Bilecik Şeyh Edebali Üniversitesi Fen Bilimleri Dergisi
Bilecik Seyh Edebali University Journal of Science
11(2), 325-334, 2024
DOİ: 10.35193/bseufbd.1313160
e-ISSN: 2458-7575 (https://dergipark.org.tr/tr/pub/bseufbd)

Araştırma Makalesi - Research Article

Groupoid and Semigroup Construction on Isosceles Triangular Numbers

İkizkenar Üçgensel Sayılar Üzerindeki Grupoid ve Yarıgrup Yapıları

Ahmet Emin[1*], Ümit Sarp[2]

Geliş / Received: 12/06/2023 ***Revize / Revised: 20/10/2023*** ***Kabul / Accepted: 26/10/2023***

ABSTRACT

Basic information about figurative numbers is provided. Then, information about isosceles triangular numbers, one of the two-dimensional figurative numbers, is given. It also includes information about algebraic structures and their definitions. Additionally, a binary operation that includes k -isosceles triangular numbers is presented, and the study investigates whether the algebraic structures defined with this operation form a groupoid or semigroup. Also, two examples are given that satisfy the results at the end of the paper.

Keywords-* Isosceles *Triangular Number, Binary Operation, Groupoid, Semigroup, Monoid

ÖZ

Figüratif sayılar hakkında temel bilgiler verilmektedir. Daha sonra iki boyutlu figüratif sayılardan biri olan ikizkenar üçgen sayıları hakkında bilgi verilmiştir. Ayrıca cebirsel yapılar ve tanımları hakkında bilgi içerir. Ek olarak k -ikizkenar üçgensel sayıları içeren bir ikili işlem sunulmuş ve bu ikili işlem ile tanımlanan yapıların bir grupoid veya yarı grup oluşturup oluşturmadığı araştırılmıştır. Ayrıca, makalenin sonunda sonuçları sağlayan iki adet örnek verilmiştir.

Anahtar Kelimeler-* *İkizkenar Üçgensel Sayı, İkili İşlem, Grupoid, Yarıgrup, Monoid

[1]*Corresponding Author Contact: ahmetemin@karabuk.edu.tr (https://orcid.org/0000-0001-7791-7181)
Department of Mathematics, Faculty of Science, Karabuk University, Karabük, Türkiye
[2]Contact: umitsarp@ymail.com (https://orcid.org/0000-0002-1260-785X)
Continuing Education Application and Research Center, İzmir Katip Çelebi University, Izmir, Türkiye

I. INTRODUCTION AND PURPOSE

Figurate numbers are numbers that represent geometric shapes such as triangles and squares, formed by arranging specific points in a plane or space according to a certain rule. There are many types of figurate numbers, such as polygonal numbers, centered polygonal numbers, L-shape numbers, and solid numbers. Among these, polygonal numbers are the most well-known and widely studied. In particular, the beauty and elegance of triangular numbers have captured the attention of scientists for thousands of years.

Points positioned in the plane as a regular triangle with equal intervals represent triangular numbers. They are formed by starting with a single point and adding the same common difference, which is always one, to each successive point.

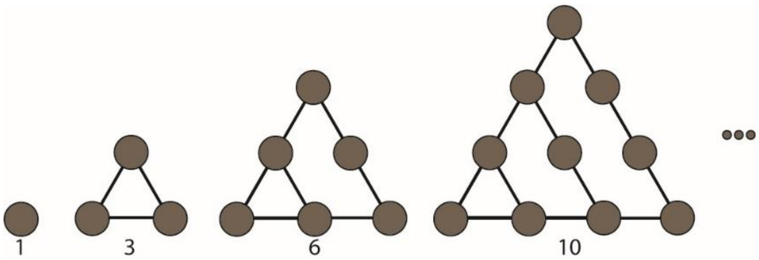

Figure 1 . Triangular numbers

The triangular numbers can be represented by $S_3(n) = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$. For further knowledge on figurate numbers and triangular numbers, one can refer to book by Deza et al. [1].

Isosceles triangular numbers are a novel type of figurate numbers that were first defined and introduced by Jitman et al. in [2,3]. Isosceles triangular numbers are a new type of figurate numbers that can be represented by a pattern of points arranged in the shape of an isosceles triangle. The nth $k-$isosceles triangular number is represented by $\Delta(n,k)$. It's worth noting that regular triangular numbers are a specific case of isosceles triangular numbers. If $k=1$, then the isosceles triangular number becomes a triangular number, i.e., $\Delta(n,1) = S_3(n)$.

