FUZZY DİZİSEL YEREL ENTROPİ FONKSİYONU ÜZERİNE

İsmail TOK*

Özet

Bu makalede,ilk olarak,fuzzy dinamik sisteminin bazı temel özellikleri verilerek, fuzzy dizisel entropi fonksiyonunun bazı özellikleri detaya girilmeksizin incelenmektedir.Daha sonra, fuzzy dinamik sisteminin dizisel yerel entropi fonksiyonu tanımlanıyor.Son olarak da, bu fonksiyonun bazı önemli özellikleri ispatlanmaktadır.

Anahtar kelimeler. Fuzzy olasılık ölçüm uzayı,fuzzy olasılık ölçümünü koruyan σ-homomorfizma,fuzzy topolojik dinamik sistemi,fuzzy tam sistemi , fuzzy faktörü, fuzzy dizisel entropi fonksiyonu, fuzzy dizisel yerel entropi fonksiyonu.

On The Fuzzy Sequence Local Entropy Function

Abstract

In this paper,we first give the basic properties of the fuzzy dynamical system and investigate some properties of fuzzy sequence entropy function without going into details.After that,we define the fuzzy sequence local entropy function of fuzzy dynamical system.Finally,we prove some important properties of this function.

Key words. Fuzzy probability measure space,fuzzy probability measure- preserving σ-homomorphism,fuzzy dynamical system,fuzzy complete system,fuzzy factor,fuzzy sequence entropy function,fuzzy sequece local entropy function.

---

* *Prof. Dr.,İstanbul Aydın Üniversitesi Fen-Edebiyat Fakültesi.*

1-INTRODUCTION

In his classical paper (5),Kushnirenko first introduced the concept of non-fuzzy sequence
entropy function and investigated some fundamental properties of this function. After that,Hulse (3)and Newton (6) obtained some results relating to this function.

The author has proved some basic properties of fuzzy entropy function in (8) and stated some results relating to the fuzzy sequence entropy function in (9).In another its work (10),the author investigated recently some important properties of fuzzy local entropy function.

In this work,we shall give the definition of fuzzy sequence local entropy function and investigate some basic properties of this function.

It is the purpose of this paper is to give a fuzzy sequence local entropy function of fuzzy dynamical system and show the important results relating to this function.

2-DEFINITIONS

2.1 Definition. Followig Zadeh (11),a pair (X,F) is called a fuzzy set.Where X is an arbitrary non-empty set and $A : X \to [0,1]$ is a membership function.That is,a fuzzy set is characterized by a membership function A from X to the closed unit interval $I = [0,1]$.Thus,we identify a fuzzy set its membership function A.In this connection,A(x),is interpreted as the degree of membership of a point $x \in X$.The family of all fuzzy sub-sets is called a fuzzy class and will be denoted by **F.**

2.2 Definition.i) The family $A_1,\ldots,A_n$ of fuzzy sub-sets is called disjoint, if $(\bigvee_{i=1}^{j} A_i) \wedge A_{j+1} = \varnothing$, for each $j = 1,2,\ldots,n-1$.

ii) A family $\mathbf{P} = \{ A_1,\ldots,A_n \}$ of disjoint fuzzy sub-sets is called a finite fuzzy partition if $X = \bigvee_{i=1}^{n} A_i$.

iii) Let A and B be two fuzzy sub-sets of X.Then,we define the product and difference of fuzzy sets A and B by for each $x \in X,(A.B)(x) = A(x) . B(x)$

and $(A - B)(x) = \max\{A(x) - B(x), 0\}$.The complement of A is the fuzzy set $\overline{A}$ defined by $\overline{A}(x) = 1 - A(x)$ for each $x \in X$.

