THEORETICAL AND NUMERICAL ANALYSIS OF DESCENDING AND ELEVATED PENDULUM

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Abstract

This study is devoted to the detailed investigation of the motion of descending and elevated pendulum from the engineering point of view. On the assumption that the base velocity is constant, the change in the angle θ and the stress T in the rope with time are determined for varying values of initial angular velocity $\dot{\theta}$ and base velocity v_0 for both cases. The results obtained for the values of stress T reveal that the design of a heavy load lifting devices must be made by taking the dynamic value of T into account. Results for the variation of θ show that the previous results obtained in several woks can not be considered to be correct and satisfactory. The case of accelerated base motion is also discussed.

1. INTRODUCTION

The problem of a simple pendulum has received great attention for several centuries due to its usage in pendulum clocks. This problem was first formulated by Euler in the way

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \tag{1}$$

$$w = \sqrt{\frac{g}{L}} \qquad T_p = 2\pi\sqrt{\frac{L}{g}} \tag{2}$$

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Here, w and T_p are the frequency and the period, respectively . In this paper, we aim to solve the problem of a pendulum whose root is released or elevated at a constant velocity v_0 . As can be expected, this problem will not be as easy as in the case of a simple pendulum with constant length L. The first part of the present system was also described in a classic short story by Edgar Allen Poe [1]. Several mathematicians carried out studies to solve the problem and to reveal whether sweep and curvilinear velocities in the problem increases during motion, as the writer claimed [2, 3] . However, although quite valuable, none of these studies can be considered as an engineering work since most of the works neither include detailed analysis nor insert the initial conditions into the problem for a correct evaluation of the issue. On the other hand, in the case of lifts used in civil engineering, load and rope wounded on or released from a drum by the electric motor constitutes such a system, and, in order to determine the change of force and thereby the correct value of the moment on the motor for design purpose, the dynamics of the motion of the load must be studied in great detail.

2. ANALYSIS

2.1. ANALYSIS FOR DESCENDING PENDULUM

Let us consider a pendulum that consists of a particle P of mass m suspended by a light string or a rope of variable length $r=l = L+v_0t$. The whole system is in the vertical plane. We use polar coordinates for the problem. The equation of motion in the tangential direction gives

$$\nwarrow \quad +\sum F_\theta = ma_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta}) \tag{3}$$

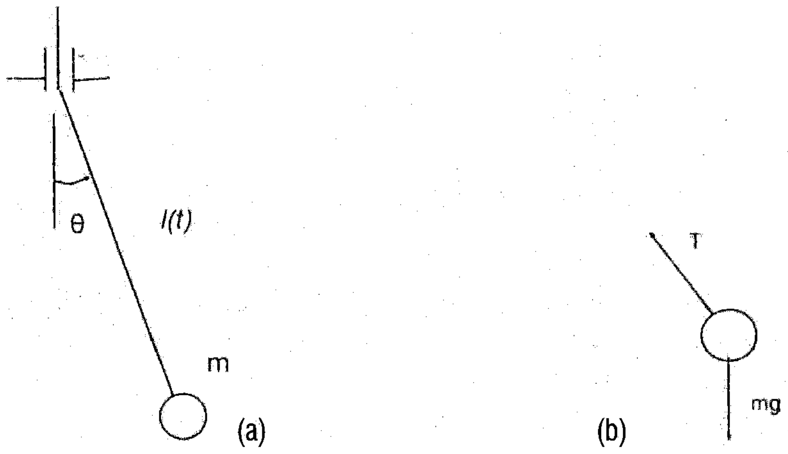


Fig. 1. Descending pendulum, and elevated pendulum with base the velocity reversed.

or $$-g\sin\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} \tag{4}$$

Realizing the equalities $r = L + v_0 t$, $\dot{r} = v_0$, $\ddot{r} = 0$ for the present case, we have

$$(L + v_0 t)\ddot{\theta} + 2v_0\dot{\theta} + g\sin\theta = 0 \tag{5}$$

The equation of motion in the radial direction results in

$$T - mg\cos\theta = -m(\ddot{r} - r\dot{\theta}^2)$$

or

$$T = m\left[g\cos\theta + (L + v_0 t)\dot{\theta}^2\right] \tag{6}$$

In the case of small displacements, we obtain

$$(L+v_0t)\ddot{\theta}+2v_0\dot{\theta}+g\theta=0 \tag{7}$$

and

$$T=m\left[g+(L+v_0t)\dot{\theta}^2\right] \tag{8}$$

In order to solve Eq.(7), we will follow a more general method that is different than that in [2] since it is not always possible to find out a transformation when the change of length l is not linear. We make the change of variable in the form

$$s = L + v_0t \tag{9}$$

Taking care that

$$\dot{\theta}=v_0\frac{d\theta}{ds},\quad \ddot{\theta}=v_0^2\frac{d^2\theta}{ds^2} \tag{10}$$

We arrive at the equation

$$s\frac{d^2\theta}{ds^2}+2\frac{d\theta}{ds}+\frac{g}{v_0^2}\theta=0 \tag{11}$$

Without proof, we will say that the solution of Bessel differential equation of the type

$$s^2\frac{d^2\theta}{ds^2}+(2\alpha+1)s\frac{d\theta}{dt}+(\alpha^2-\beta^2p^2+\beta^2\gamma^2s^{2\beta})\theta=0 \tag{12}$$

has the form [5]

$$\theta=s^{-\alpha}y=C_1s^{-\alpha}J_p(\gamma t^{\beta})+C_2s^{-\alpha}Y_p(\gamma t^{\beta}) \tag{13}$$

if p =n is a positive integer or zero. Comparing Eq.(11) with Eq.(12), we conclude that

$$\theta=\frac{1}{\sqrt{s}}\left[C_1J_1\left(\frac{2\sqrt{gs}}{v_0}\right)+C_2Y_1\left(\frac{2\sqrt{gs}}{v_0}\right)\right] \tag{14}$$

or, in terms of time t,

$$\theta = \frac{1}{\sqrt{L+v_0 t}}\left[C_1 J_1\left(\frac{2\sqrt{g(L+v_0 t)}}{v_0}\right) + C_2 Y_1\left(\frac{2\sqrt{g(L+v_0 t)}}{v_0}\right)\right] \tag{15}$$

Realizing the identities [4]

$$J_1 \cong \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{3\pi}{4}\right) \tag{16}$$

$$Y_1(x) \cong \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{3\pi}{4}\right)$$

the periodicity of θ in Eq.(15) can easily be seen.

