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Ring Characterizations with Mutually SS-Supplemented Modules

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Abstract
In this text, the notion of mutually ss-supplemented modules is characterized with the help of semiperfect rings. For this, mutually ss-supplemented modules were first classified according to certain properties. These features can be listed as refinable modules, distributive modules, fully invariant submodules, and (π-) projective modules. It was determined that every submodule of an amply ss-supplemented module is mutually ss-supplemented. It was shown that $C=\bigoplus_{\varrho\in\Lambda} C_\varrho$ is a mutually ss-supplemented module in which each submodule of C is a fully invariant submodule, for the family of mutually ss-supplemented modules $\{C_\varrho\}_{\varrho\in\Lambda}$.

1. Introduction

In this study, we refer the reader to references [1], [2], [3], and [10] to understand the basic algebraic properties of module theory. We will take all the rings as unitary and associative. We will also use all modules as unitary left S-modules. E is called a *submodule* of C if, for each $c\in E$ and $s\in S$, E is a subgroup of the module C and $sc\in E$. This is denoted as $E\leq C$. Obviously, 0 and C are submodules of C. Here, these submodules are said to be *trivial submodules* of C. Submodules other than trivial are said to be *proper submodules* [2].

If non-zero module C has no submodule except trivial submodules, C is said to be *simple* [3]. A module C is said to be *semisimple* if C is written in the form of a sum of simple modules. The necessary and sufficient condition for semisimple module C is this: each submodule of C is a direct summand in C [3]. The property of semisimplity of a module is preserved under submodules, direct summands and arbitary sums [3]. Let B be a proper submodule of C. If C has no proper submodule that includes B in C, then B is said to be a *maximal submodule* of C [3]. Let C be a module and B a proper submodule of C. If C has no any proper submodule D of C provided that $B+D=C$, then B is said to be a *small submodule* of C and denoted as $B\ll C$ [1], [3]. Here, if $B+D=C$, then $D=C$. A module C is said to be *hollow*, if each proper submodule F of C is small. A module C is said to be *local* if C has a proper submodule which includes whole proper submodules of C [1],[3]. The necessary and sufficient condition for a local module C is this: C is hollow and $Rad(C)\neq C$ [1], [3].
Let B, B' be submodules of C. A submodule B' is said to be a *supplement* of B in C, if B' is a minimal element of the submodules D of C with $C=B+D$. Here B' is a supplement of B in C in this case for $C=B+B'$ and $B\cap B'\ll B'$ [1]. An epimorphism $P\underset{\mu}{\rightarrow} B\rightarrow 0$ is said to be a *projective cover* of B if P is projective and $\ker(\mu)\ll P$. In [7], a submodule U has a supplement in a projective module C, which is a direct summand in C in this case for C/U possesses a projective cover. A module C is said to be *semiperfect* if each factor module of C possesses a projective cover [1]. The set $Soc(C)=\sum\{D\leq C \mid D \text{ is a simple submodule of } C\}$ is defined in this way that is a submodule of C. A submodule B' is said to be a *mutual supplement* of B in C if, $C=B+B'$, $B\cap B'\ll B$ and $B\cap B'\ll B'$ by [7]. $Rad(C)$ is the intersection of whole maximal submodules of C. The impression $Rad(C)$ is shown by the sum of each submodule of C. If $Rad(C)=C$, then C is said to be a *radical module*. The radical submodule of a

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semisimple module is zero [1]. $Soc_s(C)=\sum\{D \leq C \mid D\ is\ both$ simple and small submodule of $C\}$. So $Soc_s(C)\subseteq Soc(C)$ and $Soc_s(C)\subseteq Rad(C)$ [4]. A module C is said to be *ss-supplemented* if, for each submodule F of C, there is a supplement L of F in C provided that $F\cap L$ is semisimple, termed ss-supplements [4]. A submodule F of a module C has *ample ss-supplements* in C if each submodule L of C such that $C = F + L$ contains an ss-supplement of F in C. A module C is said to be *amply ss-supplemented* provided that each submodule of C has ample ss-supplements in C [4]. A module C is said to be *strongly local* if $Rad(C)$ is semisimple [4]. Following [5], a module C is said to be $\oplus_{ss}$*-supplemented* if every submodule of C is a ss-supplement that is a direct summand of C. It is clear that every $\oplus_{ss}$-supplemented module is ss-supplemented.