2.3 Definition. The fuzzy class **F** is called a fuzzy σ-algebra on X,if it satisfies the following conditions;

i) $X \in \mathbf{F}$.

ii) If $A,B \in \mathbf{F}$, then $\overline{A} \in \mathbf{F}$,$A.B \in \mathbf{F}$ and $A - B \in \mathbf{F}$.

iii) If $A_n \in \mathbf{F}$, $n \in N$, then $\sup_n A_n = \bigvee_n A_n \in \mathbf{F}$.

In this case,the pair (X,**F**) is a fuzzy measurable space and the elements of **F** are fuzzy measurable sets.For more details,we refer to (7).

2.4 Definition. Let (X,**F**) and (Y,$\mathbf{F}_1$) be two fuzzy measurable spaces.One says that the transformation T from (X,**F**) to (Y,$\mathbf{F}_1$) is fuzzy measurable,if for each $A \in \mathbf{F}_1$,then $T^{-1}(A) \in \mathbf{F}$. For more properties of these transformations,See,(7) and (8).

2.5 Definition. Let (X,**A**,m) be a classical probability measure space.See, (1).A fuzzy probability measure is a fuzzy measurable mapping μ from the fuzzy measurable space (X,**F**) to [0,1] defined by $\mu(A) = \int A \, dm$ fulfilling the following conditions;

i) $\mu(\emptyset) = 0$ and $\mu(X) = 1$.

ii) $\mu(A) \geq 0$ for each $A \in \mathbf{F}$.

iii) If $(A_n)_{n \in N}$ is a disjoint sequence of fuzzy sets and $A_n \in \mathbf{F}$ for every $n \in N$,then we have; $\mu(\bigvee_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ (fuzzy σ-additivity).

The triple (X,**F**,μ) is a fuzzy probability measure space.The elements of **F** are called fuzzy events.For more details,we refer to (2) and (7).

2.6 Definition. Let **(X,F**,μ) be a fuzzy probability measure space.A system $\mathbf{P}=\{A_1,\ldots,A_n\}$ with $A_i \in \mathbf{F}$ for each $i = 1,\ldots,n$ is called a complete system of fuzzy events if **P** is a fuzzy partition of X.Let **P** and **Q** be two fuzzy complete systems.If $\mu(\mathbf{P} \wedge \mathbf{Q}) = \mu(\mathbf{P})\,\mu(\mathbf{Q})$,then **P** and **Q** are called independent.

2.7 Definition. Let $(X,\mathbf{F},\mu)$ be a fuzzy probability measure space. The mapping $T : (X,\mathbf{F}) \rightarrow (X,\mathbf{F})$ is called a σ-homomorphism if it satisfies the following properties;

i) $T(\overline{A}) = \overline{T(A)}$ for every $A \in \mathbf{F}$.

ii) $T(\bigvee_n A_n) = \bigvee_n T(A_n)$ for any fuzzy sequence $(A_n)_{n\in N} \subset \mathbf{F}$.

The fuzzy σ-homomorphism $T : (X,\mathbf{F}) \rightarrow (X,\mathbf{F})$ is a fuzzy probability measure-preserving if it fulfills the condition $\mu(TA) = \mu(A)$ for each $A \in \mathbf{F}$.

The quadruple $(X,\mathbf{F},\mu,T)$ is called a fuzzy dynamical system. One will write briefly (X,T) instead of $(X,\mathbf{F},\mu,T)$ for convenience. For more properties of this system, See, (4) and (8).

3.FUZZY SEQUENCE ENTROPY FUNCTION

3.1 Theorem. Let (X,T) be a fuzzy dynamical system. If $\mathbf{P}$ is a fuzzy complete system with $H_{\mu,f}(\mathbf{P}) < \infty$, $D = (t_n)_{n\geq 1}$ is a sequence of integers with $t_1 = 0$ and T is an invariant fuzzy σ-homomorphism, then $\lim_{n\to\infty} \sup \frac{1}{n} H_{\mu,f}(\bigvee_{i=1}^{n} T^{t_i}P)$ exists.

Where the quantity $H_{\mu,f}(\mathbf{P})$ is a fuzzy entropy function of fuzzy complete system $\mathbf{P}$. For more details, See, (8) and (9).