Differentiating Eq.(15) with respect to time t, we have

$$\dot{\theta} = c_1\left[\frac{-v_0}{2L^{3/2}} J_1\left(\frac{2}{v_0}\sqrt{gs}\right) + \frac{\sqrt{g}}{2L}\left(J_0\left(\frac{2}{v_0}\sqrt{gs}\right) - J_2\left(\frac{2}{v_0}\sqrt{gs}\right)\right)\right]$$

$$+c_2\left[\left[\frac{-v_0}{2L^{3/2}} Y_1\left(\frac{2}{v_0}\sqrt{gs}\right) + \frac{\sqrt{g}}{2L}\left(Y_0\left(\frac{2}{v_0}\sqrt{gs}\right) - Y_2\left(\frac{2}{v_0}\sqrt{gs}\right)\right)\right]\right] \tag{17}$$

where

$$a = \frac{-v_0}{2L^{3/2}} J_1\left(\frac{2}{v_0}\sqrt{gs}\right) + \frac{\sqrt{g}}{2L}\left(J_0\left(\frac{2}{v_0}\sqrt{gs}\right) - J_2\frac{2}{v_0}\sqrt{gs}\right) \tag{18}$$

$$b = \frac{-v_0}{2L^{3/2}} Y_1\left(\frac{2}{v_0}\sqrt{gs}\right) + \frac{\sqrt{g}}{2L}\left(Y_0\left(\frac{2}{v_0}\sqrt{gs}\right) - Y_2\frac{2}{v_0}\sqrt{gs}\right) \tag{19}$$

Let us assume that the initial conditions at $t = 0$ are given as

$$\theta = \theta_0$$
$$\dot{\theta} = \dot{\theta}_0 \qquad (20)$$

Substituting these two conditions into Eq.(15) and Eq.(17) respectively gives

$$\theta_0 = \frac{1}{\sqrt{L}}\left[c_1 J_1(z) + c_2 Y_1(z)\right]$$

$$\dot{\theta}_0 = c_1\left[\frac{-v_0}{2L^{3/2}}J_1(z) + \frac{\sqrt{g}}{2L}(J_0(z) - J_2(z))\right] + c_2\left[\frac{-v_0}{2L^{3/2}}Y_1(z) + \frac{\sqrt{g}}{2L}(Y_0(z) - Y_2(z))\right] \qquad (22)$$

where $z = \frac{2}{v_0}\sqrt{gL}$. Solving Eq.(21) and Eq.(22) simultaneously gives

$$c_1 = \frac{\dot{\theta}_0 - \dfrac{\bar{b}\sqrt{L}\theta_0}{Y_1(z)}}{\bar{a} - \dfrac{\bar{b}J_1(z)}{Y_1(z)}}, \qquad c_2 = \frac{1}{Y_1(z)}\left(\sqrt{L}\theta_0 - c_1 J_1(z)\right) \qquad (23)$$

Here $\bar{b}$, $\bar{a}$. are given by

$$\bar{a} = \left[\frac{-v_0}{2L^{3/2}}J_1(z) + \frac{\sqrt{g}}{2L}(J_0(z) - J_2(z))\right]$$

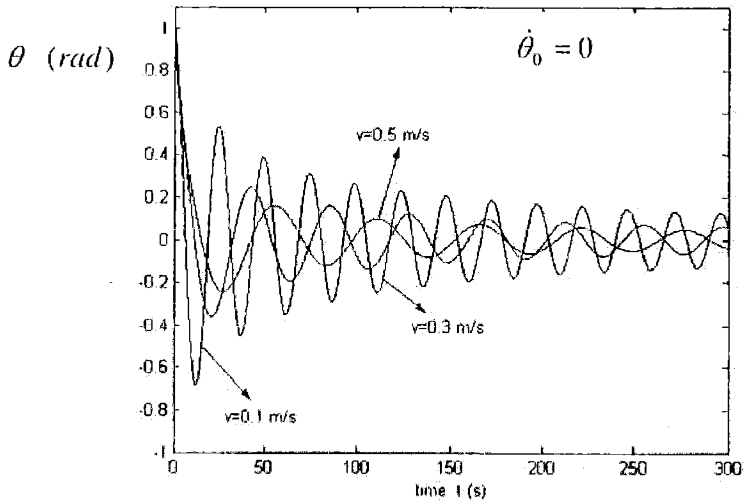
$$\bar{b} = \left[\frac{-v_0}{2L^{3/2}}Y_1(z) + \frac{\sqrt{g}}{2L}(Y_0(z) - Y_2(z))\right] \qquad (24)$$

The problem is now completely solved in terms of the initial conditions.