2. Material and Method

In [7], the notion of mutually ss-supplemented modules is defined as a strong notion of ss-supplemented modules and is served relevant attributions about these modules.

Following [7], we give the following facts.

Lemma 2.1: [7, Lemma 2.2] Let $\boldsymbol{B}, \boldsymbol{B'}$ be submodules of the module $\boldsymbol{C}$. Then the following statements are equivalent.
$(\boldsymbol{i})$ $\boldsymbol{B}$ and $\boldsymbol{B'}$ are mutual ss-supplements in $\boldsymbol{C}$;
(ii) $\boldsymbol{C}=\boldsymbol{B}+\boldsymbol{B'}$, $\boldsymbol{B}\cap\boldsymbol{B'}\subseteq \boldsymbol{Rad}(\boldsymbol{B})$, $\boldsymbol{B}\cap\boldsymbol{B'}\subseteq \boldsymbol{Rad}(\boldsymbol{B'})$ and $\boldsymbol{B}\cap\boldsymbol{B'}$ is semisimple;
(iii) $\boldsymbol{C}=\boldsymbol{B}+\boldsymbol{B'}$, $\boldsymbol{B}\cap\boldsymbol{B'}\ll\boldsymbol{B}$, $\boldsymbol{B}\cap\boldsymbol{B'}\ll\boldsymbol{B'}$ and $\boldsymbol{B}\cap\boldsymbol{B'}$ is semisimple;

A module $\boldsymbol{C}$ is said to be *mutually ss-supplemented* if every submodule $\boldsymbol{D}$ of $\boldsymbol{C}$ has an ss-supplement $\boldsymbol{B}$ in $\boldsymbol{C}$ and there exists a submodule $\boldsymbol{B'}$ of $\boldsymbol{C}$ such that $\boldsymbol{B}$ and $\boldsymbol{B'}$ are mutual ss-supplements in $\boldsymbol{C}$. It is clear that $\oplus_{\boldsymbol{ss}}$-supplemented module is mutually ss-supplemented [7].

Lemma 2.2: [7, Lemma 2.9] Let $\boldsymbol{D}$ and $\boldsymbol{E}$ be submodules of the module $\boldsymbol{C}$ in which $\boldsymbol{D}$ is mutually ss-supplemented. If $\boldsymbol{D}+\boldsymbol{E}$ has a mutual ss-supplement in $\boldsymbol{C}$, then $\boldsymbol{E}$ has a mutual ss-supplement in $\boldsymbol{C}$.

Theorem 2.3: [7, Theorem 2.8] Let $\boldsymbol{C}$ be a module with $\boldsymbol{Rad}(\boldsymbol{C})\ll\boldsymbol{C}$. Then the following statements are equivalent:
(i) $\boldsymbol{C}$ is mutually ss-supplemented;
(ii) Every submodule of $\boldsymbol{C}$ has a mutual supplement in $\boldsymbol{C}$ and $\boldsymbol{Rad}(\boldsymbol{C})$ has a ss-supplement in $\boldsymbol{C}$;
(iii) Every submodule of $\boldsymbol{C}$ has a mutual supplement in $\boldsymbol{C}$ and $\boldsymbol{Rad}(\boldsymbol{C})\subseteq\boldsymbol{Soc}(\boldsymbol{C})$.
Proof: (i)⟹(ii) Let $\boldsymbol{C}$ be a mutually ss-supplemented. Then every submodule of $\boldsymbol{C}$ has mutual ss-supplement in $\boldsymbol{C}$. Then $\boldsymbol{Rad}(\boldsymbol{C})$ is so.
(ii)⟹(iii) Since $\boldsymbol{Rad}(\boldsymbol{C})\ll\boldsymbol{C}$, $\boldsymbol{C}$ is a unique ss-supplement of $\boldsymbol{Rad}(\boldsymbol{C})$ in $\boldsymbol{C}$ by the hypothesis. So $\boldsymbol{Rad}(\boldsymbol{C})\subseteq\boldsymbol{Soc}(\boldsymbol{C})$.
(iii)⟹(i) By Lemma 2.1.

