) \end{bmatrix},$$

then $rank(M_3)=1$.

Case 2: Suppose that $h_{\mathbf{w}_0,v_0}$ has A_2-singularity at x_0.

We define a 2×3 matrix M_4 by

$$M_4 = \begin{bmatrix} -1 & \left(-\frac{y''(x_0)}{\kappa(x_0)}\right) & \left(\frac{z''(x_0)}{\kappa(x_0)}\right) \\ 0 & \left(-\frac{y''(x_0)}{\kappa(x_0)}\right)' & \left(\frac{z''(x_0)}{\kappa(x_0)}\right)' \end{bmatrix}$$

By the Case 1 of Proposition 3.4, the second row of M_4 does not vanish. So, M_4 is of rank 2.

Case 3: Suppose that $h_{\mathbf{w}_0,v_0}$ has A_3- singularity at the point x_0. Let M_5 be defined as

$$M_5 = \begin{bmatrix} -1 & \left(-\frac{y''(x_0)}{\kappa(x_0)}\right) & \left(\frac{z''(x_0)}{\kappa(x_0)}\right) \\ 0 & \left(-\frac{y''(x_0)}{\kappa(x_0)}\right)' & \left(\frac{z''(x_0)}{\kappa(x_0)}\right)' \\ 0 & \left(-\frac{y''(x_0)}{\kappa(x_0)}\right)'' & \left(\frac{z''(x_0)}{\kappa(x_0)}\right)'' \end{bmatrix}$$

By the Case 2 Proposition 3.4, M_5 is non-singular and hence M_5 has full rank.

Corollary 3.7: The proof of Theorem 3.1 follows Proposition 3.2, 3.4, 3.5 and Theorem 3.3.

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