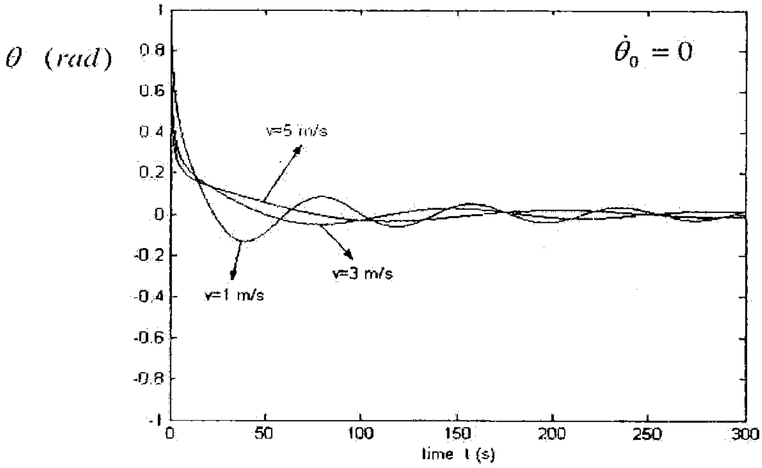
2.1.1. APPLICATION FOR DESCENDING PENDULUM

As an application of the method, let us assume that $L = 1$, $g = 9.81\ m/s^2$,

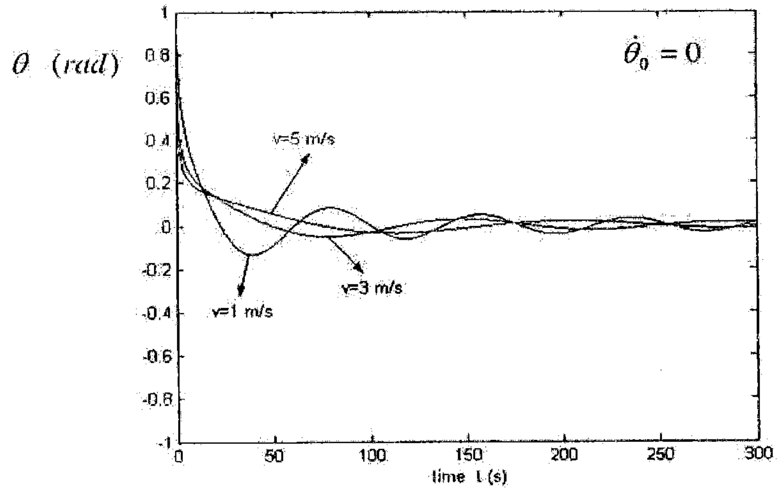
$\theta_0=\pi/3$) and plot the change of θ with various $\dot{\theta}_0$ and v_0. In the calculations, $J_1(z)$, $J_2(z)$, $Y_1(z)$, $Y_2(z)$ taking place in the constants c_1 and c_2 were calculated using matlab function, besselj(1:1,(m:.1:n)), besselj(2:2,(m:. 1 :n)), bessely(1:1,(m:.1:n)) and bessely(2:2,(m:.1:n)). Here, m and n are the limits of the interval (0,x). $J_1(s)$, $J_2(s)$, $Y_1(s)$, $Y_2(s)$ were also calculated using these functions. Since matlab recognize only the $J_c(x)$, $Y_c(x)$ for (0,x), we have used the transformation $x=2\sqrt{L+v_0t}/v_0$ to obtain $J_c(2\sqrt{L+v_0t}/v_0)$, $Y_c(2\sqrt{L+v_0t}/v_0)$ in (0,t). The results are shown in Figs.(2)-(6). In all of the figures, it is seen that the base velocity causes the vibration decaying in time, and the amplitudes of the vibration decreases for increasing values of v_0, as is expected. Again, as we expect, oscillation angle θ increases for increasing value of initial angular velocity $\dot{\theta}_0$ as shown in Figs.3(a), (b),(c) and Fig.4(a), (b), (c). When we compare these results with that of [2] we see that the results evaluated by Brinchaus and Zhu by using ode45 solver of matlab do not precisely coincide. In fact, that work do not include any information on the constant and the results states nothing on the history of the motion for its users. We also see from Fig.2(c), 3(c) and 4(c) that , for sufficiently large values of v_0, we don't have large differences among θ's obtained for different values of L and v_0.


θ (rad)
$\dot{\theta}_0 = 0$
v=0.5 m/s
v=0.3 m/s
v=0.1 m/s
time t (s)

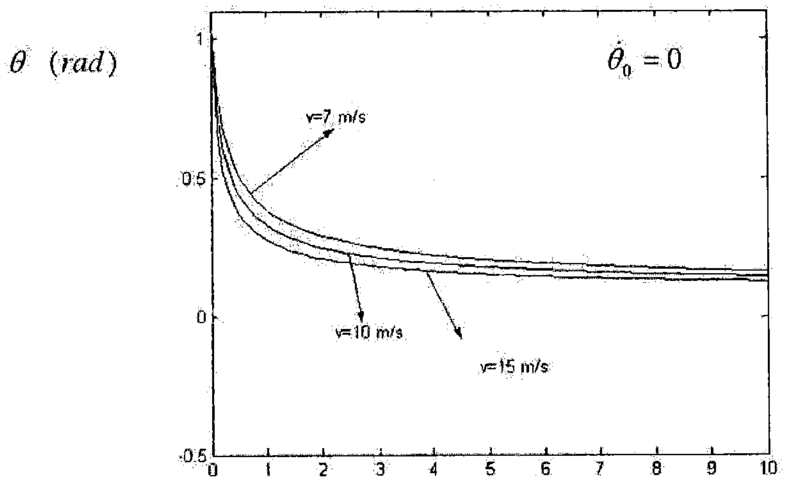
(a)


θ (rad)
$\dot{\theta}_0 = 0$
v=5 m/s
v=3 m/s
v=1 m/s
time t (s)

(b)

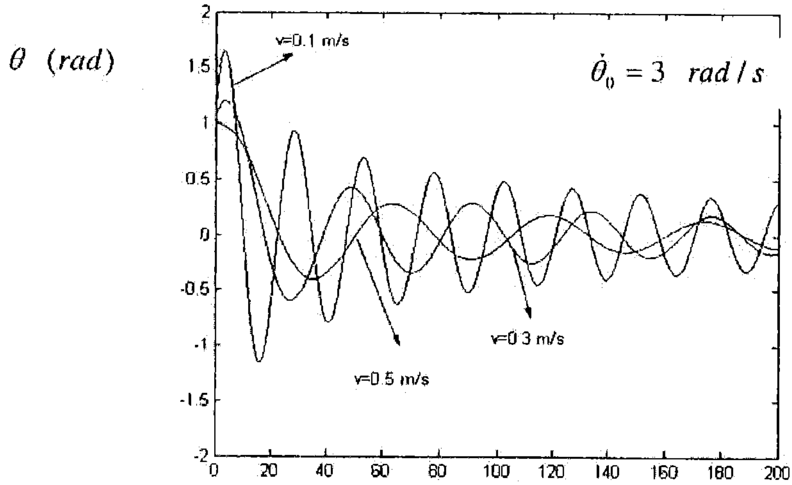

θ (rad)
$\dot{\theta}_0 = 0$
v=5 m/s
v=3 m/s
v=1 m/s
time t (s)