Additionally, for convenience, if $n=0$, then $\Delta(0,k)$ is used in some contexts and it is defined to be 0. Some summation formulas for isosceles triangular numbers are derived [2,3].

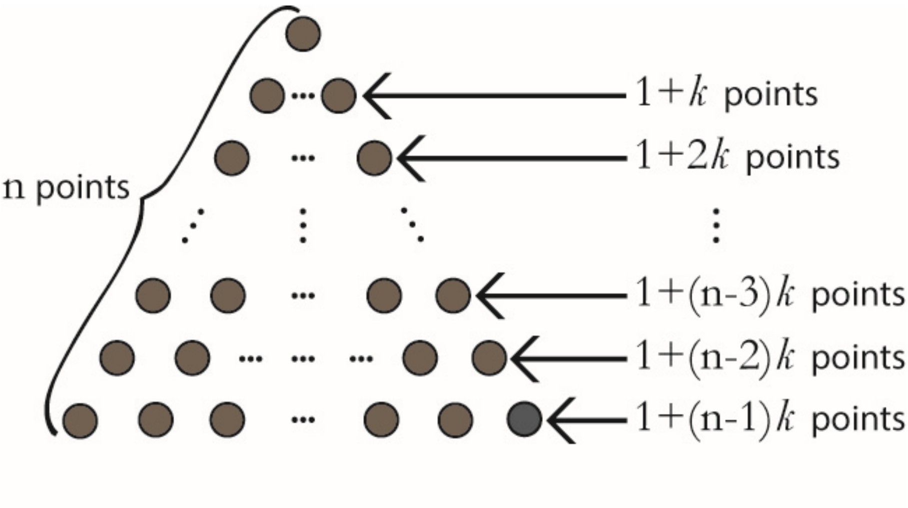

Figure 2. Isosceles triangular numbers

The concept of a binary operation is crucial in many algebraic structures. It takes on different names depending on the binary operation defined on the algebraic structures. Groupoids, semigroups, and monoids are a few examples. Specifically, Sparavigna has established that it is a groupoid with binary operators defined on some polygonal numbers in [4,5]. Additionally, Emin has defined some algebraic structure on some polygonal and figurate numbers in [6,7]. By utilizing similar methods as in these papers, we will present a general binary operation that includes k-isosceles triangular numbers. Additionally, it will be examined whether the structures a semigroup or not. Other algebraic properties may be studied with the binary operation defined on this new algebraic structure.

Algebraic structures allow the study of different properties of mathematical objects. It helps to better understand mathematical systems by revealing the relationships of mathematical structures with each other. For this reason, algebraic structures have an important place in mathematics.

It is possible to see algebraic structures in mathematics as well as in other branches of science. Algebraic structures can be used in disciplines such as Physics, Engineering and computers. This can be help to contribute to the modeling and solving of problems.

Various algebraic structures are defined in order to make the transition between geometry and algebra. It is also possible to express figurate numbers geometrically. For this reason, if an algebraic structure can be defined on figurate numbers, its relationship with other mathematical systems can be examined algebraically.

The real motivation behind this paper is to explore the properties of algebraic structures defined with isosceles triangular numbers and determine whether they form a groupoid or semigroup, supported by illustrative examples.

II. PRELIMINARIES

Definition 2.1. A groupoid $(A,\bullet)$ is an algebraic structure on a set with a binary operator. The only restriction on the operator is closure $\bullet: A\times A\to A$ (i.e., applying the binary operator to two elements of a given set returns a value which is itself a member of A). Associativity, commutativity, etc., are not required [8].

$(A,\bullet)$ is called a *semigroup* if the operation $\bullet$ is satisfies properties of associative, that is, if for all $x,y,z\in A$;

$$(x\bullet y)\bullet z = x\bullet(y\bullet z). \tag{1}$$

If $x\bullet y = y\bullet x$ equality is provided for all $x,y\in A$, A is called a *commutative semigroup*.

$(A,\bullet)$ is called a *monoid* if the operation $\bullet$ is satisfies the properties of identity, that is, there is an $e\in A$, for all $x\in A$;

$$x\bullet e = e\bullet x = x. \tag{2}$$

A semigroup is an algebraic structure that is associative and closed, and a semigroup with an identity element is called a monoid.