Proof. See, the Theorem 3.8 of (9).

3.2 Definition. The quantity $h_{\mu,f,D}(T,\mathbf{P}) = \lim_{n\to\infty} \sup \frac{1}{n} H_{\mu,f}(\bigvee_{i=1}^{n} T^{t_i}P)$ is called a fuzzy sequence entropy function of T with respect to the fuzzy complete system $\mathbf{P}$.

3.3 Proposition. Let $\mathbf{P}$ and $\mathbf{Q}$ be two fuzzy complete systems of (X,T) with $H_{\mu,f}(\mathbf{P}) < \infty$ and with $H_{\mu,f}(\mathbf{Q}) < \infty$ and $D = (t_n)_{n\geq 1}$ be a sequence of integers with $t_1 = 0$. Then,

i) $h_{\mu,f,D}(T,\mathbf{P}) \geq 0$.

ii) $h_{\mu,f,D}(T,\mathbf{P}) \leq H_{\mu,f}(\mathbf{P})$.

iii) If $\mathbf{P} \subset \mathbf{Q}$, then $h_{\mu,f,D}(T,\mathbf{P}) \leq h_{\mu,f,D}(T,\mathbf{Q})$.

iv) $h_{\mu,f,D}(T,\mathbf{P} \vee \mathbf{Q}) \leq h_{\mu,f,D}(T,\mathbf{P}) + h_{\mu,f,D}(T,\mathbf{Q})$.With finite fuzzy entropy the equality holds if and only if **P** and **Q** are independent.

v) $h_{\mu,f,D}(T,\mathbf{P}) = h_{\mu,f,D}(T,T\mathbf{P})$.

Proof. See,the Proposition 3.11 of (9).

3.4 Definition. The quantity

$$h_{\mu,f,D}(T) = \sup_{P} \{h_{\mu,f,D}(T,\mathbf{P}) : \mathbf{P} \text{ is a fuzzy complete system of } X \text{ with } H_{\mu,f}(\mathbf{P}) < \infty\}$$

is called a fuzzy sequence entropy function of fuzzy dynamical system (X,T). Where the supremum is taken over all fuzzy complete system of X with the finite fuzzy entropies.

3.5 Corollary. Let $D = (t_n)_{n\geq 1}$ be a sequence of integers with $t_1 = 0$ and $(\mathbf{P}_k)_{k\geq 1}$ be a family of fuzzy complete systems of (X,T) such that $\mathbf{P}_1 \subset \mathbf{P}_2 \subset \ldots$ and

$\bigvee_{i=1}^{n} \mathbf{P}_i = X$.Then, $h_{\mu,f,D}(T) = \lim_{k\to\infty} h_{\mu,f,D}(T,\mathbf{P}_k)$.

Proof. See,the Lemma 3.12 of (9).

3.6 Definition. Let **P** be a finite fuzzy complete system of (X,T) with $H_{\mu,f}(\mathbf{P}) < \infty$ and $D = (t_n)_{n\geq 1}$ be a sequence of integers with $t_1 = 0$.

If $\bigvee_{i=-\infty}^{\infty} T^{ti}\mathbf{P} \equiv \mathbf{F}$,then the fuzzy complete system **P** is called a fuzzy generator of **F** for T.

3.7 Theorem. Let **P** be fuzzy complete system which generates **F**.Consider $D = (t_n)_{n\geq 1}$ a sequence of integers with $t_1 = 0$.Then,$h_{\mu,f,D}(T) = h_{\mu,f,D}(T,\mathbf{P})$.

Proof. Write $\mathbf{P}_k = \bigvee_{i=-k}^{k} T^{ti}\mathbf{P}$.Clearly $(\mathbf{P}_k)_{k\geq 1}$ is an increasing sequence of fuzzy complete systems of (X,T) which generates **F**.Thus,one has only to apply the Corollary 3.5.