(b)

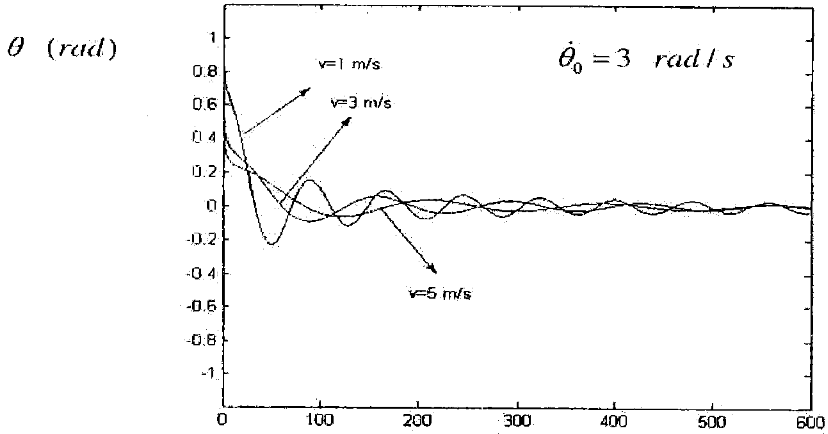

θ (rad)
$\dot{\theta}_0 = 0$
v=7 m/s
v=10 m/s
v=15 m/s

(c)

Fig.2.Variation of angle θ with time t for various values of base velocity v_0: $\dot{\theta}_0 = 0$.



(a)



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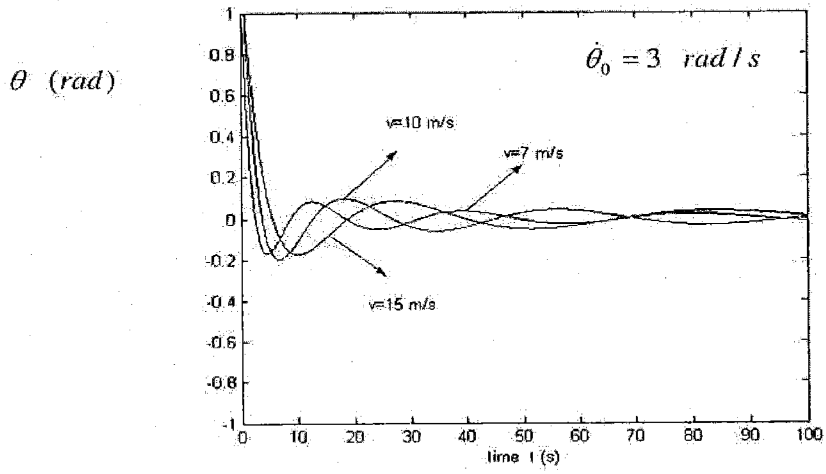
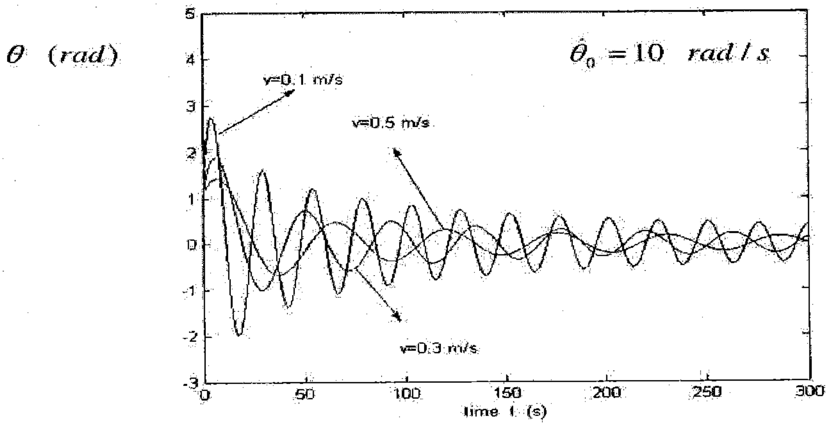
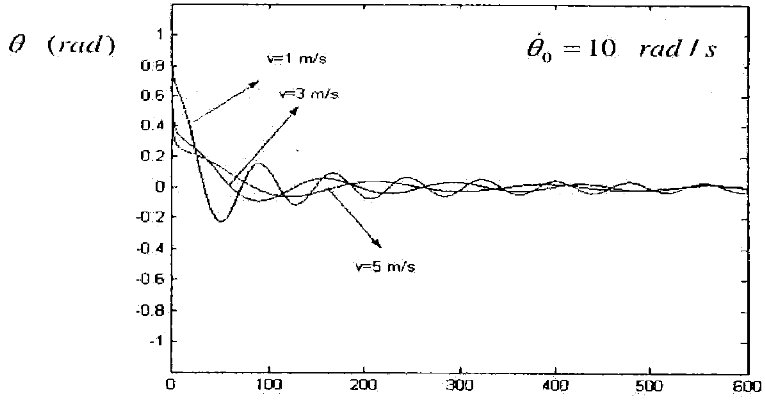


(c)

Fig .3. Variation of angle θ with time t for various values of base velocity v_0 and $\dot{\theta}_0 = 3$



(a)