If A does not have an identity element, then adding the identity element to A allows for the conversion of this algebraic structure into a monoid. As defined in Eq. (2), $(A\cup\{e\},\bullet)$ became a monoid.

Example 2.1. The sets N, Z, Q, R and Z_n are all semigroups with the respect to the addition and multiplications operations.

It is clear that a monoid is a semigroup. Semigroups, then, provide some monoid concepts exactly. For example, the order of a semigroup is equal to the number of members of the semigroup, as in monoids. The N, Z, Q, R and Z_n semigroups satisfy certain properties of these structures' monoid counterparts. Also, they have the property of commutative. However, not all semigroups have to be commutative.

A semigroup $(A=\{e\},\bullet)$ consisting of only one element that satisfies the property of $e\bullet e = e$ is called a *trivial semigroup*. Let T be a subset of A $(T\neq\varnothing)$. If the set T provides the closure property according to the operation defined in Eq. (1), then T is called a *sub-semigroup* of the semigroup $(A,\bullet)$.

Definition 2.2. For $n,k\in N$, the nth k-isosceles triangular number formula is as follows [2];

$$\Delta(n,k) := n + \frac{n(n-1)}{2}k. \tag{3}$$

For $n,k\in N$, $k-$isosceles triangular number $\Delta(n,k)$ is obtained as the sum of the first n elements of the sequence [3]. So, it holds;

$$\Delta(n,k)=1+(1+k)+(1+2k)+\ldots+(1+(n-1)k)$$
$$=\left[\frac{1+(1+(n-1)k)}{2}\right]\left[\frac{(n-1)k}{k}+1\right] \quad (4)$$
$$=n+\frac{n(n-1)}{2}k.$$

For the remainder of the paper, set A will be employed as a set constituted by the elements of the $\Delta(n,k)$ sequences.

Example 2.1. For $k=1,2,3,4$ and $n\in N$, $k-$isosceles triangular numbers formulas are as follows;

$$\Delta(n,1)=n+\frac{n(n-1)}{2}\times 1=\frac{n(n+1)}{2},$$
$$\Delta(n,2)=n+\frac{n(n-1)}{2}\times 2=n^2,$$
$$\Delta(n,3)=n+\frac{n(n-1)}{2}\times 3=\frac{n(3n-1)}{2},$$
$$\Delta(n,4)=n+\frac{n(n-1)}{2}\times 4=n(2n-1).$$

The above expression implies the following recurrence formula for $k-$isosceles triangular numbers;

$$\Delta(n+1,k)=\Delta(n,k)+1+nk \quad and \quad \Delta(1,k)=1.$$

In particular, we get

$$\Delta(n+1,1)=\Delta(n,1)+1+n,$$
$$\Delta(n+1,2)=\Delta(n,2)+1+2n,$$
$$\Delta(n+1,3)=\Delta(n,3)+1+3n,$$
$$\Delta(n+1,4)=\Delta(n,4)+1+4n.$$

Example 2.2. Some figures of $k-$isosceles triangular number for $k=1,2,3$ are as follows [3];

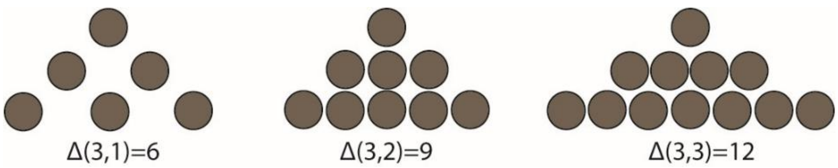

Figure 3. Some figures isosceles triangular numbers

III. SEMIGROUP CONSTRUCTION ON POLYGONAL NUMBERS

A set consisting of elements of all $k-$isosceles triangular numbers sequence $\Delta(n,k)$ is created in this section. An algebraic structure is obtained by defining binary operation on the defined set. Theorems that show necessary conditions for this algebraic structure to be groupoid and semigroup are given.