3.8 Definition. Let (X,T) and (Y,S) be two fuzzy dynamical systems.We say (Y,S) is a fuzzy factor of (X,T) if there exist $\mathbf{P} \in \mathbf{F}$ and $\mathbf{Q} \in \mathbf{F}_1$ such that

i) $\mu(\mathbf{P}) = 1$ and $\nu(\mathbf{Q}) = 1$.

ii) There exists a fuzzy probability measure- preserving function $\varphi : \mathbf{P} \to \mathbf{Q}$ such that $\varphi(T(x)) = S(\varphi(x))$ for all $x \in X$.

3.9 Proposition. Let (Y,S) be a fuzzy factor of (X,T) and $D = (t_n)_{n\geq 1}$ be a sequence of integers with $t_1 = 0$.Then,$h_{\nu,f,D}$ (S) $\leq h_{\mu,f,D}$ (T).

Proof. See,the Proposition 3.13 of (9).

3.10 Corollary.i) $h_{\mu,f,D}$ (T) ≥ 0.

ii) $h_{\mu,f,D}$ (Id) = 0.

Proof.(i) and **(ii)** are trivial from the Proposition 3.3 **(i)** and **(ii)** and Definition 3.4.

3.11 Proposition. Let (X,T) and (Y,S) be two fuzzy dynamical systems and $D = (t_n)_{n\geq 1}$ be a sequence of integers with $t_1 = 0$.
Then,$h_{\mu\times\nu,f,D}$ (T × S) = $h_{\mu,f,D}$ (T) + $h_{\nu,f,D}$ (S).Where T × S is a fuzzy σ-homomorphism defined on the fuzzy product space X ×Y with (T×S)(x , y) = (Tx,Sy) for all (x,y) ∈ X × Y

Proof. Let $(\mathbf{P}_n)_{n\geq 1}$ (resp. $(\mathbf{Q}_n)_{n\geq 1}$) be an increasing sequence of fuzzy complete systems of X (resp.Y) which generates **F** (resp. $\mathbf{F}_1$).Each $\mathbf{P}_n$ induces a fuzzy complete system $\mathbf{U}_n$ of fuzzy product space X ×Y,the elements of $\mathbf{U}_n$ being of the form A × X.Where A runs trough the elements $\mathbf{P}_n$.Similarly,$\mathbf{Q}_n$ induces a fuzzy complete system $\mathbf{V}_n$ of fuzzy product space X ×Y.It easy to see that $\mathbf{R}_n = \mathbf{U}_n \vee \mathbf{V}_n$ is an increasing sequence of fuzzy complete systems of X ×Y which generates $\mathbf{F} \times \mathbf{F}_1$
Since $\mathbf{U}_n$ and $\mathbf{V}_n$ are independent,one has from the Proposition 3.3 (iv) and Definition 3.2;

$$H_{\mu\times\nu,f}\left(\bigvee_{i=1}^{n}(T\times S)^{ti}\,\mathbf{R}_n\right)=H_{\mu\times\nu,f}\left(\bigvee_{i=1}^{n}(T\times S)^{ti}\,\mathbf{U}_n\right)+H_{\mu\times\nu,f}\left(\bigvee_{i=1}^{n}(T\times S)^{ti}\,\mathbf{V}_n\right) \quad (3\text{-}1).$$

But clearly,

$$H_{\mu\times\nu,f}\left(\bigvee_{i=1}^{n}(T\times S)^{ti}\,\mathbf{U}_n\right) = H_{\mu,f}\left(\bigvee_{i=1}^{n} T^{ti}\mathbf{P}_n\right) \quad (3\text{-}2) \text{ and}$$

$$H_{\mu\times\nu,f}\left(\bigvee_{i=1}^{n}(T\times S)^{ti}\,\mathbf{V}_n\right) = H_{\nu,f}\left(\bigvee_{i=1}^{n} S^{ti}\mathbf{Q}_n\right) \quad (3\text{-}3).$$