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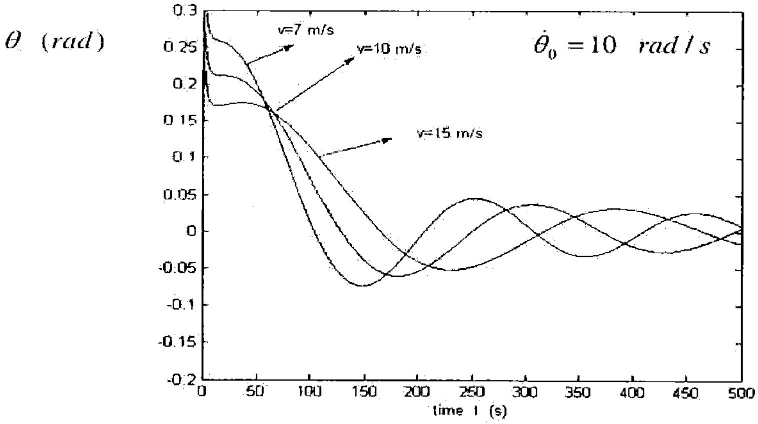
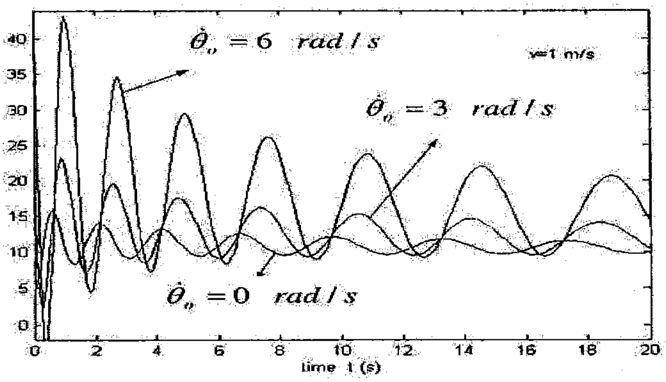


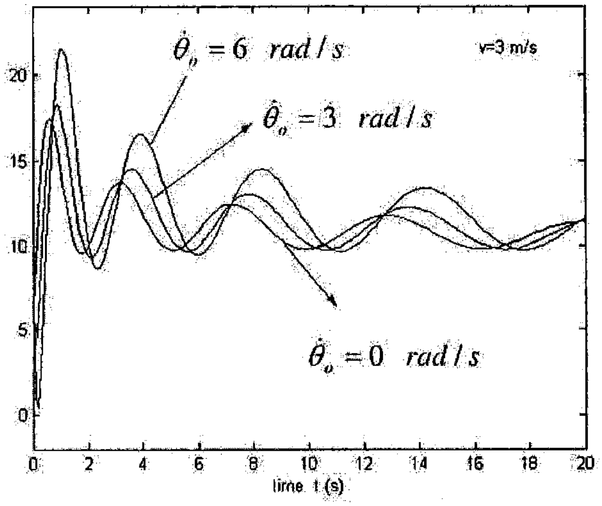
(c)

Fig .4. Variation of angle θ with time t for various values of base velocity v_0 and $\dot{\theta}_0 = 10$.

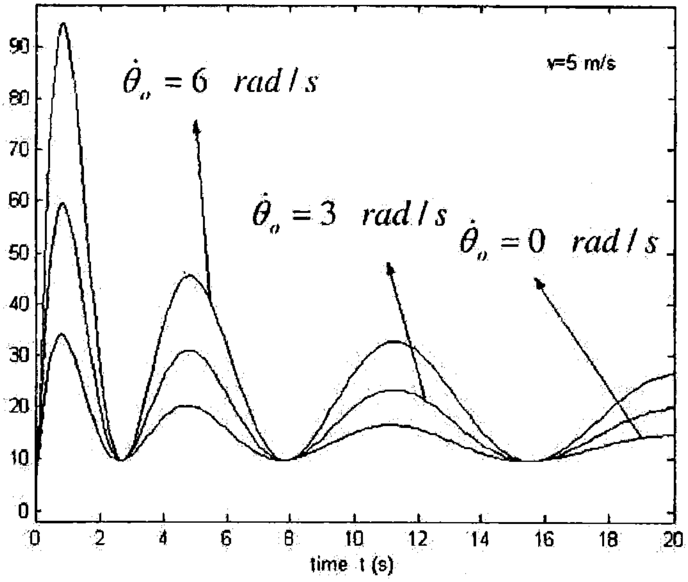
From the engineering point of view, we are mostly interested in the stresses in the rod. Figs.5(a),(b),(c) show the plots for the stress versus time. As seen from the figures, the change of stress with time for different angular velocity $\dot{\theta}_0$ and the base velocity v_0 is quite important. We can say that the intensity of the stress can intensively increases at the onset of the motion as seen in the figures for specific values of variables. Therefore, while designing the dimensions of the connecting rope for heavy load lifting devices, the dynamic value of the stress must be used.



(a)



(b)



(c)

Fig .5. The variation of stress T with lime t for various values of angular velocity $\dot{\theta}$ for v_0= 1 m/s , v_0= 3 m/s and v_0= 5 m/s

With the aid of program we have written, pendulum's sweep can also be found. Fig.6 is the plot of sweep for v_0 = 1 m/s and $\dot{\theta} = 0$. As it is claimed in [2] and [3], sweep increases with time .