Lemma 2.4: [7, Lemma 2.11] Let $\boldsymbol{C}$ be a projective module. Then $\boldsymbol{C}$ is mutually ss-supplemented if and only if every submodule of $\boldsymbol{C}$ has a mutual ss-supplement in $\boldsymbol{C}$ and $\boldsymbol{Rad}(\boldsymbol{C})\subseteq\boldsymbol{Soc}(\boldsymbol{C})$.
Proof: (⟹) Let $\boldsymbol{C}$ be a projective module. By [10, 42.3] $\boldsymbol{C}$ is semiperfect. Then we get $\boldsymbol{Rad}(\boldsymbol{C})\ll\boldsymbol{C}$ by [10, 21.6].
(⟸) The proof holds by Theorem 2.3.

Proposition 2.5: [7, Proposition 2.10] Let $\boldsymbol{C}$ be a module which is the sum of the submodules $\boldsymbol{C_1}, \boldsymbol{C_2}$. If $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$ are mutually ss-supplemented, then $\boldsymbol{C}$ is so.

Corollary 2.6: Let $\boldsymbol{C_1}, \boldsymbol{C_2}, \dots, \boldsymbol{C_m}$ be mutually ss-supplemented submodules of $\boldsymbol{C}$. Then $\boldsymbol{C_1}+\boldsymbol{C_2}+\dots+\boldsymbol{C_m}$ is mutually ss-supplemented.
Proof: Let us apply induction on $\boldsymbol{m}$. If $\boldsymbol{m}=\boldsymbol{1}$, then it is clear that $\boldsymbol{C}=\boldsymbol{C_1}$ is mutually ss-supplemented. Suppose that $\boldsymbol{C}=\boldsymbol{C_1}+\boldsymbol{C_2}+\dots+\boldsymbol{C_{k-1}}$ is mutually ss-supplemented for $\boldsymbol{m}=\boldsymbol{k}-\boldsymbol{1}$. Let us m= k and $\boldsymbol{D}$ be any submodule of $\boldsymbol{C}$. Since 0 is a mutual ss-supplement of $\boldsymbol{C}=\boldsymbol{C_1}+\boldsymbol{C_2}+\dots+\boldsymbol{C_{k-1}}+\boldsymbol{C_k}+\boldsymbol{D}$. Since $\boldsymbol{C_1}+\boldsymbol{C_2}+\dots+\boldsymbol{C_{k-1}}$ is mutually ss-supplemented, then $\boldsymbol{C_m}+\boldsymbol{D}$ has a mutual ss-supplement in $\boldsymbol{C}$. By Lemma 2.2, $\boldsymbol{D}$ has a mutual ss-supplement in $\boldsymbol{C}$. So $\boldsymbol{C}=\boldsymbol{C_1}+\boldsymbol{C_2}+\dots+\boldsymbol{C_m}$ is mutually ss-supplemented.

3. Results and Discussion

In this part, we prove that the notion of mutually ss-supplemented modules is strictly stronger than notion of ss-supplemented modules. We conclude this paper by characterizing semiperfect rings thanks to mutually ss-supplemented modules.

Proposition 3.1: Every amply ss-supplemented module is mutually ss-supplemented.

Proof. Let C be an amply ss-supplemented module and $D \leq C$. It follows that D has an ss-supplement in C, say B. So we can write $C = D + B = B + D$. Since C is amply ss-supplemented, there exists a submodule B' of D such that B' is an ss-supplement of B in C. Therefore $B \cap B'$ is semisimple and small in B. Since B is a supplement in C, $B \cap B'$ is small in B' by [9, 41.1(5)]. It means that B and B' are mutual ss-supplements in C. Hence C is mutually ss-supplemented.