Before we can construct the theorem that yields the main result of this study, we need to define a set and a binary operation on that set. So, let A denote the sequence of numbers $\Delta(n,k)$.

$$A=\left\{1,k+2,3k+3,6k+4,10k+5,15k+6,\ldots,\frac{n(n-1)}{2}k+n,\ldots\right\} \quad (5)$$

A binary operation can be found on the given set of A as,

$$\begin{aligned}\left(\Delta(n,k)+\frac{1}{8}\frac{(k-2)^2}{k}\right)^{\frac{1}{2}} &= \left(n+\frac{n(n-1)}{2}k+\frac{1}{8}\frac{(k-2)^2}{k}\right)^{\frac{1}{2}}\\ &=\left(\frac{1}{2k}\left(\frac{4n^2k^2-4nk^2+8nk+k^2-4k+4}{4}\right)\right)^{\frac{1}{2}}\\ &=\left(\frac{1}{2k}\left(\frac{2nk-k+2}{2}\right)^2\right)^{\frac{1}{2}}\\ &=\frac{1}{\sqrt{2k}}\left(nk-\frac{k-2}{2}\right).\end{aligned} \tag{6}$$

We define as follows,

$$\begin{aligned}\Delta_n &= \left(\Delta(n,k)+\frac{1}{8}\frac{(k-2)^2}{k}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2k}}\left(nk-\frac{k-2}{2}\right),\\ \Delta_m &= \left(\Delta(m,k)+\frac{1}{8}\frac{(k-2)^2}{k}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2k}}\left(mk-\frac{k-2}{2}\right),\\ \Delta_{n+m} &= \left(\Delta(n+m,k)+\frac{1}{8}\frac{(k-2)^2}{k}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2k}}\left((n+m)k-\frac{k-2}{2}\right).\end{aligned} \tag{7}$$

We use Δ_n for definition of binary operation as follows;

$$\begin{aligned}\Delta_{n+m} = \Delta_n \bullet \Delta_m &= \Delta_n + \Delta_m + \frac{k-2}{2\sqrt{2k}}\\ &= \frac{1}{\sqrt{2k}}\left(nk-\frac{k-2}{2}\right)+\frac{1}{\sqrt{2k}}\left(mk-\frac{k-2}{2}\right)+\frac{k-2}{2\sqrt{2k}}\\ &= \frac{1}{\sqrt{2k}}\left(nk-\frac{k-2}{2}+mk-\frac{k-2}{2}+\frac{k-2}{2}\right)\\ &= \frac{1}{\sqrt{2k}}\left((n+m)k-\frac{k-2}{2}\right).\end{aligned} \tag{8}$$

Therefore, we have the binary operation as follows;

$$\left(\Delta(n+m,k)+\frac{1}{8}\frac{(k-2)^2}{k}\right)^{\frac{1}{2}} = \left(\Delta(n,k)+\frac{1}{8}\frac{(k-2)^2}{k}\right)^{\frac{1}{2}} + \left(\Delta(m,k)+\frac{1}{8}\frac{(k-2)^2}{k}\right)^{\frac{1}{2}} + \frac{k-2}{2\sqrt{2k}}. \tag{9}$$

As a result, from Eq. (9), we can rewrite the defined binary operation as follows;

$$\begin{aligned}\Delta(n,k)\bullet\Delta(m,k) &= \Delta(n+m,k)\\ &= \Delta(n,k)+\Delta(m,k)+\frac{(k-2)^2}{4k}\\ &\quad +2\left(\Delta(n,k)+\frac{(k-2)^2}{8k}\right)^{\frac{1}{2}}\left(\Delta(m,k)+\frac{(k-2)^2}{8k}\right)^{\frac{1}{2}}\\ &\quad +\frac{k-2}{\sqrt{2k}}\left(\Delta(n,k)+\frac{(k-2)^2}{8k}\right)^{\frac{1}{2}}+\frac{k-2}{\sqrt{2k}}\left(\Delta(m,k)+\frac{(k-2)^2}{8k}\right)^{\frac{1}{2}}.\end{aligned} \tag{10}$$

Theorem 3.1. Let $n, k \in N$ and A be the set of sequences of numbers $\Delta(n,k)$, and let $\bullet$ be a binary operation defined on the set A. Then the algebraic structure $(A,\bullet)$ is a groupoid.