Therefore one writes,

$$H_{\mu\times\nu,f}\left(\bigvee_{i=1}^{n}(T\times S)^{ti}\,\mathbf{R}_n\right) = H_{\mu,f}\left(\bigvee_{i=1}^{n} T^{ti}\,\mathbf{P}_n\right) + H_{\nu,f}\left(\bigvee_{i=1}^{n} S^{ti}\,\mathbf{Q}_n\right) \quad (3\text{-}4).$$

Dividing the equality 3-4 by n>0 and leting n → ∞,one obtains the following result from the Theorem 3.1,

$h_{\mu\times\nu,f,D}$ (T × S,R_n) = $h_{\mu,f,D}$ (T,$\mathbf{P}_n$) + $h_{\nu,f,D}$ (S,$\mathbf{Q}_n$) (3-5).For n → ∞,one has therefore by the Corollary 3.5,
$h_{\mu\times\nu,f,D}$(T S) = $h_{\mu,f,D}$ (T) + $h_{\nu,f,D}$ (S) (3-6)

4.FUZZY SEQUENCE LOCAL ENTROPY FUNCTION

4.1 Definition. Let **P** be a finite fuzzy complete system of (X,T) with
$H_{\mu,f}$ (**P**) < ∞.Then ,the quantity
$L_{\mu,f,D}$ (T) = $h_{\mu,f,D}$ (T) - $h_{\mu,f,D}$ (T,**P**) is called a fuzzy sequence local entropy function.

4.2 Corollary. i) $L_{\mu,f,D}$ (T) ≥ 0.
ii) $L_{\mu,f,D}$ (Id) = 0.
iii) If **P** is a finite fuzzy complete system which generates **F**,
then $L_{\mu,f,D}$ (T) = 0.

Proof. (**i**) and (**ii**) are trivial from the Proposition 3.3 (**i**) and (**ii**),Corollary 3.10 (**i**) and (**ii**) and Definition 4.1.

iii) Suppose that **P** is a generating fuzzy complete system of (X,T) with $H_{\mu,f}$ (**P**) < ∞.
Write the following equality from the Definition 4.1,
$L_{\mu,f,D}$ (T) = $h_{\mu,f,D}$ (T) - $h_{\mu,f,D}$ (T,**P**) (4-1).
Since $h_{\mu,f,D}$ (T) = $h_{\mu,f,D}$ (T,**P**) (4-2) from the Theorem 3.7,we obtain thus the result $L_{\mu,f,D}$ (T) = 0 (4-3).

4.3 Proposition. If the fuzzy dynamical system (Y,S) is a fuzzy factor of (X,T),then,
$L_{\nu,f,D}$ (S) ≤ $L_{\mu,f,D}$ (T) + $h_{\mu,f,D}$ (T,**P**) - $h_{\nu,f,D}$ (S,**Q**).

Proof.

We write the following inequality from the Proposition 3.9,
$h_{\nu,f,D}$ (S) ≤ $h_{\mu,f,D}$ (T) (4-4).
Let **P** be a finite fuzzy complete system of the fuzzy dynamical system (X,T) with $H_{\mu,f}$ (**P**) < ∞.

As $h_{\mu,f,D}$ (T,**P**) ≥ 0 from the Proposition 3.3 (**i**),one has the following inequality,

$h_{\nu,f,D}$ (S) - $h_{\mu,f,D}$ (T,**P**) $\leq$ $h_{\mu,f,D}$ (T) - $h_{\mu,f,D}$ (T,**P**) (4-5).Therefore,one obtains from the Definition 4.1, $h_{\nu,f,D}$ (S) $\leq$ $L_{\mu,f,D}$ (T) + $h_{\mu,f,D}$ (T,**P**) (4-6).Let **Q** be a finite fuzzy complete system of the fuzzy factor(Y,S) with $H_{\nu,f}$ (**Q**) $< \infty$.