Using the above proposition, we get the following implications on modules.

$$\oplus_{ss}\text{-supplemented}$$
$$\Downarrow$$
amply ss-supplemented $\Longrightarrow$ mutually ss-supplemented $\Longrightarrow$ ss-supplemented

Lemma 3.2: Let C be a π-projective and ss-supplemented module. Then C is mutually ss-supplemented.
Proof. It is clear from [4, Proposition 37] and Proposition 3.1.

Corollary 3.3: Let C be an amply ss-supplemented. Then every submodule of C is mutually ss-supplemented.
Proof. It follows from [4, Corollary 36].

Recall from [6] that a module C is called *tg-supplemented* if every submodule D of C has a Rad-supplement, say B, where B is a t-summand of C, that is, $C = D + B$, $D \cap B \ll B$ and B, B' are mutual supplements in C, where B' is a submodule of C.

Theorem 3.4: A module C is mutually ss-supplemented if and only if it is tg-supplemented and $Rad(C)$ is semisimple.
Proof. (⇒) Let D be a submodule of C. Since C is mutually ss-supplemented, there exists submodules B, B' of C such that B is a ss-supplemented of D in C, and, B, B' are mutual ss-supplements in C. Therefore B is a Rad-supplement of D in C, and B, B' are mutual supplements in C. Thus C is tg-supplemented. Now, we will show that $Rad(C)$ is semisimple. Since $Rad(C)$ is the sum of all small submodules of C, it suffices to show that any small submodule of C is semisimple. Let N be a small submodule of C. Since C is ss-supplemented, it follows from [4, Lemma 13] that N is semisimple. So $N \subseteq Soc(C)$, which implies $Rad(C) \subseteq Soc(C)$. It means that $Rad(C)$ is semisimple.

(⇐) Let D be a submodule of C. By the assumption, there exist submodules B and B' of C such that B is a Rad-supplement of D in C and B, B' are mutual supplements in C. Therefore $B \cap D \subseteq Rad(C)$ and $B \cap B' \subseteq Rad(C)$. Since $Rad(C)$ is semisimple, $B \cap D$ and $B \cap B'$ are semisimple. It follows from [4, Lemma 3] that B is a ss-supplement of D in C, and B, B' are mutual ss-supplements in C. Hence C is mutually ss-supplemented.

Example 3.5: Let K be a quotient field of a Dedekind domain S. Since K/S is a non-local hollow module, the hollow module K/S is not a strongly local module. From [4, Proposition 16] K/S is not ss-supplemented, and so it is not mutually ss-supplemented.

Theorem 3.6: The following statements are given for a ring S where each left ideal has a mutually supplement:

(i) ${}_SS$ is mutually ss-supplemented,

(ii) S is semiperfect and $Rad(S) \subseteq Soc({}_SS)$,

(iii) Every S-module is mutually ss-supplemented.

Proof: By [4, Theorem 41] and Proposition 3.1.

Recall from [10] that a submodule D of C is called characteristic (or fully invariant) if $\theta(D) \leq D$ for each endomorphism θ of C.

Theorem 3.7: Let $\{C_\varrho\}_{\varrho\in\Lambda}$ be a family of mutually ss-supplemented modules $C = \oplus_{\varrho\in\Lambda} C_\varrho$ where each submodule of C is fully invariant. Then C is a mutually ss-supplemented module.