Proof. From the binary operation $\bullet$, we can derive a recursive relation for $m=1$;

$$\begin{aligned}\Delta(n,k)\bullet\Delta(1,k)&=\Delta(n+1,k)\\ &=\Delta(n,k)+\Delta(1,k)+\frac{(k-2)^2}{4k}\\ &+2\left(\Delta(n,k)+\frac{(k-2)^2}{8k}\right)^{\frac{1}{2}}\left(\Delta(1,k)+\frac{(k-2)^2}{8k}\right)^{\frac{1}{2}}\\ &+\frac{k-2}{\sqrt{2k}}\left(\Delta(n,k)+\frac{(k-2)^2}{8k}\right)^{\frac{1}{2}}+\frac{k-2}{\sqrt{2k}}\left(\Delta(1,k)+\frac{(k-2)^2}{8k}\right)^{\frac{1}{2}}.\end{aligned} \tag{11}$$

And finally, we have,

$$\Delta(n+1,k)=\Delta(n,k)+\frac{k}{2}+\sqrt{2k}\left(\Delta(n,k)+\frac{(k-2)^2}{8k}\right)^{\frac{1}{2}}. \tag{12}$$

We prove this part of the theorem by using mathematical induction on n. For $n=1,2,3$ and starting from number $\Delta(1,k)=1$, we have $k+2$, $3k+3$, $6k+4$ which are the elements of the set of A in Eq. (5). In fact, for $k\geq 1$;

$$\begin{aligned}\Delta(2,k)&=\Delta(1,k)+\frac{k}{2}+\sqrt{2k}\left(\Delta(1,k)+\frac{(k-2)^2}{8k}\right)^{\frac{1}{2}}\\ &=1+\frac{k}{2}+\sqrt{2k}\left(1+\frac{k^2-4k+4}{8k}\right)^{\frac{1}{2}}\\ &=1+\frac{k}{2}+\sqrt{2k}\,\frac{k+2}{2\sqrt{2k}}=k+2,\end{aligned}$$

$$\begin{aligned}\Delta(3,k)&=\Delta(2,k)+\frac{k}{2}+\sqrt{2k}\left(\Delta(2,k)+\frac{(k-2)^2}{8k}\right)^{\frac{1}{2}}\\ &=k+2+\frac{k}{2}+\sqrt{2k}\left(k+2+\frac{k^2-4k+4}{8k}\right)^{\frac{1}{2}}\\ &=k+2+\frac{k}{2}+\sqrt{2k}\,\frac{3k+2}{2\sqrt{2k}}=3k+3,\end{aligned}$$

$$\begin{aligned}\Delta(4,k)&=\Delta(3,k)+\frac{k}{2}+\sqrt{2k}\left(\Delta(3,k)+\frac{(k-2)^2}{8k}\right)^{\frac{1}{2}}\\ &=3k+3+\frac{k}{2}+\sqrt{2k}\left(3k+3+\frac{k^2-4k+4}{8k}\right)^{\frac{1}{2}}\\ &=3k+3+\frac{k}{2}+\sqrt{2k}\,\frac{7k+2}{2\sqrt{2k}}=6k+4.\end{aligned}$$

We assume that Eq. (12) holds recursively for $n+1$.

For $k \geq 1$, $\Delta(n+1,k) = \Delta(n,k) + \frac{k}{2} + \sqrt{2k}\left(\Delta(n,k) - \frac{(k-2)^2}{8k}\right)^{\frac{1}{2}}$ is the element of the set of A. Also, from Eqs. (7) and (12), we have

$$\begin{aligned}
\Delta(n+1,k) + \frac{k}{2} + \sqrt{2k}\left(\Delta(n+1,k) - \frac{(k-2)^2}{8k}\right)^{\frac{1}{2}} &= \Delta(n,k) + \frac{k}{2} + \sqrt{2k}\left(\Delta(n,k) - \frac{(k-2)^2}{8k}\right)^{\frac{1}{2}} + \frac{k}{2} + \sqrt{2k}\left(\Delta(n+1,k) - \frac{(k-2)^2}{8k}\right)^{\frac{1}{2}} \\
&= \Delta(n,k) + \frac{k}{2} + \sqrt{2k}\frac{1}{\sqrt{2k}}\left(nk - \frac{k-2}{2}\right) + \frac{k}{2} + \sqrt{2k}\frac{1}{\sqrt{2k}}\left((n+1)k - \frac{k-2}{2}\right) \\
&= \Delta(n,k) + k + nk + (n+1)k - k + 2 \\
&= \Delta(n,k) + (2n+1)k + 2 \\
&= n + \frac{n(n-1)}{2}k + (2n+1)k + 2 \\
&= n + 2 + \frac{(n+2)(n+1)}{2}k \\
&= \Delta(n+2,k)
\end{aligned}$$

Therefore $\Delta(n+2,k)$ is element of the set of A. Thus, the algebraic structure $(A,\bullet)$ satisfies the properties of closure which gives us the theorem that $(A,\bullet)$ is a groupoid.