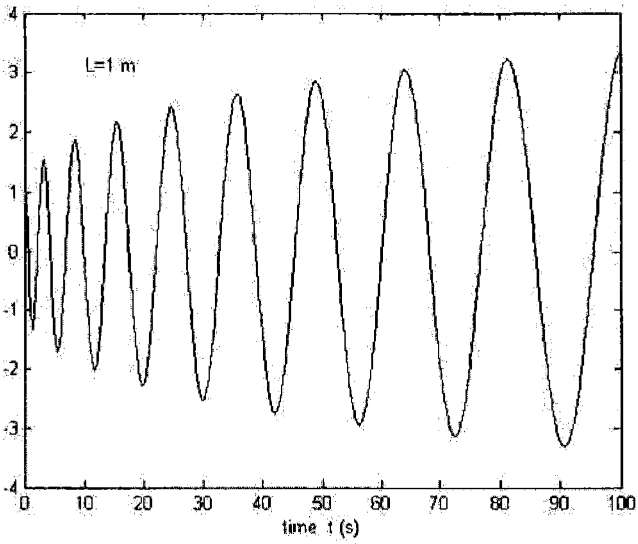


Fig.6. Variation of pendulum's sweep with time t for the values of v_0=1 m/s and $\dot{\theta}=0$

2.2. ANALYSIS FOR THE PENDULUM ELEVATED

Il the pendulum is elevated such that $l=L-v_0t$, instead of being descended, then the equation of motion takes the form

$$\bar{s}\frac{d^2\theta}{d\bar{s}^2}+2\frac{d\theta}{d\bar{s}}+\frac{g}{v_0^2}\theta=0 \tag{25}$$

where $\bar{s}=L-v_0t$. Comparing Eq.(25) with Eq.(12), we can conclude that the solution must be of the form

$$\theta=\frac{1}{\sqrt{L-v_0t}}\left[C_3J_1\left(\frac{2\sqrt{g(L-v_0t)}}{v_0}\right)+C_4Y_1\left(\frac{2\sqrt{g(L-v_0t)}}{v_0}\right)\right]$$

Satisfaction of the initial conditions (25) by Eq.(26) results in

$$\theta_0 = \frac{1}{\sqrt{L}} \left[C_3 J_1(z) + C_4 Y_1(z) \right] \tag{27}$$

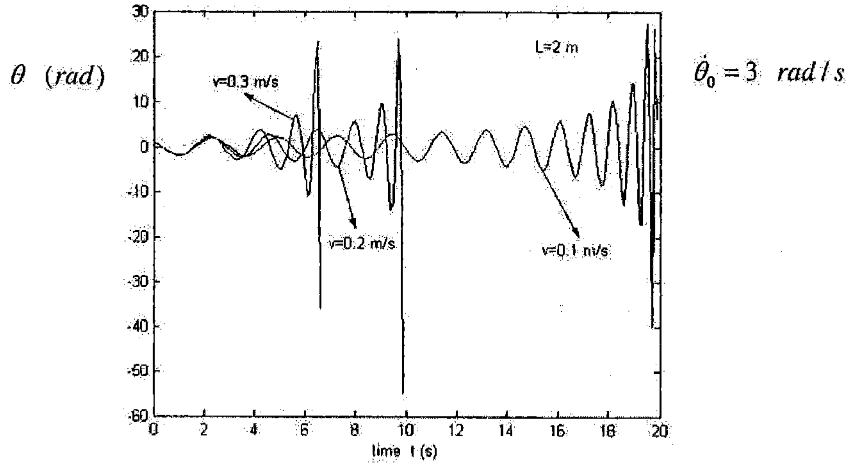
$$\dot{\theta} = c_3 \left[\frac{-v_0}{2L^{3/2}} J_1(z) + \frac{\sqrt{g}}{2L} \left(J_0(z) - J_2(z) \right) \right] + c_4 \left[\frac{-v_0}{2L^{3/2}} Y_1(z) + \frac{\sqrt{g}}{2L} \left(Y_0(z) - Y_2(z) \right) \right] \tag{28}$$

where $z = \frac{2}{v_0}\sqrt{gL}$. Solving Eq.(26) and Eq.(27) simultaneously gives

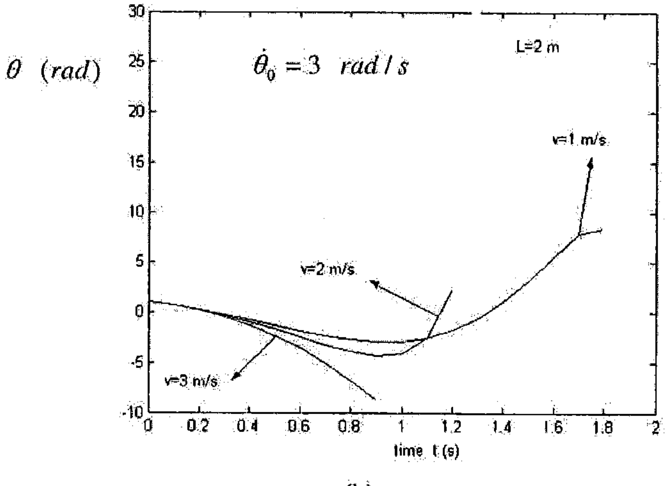
$$c_3 = \frac{\dot{\theta} - \dfrac{\bar{b}\sqrt{L}\theta_0}{Y_1(z)}}{\bar{a} - \dfrac{\bar{b}J_1(z)}{Y_1(z)}}, \qquad c_4 = \frac{1}{Y_1(z)} \left(\sqrt{L}\theta_0 - c_1 J_1(z) \right) \tag{29}$$

where $\bar{b}$, $\bar{a}$ are given by Eqs.(24)

We now apply these results to a specific problem, as before. We again assume $L = 1$ m , $g = 9.81$ m/s^2 , $\theta_0 = \pi/3$ and plot the change of θ with various $\dot{\theta}_0$ and v_0 .