As $h_{\nu,f,D}$ (S,**Q**) ≥ 0 from the Proposition 3.3 (**i**),we can also write the following inequality,
$h_{\nu,f,D}$ (S) - $h_{\nu,f,D}$ (S,**Q**) $\leq$ $L_{\mu,f,D}$ (T) + $h_{\mu,f,D}$ (T,**P**) - $h_{\nu,f,D}$ (S,**Q**) (4-7).
Hence,we obtain the result from the Definition 4.1,
$L_{\nu,f,D}$ (S) $\leq$ $L_{\mu,f,D}$ (T) + $h_{\mu,f,D}$ (T,**P**) - $h_{\nu,f,D}$ (S,**Q**) (4-8).

4.4 Proposition. Let (X,T) and (Y,S) be two fuzzy dynamical systems.Then, $L_{\mu\times\nu,f,D}$ (T $\times$ S) = $L_{\mu,f,D}$ (T) + $L_{\nu,f,D}$ (S).Where T $\times$ S is a fuzzy σ-homomorphism defined on the fuzzy product space X $\times$Y with (T$\times$S)(x , y) = (Tx,Sy) for all (x,y) $\in$ X $\times$ Y

Proof. Let **P** and **Q** be two finite fuzzy complete systems of X and Y respectively with $H_{\mu,f}$ (**P**) $< \infty$ and $H_{\nu,f}$ (**Q**) $< \infty$.Then,we can write the following equalities by Proposition 3.11;

$h_{\mu\times\nu,f,D}$ (T$\times$S,**P**$\times$**Q**) = $h_{\mu,f,D}$ (T,**P**) + $h_{\nu,f,D}$(S,**Q**) (4-9) and
$h_{\mu\times\nu,f,D}$ (T$\times$S) = $h_{\mu,f,D}$ (T) + $h_{\nu,f,D}$ (S) (4-10).

Therefore we have,
$h_{\mu\times\nu,f,D}$ (T$\times$S)-$h_{\mu\times\nu,f,D}$ (T$\times$S,**P**$\times$**Q**)= $h_{\mu,f,D}$(T)-$h_{\mu,f,D}$(T,**P**)+$h_{\nu,f,D}$(S)-$h_{\nu,f,D}$(S,**Q**) (4.11).

Hence,we obtain the result from the Definition 4.1,
$L_{\mu\times\nu,f,D}$ (T$\times$S) = $L_{\mu,f,D}$ (T) + $L_{\nu,f,D}$ (S) (4-12).

REFERENCES

(1) **COHN D.,L.,**Measure theory,Boston,1980.
(2) **DUSMITRESCU D.,**Fuzzy measures and entropy of fuzzy partitions,J. Math. Anal. Appl.,176,(1993),359-373.
(3) **HULSE P.,**Sequence entropy and subsequence generators,J. London Math. Soc.(2),(1982),
441-450.
(4) **KLOEDEN P.,E.,**Fuzzy dynamical systems,Fuzzy Sets and Sys.,7,(1982),275-296.
(5) **KUSHENIRENKO A.,G.,**On metric invariants of entropy type,Russian Math.Surveys,4,(1970),
119-128.

(6) **NEWTON D.**,On sequence entropy I,Math. Sys. Theory,4(2),(1970),119-125.

(7) **TOK I.**,On the fuzzy measure space,Ege Univ. J. Sc. Fac. Series A,8(1),(1985),89-100.

(8) **TOK I.**,Entropy of fuzzy dynamical system,Ege Univ. J.Sc. Fac. Series A,8(1),(1985),101-109.

(9) **TOK I.**,On the fuzzy sequence entropy function,D.A.D. II,Int.Math. Polish Acad.Sc.Varsawa,1995

(10) **TOK I.**,On the fuzzy local entropy function, Ege Univ. J. Sc. Fac. Series A,44,2005.

(11) **ZADEH L.,A.**,Fuzzy sets,Inform. Control,8,(1965),338-353.