□

The following theorem presents a necessary condition for the algebraic structure $(A,\bullet)$ to be a semigroup, which is the central result of this paper.

Theorem 3.2. Let $n, k \in N$ and A be the set of sequences of numbers $\Delta(n,k)$, and let $\bullet$ be a binary operation defined on the set A. Then the algebraic structure $(A,\bullet)$ is a semigroup.

Proof. From Theorem 3.1., it follows that the algebraic structure is a groupoid. Now, we need to demonstrate that it satisfies the properties of associativity, as established by the construction of the binary operation.

For $n, m \in N$ and $k \geq 1$ using Δ_n, Δ_m and Δ_{n+m} the following is obtained.

$$\begin{aligned}
(\Delta_n \bullet \Delta_m) \bullet \Delta_p &= \Delta_{n+m} + \Delta_p + \frac{k-2}{2\sqrt{2k}} \\
&= \frac{1}{\sqrt{2k}}\left((n+m)k - \frac{k-2}{2}\right) + \frac{1}{\sqrt{2k}}\left(pk - \frac{k-2}{2}\right) + \frac{k-2}{2\sqrt{2k}} \\
&= \frac{1}{\sqrt{2k}}\left((n+m+p)k - \frac{k-2}{2}\right) + \frac{k-2}{2\sqrt{2k}} \\
&= \frac{1}{\sqrt{2k}}\left(nk - \frac{k-2}{2}\right) + \frac{1}{\sqrt{2k}}\left((m+p)k - \frac{k-2}{2}\right) + \frac{k-2}{2\sqrt{2k}} \\
&= \Delta_n + \Delta_{m+p} + \frac{k-2}{2\sqrt{2k}} \\
&= \Delta_n \bullet (\Delta_m \bullet \Delta_p).
\end{aligned} \tag{13}$$

So, we obtain $(\Delta_n \bullet \Delta_m) \bullet \Delta_p = \Delta_n \bullet (\Delta_m \bullet \Delta_p)$ which gives us that $(A,\bullet)$ satisfies the properties of associativity.

□

Considering the starting point as $\Delta(0,k) = 0$ and using Theorem 3.2., we can deduce the following corollary that provides the conditions for $(A,\bullet)$ to be a monoid.

Corollary 3.1. Let A be the set of sequences of numbers $\Delta(n,k)$, and let $\bullet$ be a binary operation defined on the set A. If $\Delta(0,k)=0\in A$, then the algebraic structure $(A,\bullet)$ is a monoid.

Proof. From the Theorem 3.2., we have demonstrated that $(A,\bullet)$ is a semigroup. Demonstrating that the algebraic structure $(A,\bullet)$ is a monoid requires showing that it has an identity element. Let $\Delta(0,k)=0\in A$ and from Eq. (10);

$$\Delta(n,k)\bullet\Delta(0,k)=\Delta(n+0,k)=\Delta(n,k) \quad \text{and} \quad \Delta(0,k)\bullet\Delta(n,k)=\Delta(0+n,k)=\Delta(n,k).$$

Thus, we have,

$$\Delta(n,k)\bullet\Delta(0,k)=\Delta(0,k)\bullet\Delta(n,k)=\Delta(n,k).$$

The corollary 3.1. demonstrates that the pair $(A,\bullet)$ satisfies the identity properties.