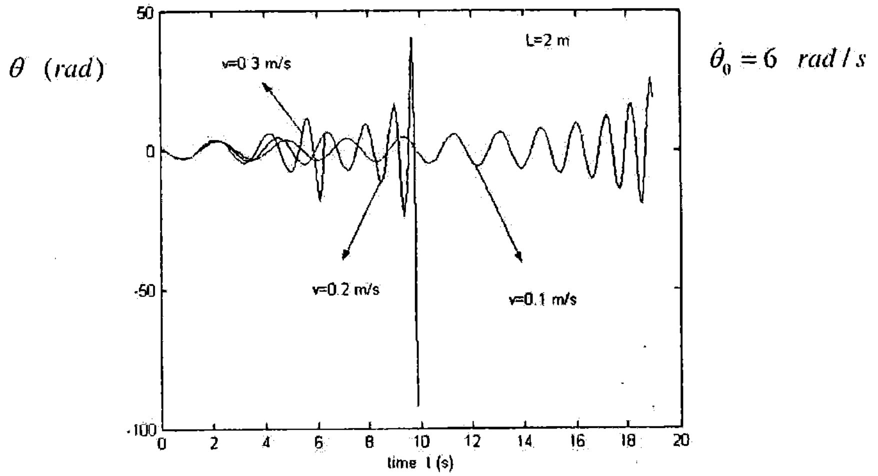


(a)

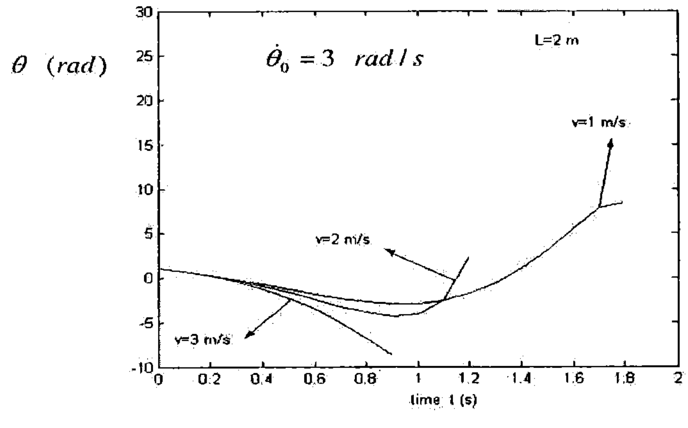


(b)

Fig.7.Variation of angle θ with time t for various values of base velocity v_0: $\dot{\theta}_0 = 0$ rad / s.

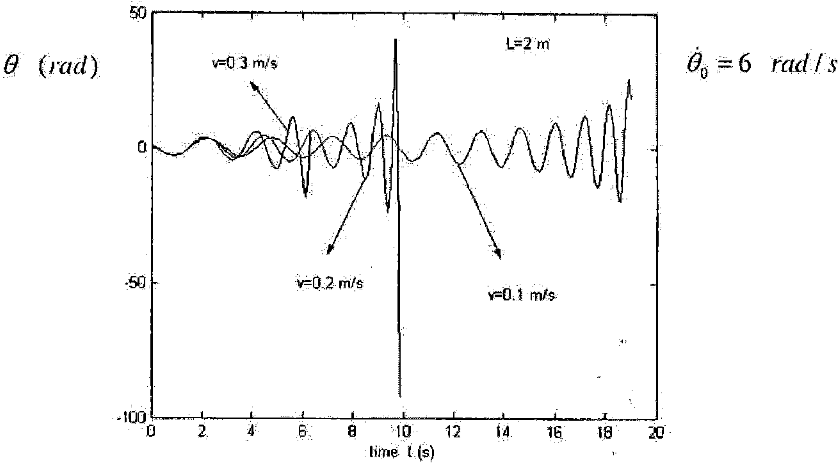


(a)

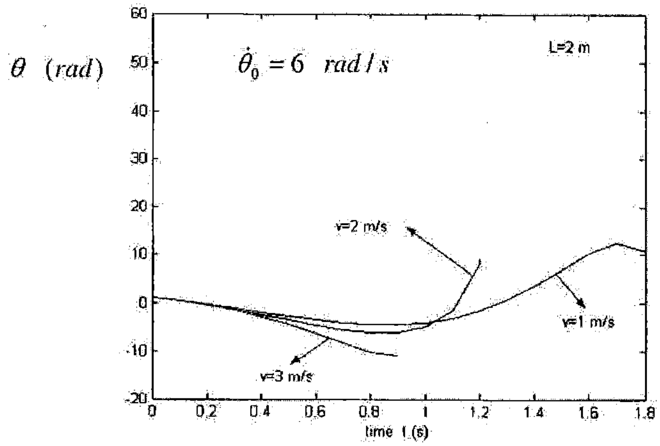


(b)

Fig.8. Variation of angle θ with time t for various values of base velocity v_0: $\dot{\theta}_0 = 3$ rad/s .



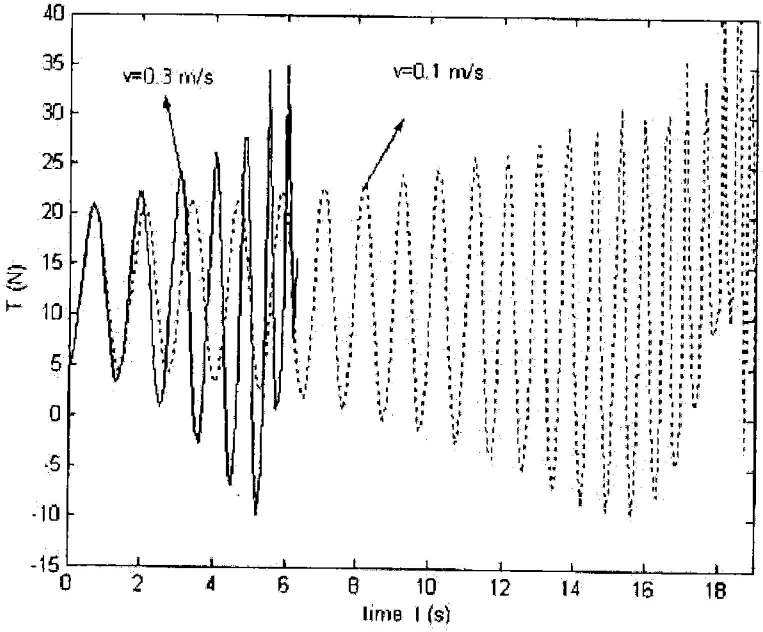
(a)



(b)