Proof: Let D be any submodule of C. By hypothesis, since $D = \oplus_{\varrho\in\Lambda} (D \cap C_\varrho)$, then $\oplus_{\varrho\in\Lambda} (C_\varrho/(D \cap C_\varrho)) \cong \oplus_{\varrho\in\Lambda} C_\varrho / \oplus_{\varrho\in\Lambda} (D \cap C_\varrho) = C/D$. Since C_ϱ is mutually ss-supplemented for each $\varrho \in \Lambda$, C_ϱ has such submodules K_ϱ and T_ϱ where K_ϱ is an ss-supplement of $D \cap C_\varrho$, and K_ϱ and T_ϱ are mutual ss-supplements of C_ϱ. Hence it is obvious that $(D \cap C_\varrho) \cap K_\varrho = D \cap K_\varrho$ is semisimple for each $\varrho \in \Lambda$. Let $\oplus_{\varrho\in\Lambda} K_\varrho = K$ and $\oplus_{\varrho\in\Lambda} T_\varrho = T$. Let $C = \oplus_{\varrho\in\Lambda} C_\varrho = \oplus_{\varrho\in\Lambda} (D \cap C_\varrho) + \oplus_{\varrho\in\Lambda} K_\varrho = D + K$ and $D \cap K = \oplus_{\varrho\in\Lambda} (D \cap C_\varrho) \cap \oplus_{\varrho\in\Lambda} K_\varrho \subseteq \oplus_{\varrho\in\Lambda} ((D \cap C_\varrho) \cap K_\varrho) = \oplus_{\varrho\in\Lambda} (D \cap K_\varrho) \ll K$. Since $D \cap K_\varrho$

is semisimple for each $\varrho \in \Lambda$, by [3], $D \cap K$ is semisimple. Then $D \cap K \ll K$ and since $D \cap K$ is semisimple, $D \cap K \subseteq Soc_s(K)$. By similar operations, it can be shown that $K \cap T$ are mutual ss-supplemented in C by using K_ϱ and T_ϱ to be mutual ss-supplements in C_ϱ for every $\varrho \in \Lambda$, so C is mutually ss-supplemented.

Recall from [8] that a module C is called *duo* if each submodule is fully invariant.

Corolary 3.8: Let $\{C_\varrho\}_{\varrho\in\Lambda}$ be the class of mutually ss-supplemented modules and $C = \bigoplus_{\varrho\in\Lambda} C_\varrho$ where C is a duo-module. Then C is a mutually ss-supplemented module.

Proposition 3.9: Let the module C be π-projective mutually ss-supplemented module, then C is a $\bigoplus_{ss}$-supplemented module.

Proof: Let D be a submodule of C. According to the hypothesis, C has such submodules L and L' provided that L is a ss-supplement of D and L, L' are mutual supplements of C. Since C is a π-projective module, it follows from [10, 41.14(2)] that $L \cap L' = 0$ and so $C = L \oplus L'$. Then C is a $\bigoplus_{ss}$-supplemented module.

Proposition 3.10: Let S be a semisimple ring. Then S-module C is mutually ss-supplemented if and only if every submodule of C has a mutual ss-supplement in C.

Proof: Recall from [9, Proposition 4.5] that the ring S is semisimple if and only if every S-module is projective. The proof follows from Lemma 2.4.

Recall from [11, 8.3] that a module C is called *refinable* if for each submodule D, K of C with $D + K = C$, there exists a direct summand D' of C with $D' \subset D$ and $D' + K = C$.

Proposition 3.11: Every refinable mutually ss-supplemented module is $\bigoplus_{ss}$-supplemented.

Proof: Let D be any submodule of refinable mutually ss-supplemented module C. Since C is a mutually ss-supplemented module, there is such a submodule K of C with $C = D + K, D \cap K \ll K, D \cap K \ll D$ and $D \cap K$ is semisimple. It is also $D \cap K \ll C$. Since C is refinable, there is a direct summand L of C so that $L \subseteq K$ and $C = D + L$. Then $D \cap L \ll L$. It follows from [3] that $D \cap L \leq D \cap K$ is semisimple. Since $C = D + L$, $D \cap L \ll L$ and C has a direct summand L provided that $D \cap L$ is semisimple, as required.

4. Conclusion and Suggestions

Although the module has been made in theory in recent years, it is mentioned in the article named mutually ss-supplemented modules published in the reference [7] in the concept of mutually ss-supplement submodule, which is a special form of the ss-supplement submodule concept in the article in the reference [4], which has led to many studies with many references. Expressed as the characterization of the semiperfect rings of the data. Apart from this, special theorems have been developed to reach amply mutually ss-supplemented modules.

Contributions of the authors

There is no conflict of interest between the authors.

Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

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