$\square$

Example 3.1. The $\Delta(n,3)$ is referred to as the $3-$isosceles triangular numbers, which are integers of the form:

$$\Delta(n,3)=1+(1+3)+(1+6)+\cdots(1+(n-1)3)=\frac{n(3n-1)}{2}.$$

Let $A=\{1,5,12,22,35,51,70,92,\ldots\}$ denote the sequence of numbers $\Delta(n,3)$. We can define a binary operation on the set A;

$$\left(\Delta(n,3)+\frac{1}{24}\right)^{\frac{1}{2}}=\frac{1}{\sqrt{6}}\left(3n-\frac{1}{2}\right)$$

and,

$$P_n=\left(\Delta(n,3)+\frac{1}{24}\right)^{\frac{1}{2}}=\frac{1}{\sqrt{6}}\left(3n-\frac{1}{2}\right),$$

$$P_m=\left(\Delta(m,3)+\frac{1}{24}\right)^{\frac{1}{2}}=\frac{1}{\sqrt{6}}\left(3m-\frac{1}{2}\right),$$

$$P_{n+m}=\left(\Delta(n+m,3)+\frac{1}{24}\right)^{\frac{1}{2}}=\frac{1}{\sqrt{6}}\left(3(n+m)-\frac{1}{2}\right).$$

We use P_n for definition of binary operation:

$$\begin{aligned}
P_{n+m}&=P_n\bullet P_m\\
&=P_n+P_m+\frac{1}{2\sqrt{6}}\\
&=\frac{1}{\sqrt{6}}\left(3n-\frac{1}{2}\right)+\frac{1}{\sqrt{6}}\left(3m-\frac{1}{2}\right)+\frac{1}{2\sqrt{6}}\\
&=\frac{1}{\sqrt{6}}\left(3n-\frac{1}{2}+3m-\frac{1}{2}+\frac{1}{2}\right)\\
&=\frac{1}{\sqrt{6}}\left(3(n+m)-\frac{1}{2}\right).
\end{aligned}$$

As a result, we can show the defined binary operation as follows;

$$\begin{aligned}\Delta(n,3)\bullet\Delta(m,3)&=\Delta(n+m,3)\\&=\Delta(n,3)+\Delta(m,3)+\frac{1}{12}\\&+2\left(\Delta(n,3)+\frac{1}{24}\right)^{\frac{1}{2}}\left(\Delta(m,3)+\frac{1}{24}\right)^{\frac{1}{2}}\\&+\frac{1}{\sqrt{6}}\left(\Delta(n,3)+\frac{1}{24}\right)^{\frac{1}{2}}+\frac{1}{\sqrt{6}}\left(\Delta(m,3)+\frac{1}{24}\right)^{\frac{1}{2}}.\end{aligned}$$

For $m=1$, recursive relation is obtained;

$$\Delta(n+1,3)=\Delta(n,3)+\frac{3}{2}+\sqrt{6}\left(\Delta(n,3)-\frac{1}{24}\right)^{\frac{1}{2}}.$$

For $\Delta(1,3)=1$, we get $5,12,22,35,51,70,92,\ldots$ which are the elements of the set of A. From the Theorem 3.2., the algebraic structure $(A,\bullet)$ is a semigroup. Also, from Corollary 3.1., if the $\Delta(0,3)=0\in A$, then the algebraic structure $(A,\bullet)$ is a monoid.

Example 3.2. The $\Delta(n,6)$ is known as the $6-$isosceles triangular numbers, which are integers that can be expressed in the form:

$$\Delta(n,6)=1+(1+6)+(1+12)+\cdots(1+(n-1)6)=n(3n-2).$$

Let $B=\{1,8,21,40,65,96,133,176,\ldots\}$ represent the sequence of numbers $\Delta(n,6)$. We can define a binary operation on the set B;

$$\left(\Delta(n,6)+\frac{1}{3}\right)^{\frac{1}{2}}=\frac{1}{\sqrt{3}}(3n-1)$$

and,

$$\begin{aligned}O_n&=\left(\Delta(n,6)+\frac{1}{3}\right)^{\frac{1}{2}}=\frac{1}{\sqrt{3}}(3n-1),\\O_m&=\left(\Delta(m,6)+\frac{1}{3}\right)^{\frac{1}{2}}=\frac{1}{\sqrt{3}}(3m-1),\\O_{n+m}&=\left(\Delta(n+m,6)+\frac{1}{3}\right)^{\frac{1}{2}}=\frac{1}{\sqrt{3}}(3(n+m)-1).\end{aligned}$$

We use O_n for definition of binary operation:

$$\begin{aligned}O_{n+m}&=O_n\bullet O_m\\&=O_n+O_m+\frac{1}{\sqrt{3}}\\&=\frac{1}{\sqrt{3}}(3n-1)+\frac{1}{\sqrt{3}}(3m-1)+\frac{1}{\sqrt{3}}\\&=\frac{1}{\sqrt{3}}(3n-1+3m-1+1)\\&=\frac{1}{\sqrt{3}}(3(n+m)-1).\end{aligned}$$