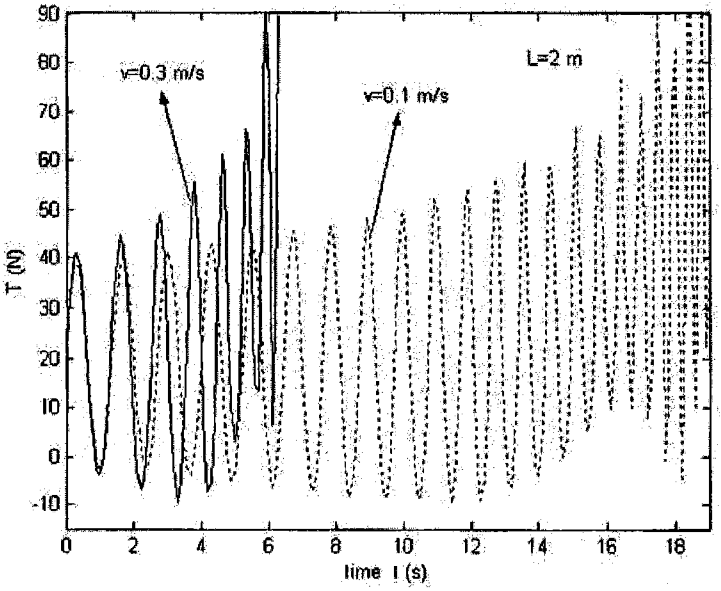
Fig .9. Variation of angle θ with time t for various values of base velocity v_0: $\dot{\theta}_0 = 6 \quad rad/s$.

The strength T in the rope is plotted in Fig.11(a) and (b) for different values of the velocity v_0: L = 2m . The intensity of T increases with increasing values v_0 and $\dot{\theta}_0$.



(a)

Fig. 10. Variation of stress T with time t for various values of base velocity v_0, $\dot{\theta}_0 = 0$.



(b)

Fig .11. Variation of strength T with time t for various values of base velocity v_0: $\dot{\theta}_0 = 0$.

Sweep for the elevated pendulum for specific values is as in Fig. 12. In this case, sweep decreases with time t.

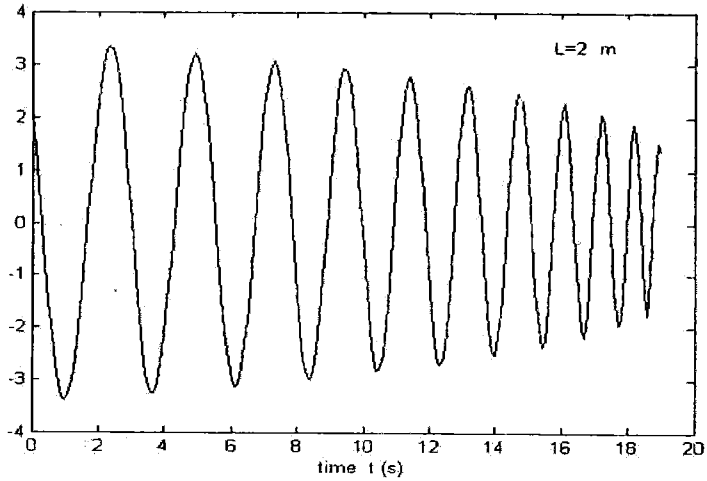


Fig. 12. Variation of pendulum's sweep with time t for v_0=0.l m/s and $\dot{\theta}_0 = 0$.

RESULTS AND DISCUSSION

In this paper, we have reanalyzed the behavior of descending pendulum, which is also known as Poe's pendulum, and studied the motion of the elevated pendulum from engineering point of view. The problem was formulated as an initial value problem and solved analytically in two cases. Just as the results can be evaluated for understanding the motion of the pendulum in both cases, but also they can be utilized in the design of lifting devices with long rope for determining the limits of the moment that the electric motor should supply. Since the strength in the rope during motion is an important value in the design of the lifting devices to determine the value of moment, we have also obtained the expression for T. In the case of descending pendulum, no comparison was made since the solution given in [2] doesn't include any information on the initial conditions. However, when we observe θ - t diagram in [2], we can say that the results are not correct and readable since the value for θ seems not to be starting with a finite value. In the present analysis, we have limited ourselves with the condition of constant base velocity since it is not always possible to find the analytical solution of the equation of motion using Bessel and Legendre functions in many other cases. Therefore, we have excluded the case of accelerated base motion of the pendulum. However, as stated in [3], in the case of accelerated base motion of the form $v = c_0t^2$, the solution of the equation results in Bernoulli's equation and can be solved analytically. But, if the base velocity has the form , $l= a_0t^2 + b_0t + c_0$, then the equation of motion takes the form

$$(a_0t^2 + b_0t + c_0)\ddot{\theta} + 2(2a_0t + b_0)\dot{\theta} + g\theta = 0 \tag{30}$$

which can not be solved using the form in Eq.(12).However, for a special case in which$(a_0t^2 + b_0t + c_0)$ can be brought into the form $(a_0t + b_0)^2$, Eq.(30) can be brought into the form of Eq.(12).If this is the case, Eq.(30) gives

$$(a_0t + b_0)^2\ddot{\theta} + 4(a_0t + b_0)\dot{\theta} + g\theta = 0 \tag{31}$$

Making the change of variable in the form $u = a_0t + b_0$, Eq.(31) can be written in the form

$$u^2\frac{d^2\theta}{du^2} + \frac{4}{a_0}u\dot{\theta} + \frac{g}{a_0^{\;2}}\theta = 0 \tag{32}$$

Eq.(32) doesn't have a form that can be solved using Eq.(12). However, reminding in mind that the equation involves ordinary singularity, the method of series expansion may be used. This problem is left to the reader to be solved because we don't want to extend this work any more. Finally, we say that a more complex problem related with the motion of a descending chain also waits to be solved. However, it must be warned that the equation of motion in that case will be a partial differential equation of variable coefficient with moving boundary condition, which, in general, would require using computational techniques.

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