As a result of this situation, we can demonstrate the defined binary operation as follows;

$$\begin{aligned}\Delta(n,6)\bullet\Delta(m,6)=\Delta(n+m,6)\\ &=\Delta(n,6)+\Delta(m,6)+\frac{2}{3}\\ &+2\left(\Delta(n,6)+\frac{1}{3}\right)^{\frac{1}{2}}\left(\Delta(m,6)+\frac{1}{3}\right)^{\frac{1}{2}}\\ &+\frac{2}{\sqrt{3}}\left(\Delta(n,6)+\frac{1}{3}\right)^{\frac{1}{2}}+\frac{2}{\sqrt{2}}\left(\Delta(m,6)+\frac{1}{3}\right)^{\frac{1}{2}}.\end{aligned}$$

If one take $m=1$, recursive relation is obtained as follows;

$$\Delta(n+1,6)=\Delta(n,6)+3+2\sqrt{3}\left(\Delta(n,6)+\frac{1}{3}\right)^{\frac{1}{2}}.$$

Because of $\Delta(1,6)=1$, we get $8,21,40,65,96,133,176,\ldots$ which are the elements of the set of B. From the Theorem 3.2., the algebraic structure $(B,\bullet)$ is a semigroup and from Corollary 3.1., if the $\Delta(0,3)=0\in B$, then the algebraic structure $(B,\bullet)$ becomes a monoid.

As a result, Theorems 3.1., 3.2., and Corollary 3.1. demonstrate that this structure forms a monoid. The important question that needs to be addressed now is whether the structure of isosceles triangular numbers constitutes a group or not.

Remark 3.1. The algebraic structure $(A,\bullet)$ does not exhibit group properties, primarily due to the operation $\bullet$ failing to satisfy the requirements for inverses. This can be exemplified as follows:

Let $\Delta(x,1)$ be the inverse of $\Delta(3,1)$, where $\Delta(3,1)=6\in A$. So, $\Delta(3,1)\bullet\Delta(x,1)=\Delta(0,1)$ and $\Delta(3+x,1)=\Delta(0,1)\Rightarrow x=-3$. Because of that $x=-3\notin N$, $\Delta(3,1)$ doesn't have an inverse.

IV. CONCLUSION

When we interpret the results, only basic algebraic structures such as semigroups and monoids can be constructed. But being able to construct algebraic structures such as groups and rings with figurative numbers will mean many new fields of study.

As a result, in this study, polygonal numbers, which have an important place in Number Theory and have been studied for centuries, are combined with semigroup and monoid structures, which is one of the basic and important subjects of Algebra. The results show us that new examples of algebraic structures can be given by using figurative numbers. Since these structures are studied in the set of integers, they can be used in fields such as artificial intelligence, algebraic machine learning, and encryption. So, they are important. They can also lead to the discovery of new interdisciplinary research areas.

REFERENCES

[1] Deza, E., & Deza, M. M., (2012). *Figurate Numbers*, World Scientific Publishing Co. Pte. Ltd., Singapore.
[2] Jitman, S., Awachai, K., & Tanla, P., (2017). Isosceles Triangular Numbers, *Mathematical Journal-Math*, 62(692), 39-49.
[3] Jitman, S. & Punpim, J., (2021). Characterizations And Identities For Isosceles Triangular Numbers, *European Journal of Pure and Applied Mathematics*, 14(2), 380-395.
[4] Sparavigna, A. C., (2019). Groupoids of OEIS A003154 Numbers (Star Numbers or Centered Dodecagonal Numbers), *Zenodo*.
[5] Sparavigna, A. C., (2019). Groupoids of OEIS A093112 and A093069 Numbers (oblong and odd square numbers), *Zenodo*.
[6] Emin, A., (2021). Semigroup Construction on Polygonal Numbers, *Journal of Engineering Technology and Applied Sciences*, 6(3), 143-153.
[7] Emin, A., (2022). Some Algebraic Structure on Figurate Numbers, *Bitlis Eren Üniversitesi Fen Bilimleri Dergisi*, 11(2), 604-612.
[8] Rosenfeld, A., (1968). *An Introduction to Algebraic Structures*. New York: Holden-Day.