Iğdır Üniversitesi Fen Bilimleri Enstitüsü Dergisi, 13(4), 2945-2957, 202x
Journal of the Institute of Science and Technology, 13(4), 2945-2957, 202x
ISSN: 2146-0574, eISSN: 2536-4618
Mathematics **DOI: 10.21597/jist.1313391**
Research Article
Geliş tarihi / Received: 12.06.2023 **Accepted: 10.08.2023**

To Cite: Gülşen, T. & Acar, M. (2023). Self-Adjoint Sturm-Liouville Dynamic Problem via Proportional Derivative. *Journal of the Institute of Science and Technology*, 13(4), 2945-2957.

Self-Adjoint Sturm-Liouville Dynamic Problem via Proportional Derivative

Tuba GÜLŞEN[1*], Mehmet ACAR[1]

Highlights:

- Proportional derivative calculus on a time scale and its properties
- The equivalents of important theorems related to spectral theory in the Sturm-Liouville dynamic equation with proportional derivative

Keywords:

- Time Scale Calculus
- Proportional derivative
- Spectral theory
- Sturm-Liouville equation

ABSTRACT:

The concept of a conformable derivative on time scales is a relatively new development in the field of fractional calculus. Traditional fractional calculus deals with derivatives and integrals of non-integer order on continuous time domains. However, time scale calculus extends these concepts to more general time domains that include both continuous and discrete points. The conformable derivative on time scales has several properties that make it advantageous in certain applications. For example, it satisfies a chain rule and has a simple relationship with the conformable integral, which facilitates the development of differential equations involving fractional order dynamics. It also allows for the analysis of systems with both continuous and discrete data points, making it suitable for modeling and control applications in various fields, including physics, engineering, and finance. In this study, the Sturm-Liouville problem and its properties are examined on an arbitrary time scale using the proportional derivative, a more general form of the fractional derivative. Important spectral properties such as self-adjointness, Green formula, Lagrange identity, Abel formula, and orthogonality of eigenfunctions for this problem are expressed in proportional derivatives on an arbitrary time scale.

[1]Tuba GÜLŞEN (**Orcid ID:** 0000-0002-2288-8050), Mehmet ACAR (**Orcid ID:** 0000-0003-1280-8034), Firat University, Faculty of Science, Department of Mathematics, Elazig, Türkiye.

***Sorumlu Yazar/*Corresponding Author*:** Tuba GÜLŞEN, e-mail: tyalcin@firat.edu.tr

This work is produced from Mehmet ACAR's master's thesis.

INTRODUCTION

Fractional computation means the differentiation and integration of an integer order. Although it lacks several characteristics offered for fractional derivatives (Ortigueira and Machado, 2015), the conformable derivative was initially known as the conformable fractional derivative (Katugampola, 2014; Khalil et al., 2014; Abdeljewad, 2015). It is advisable to think about the proportional derivative on its own, free of the fractional derivative theory, even if the more broad definition of the proportional derivative provided in definition 1 below satisfies some of the features of the fractional derivative. The conformable derivative is a specific case of the proportional derivative.

Conformable fractional derivatives have different meanings depending on the time scale (Gulsen et al., 2017; Gülşen et al., 2018; Yilmaz et al., 2022). It's interesting to note that the conformable fractional derivative operator T_α in (Benkhettou et al., 2016) is defined as $\alpha \in (0, 1]$ in the form of $|[f(\sigma(t) - f(s)]t^{1-\alpha} - T_\alpha f(t)[\sigma(t) - s]| \le \varepsilon\ |\sigma(t) - s|$;

whereas in (Benkhettou et al., 2015) it is specified as

$$|[f(\sigma(t) - f(s)] - T_\alpha(f^\Delta)(t)[\sigma(t)^\alpha - s^\alpha]| \le \varepsilon\, |\sigma(t)^\alpha - s^\alpha|.$$

The truth that $T_0(f) \neq f$ and T_α is not proportional in accordance with definition 1 is evident from both definitions. As a consequence, a new conformable derivative was discovered under the name of the proportional derivative described in (Anderson and Ulness, 2015), and a prospective definition for the proportional derivative on a time scale was discovered in (Segi Rahmat, 2019). When $\alpha = 1$, the Hilger derivative replaces the proportional derivative of a function of order $\alpha \in [0,1]$ defined on a time scale.

In this research will be used the proportional derivative to analyze the Sturm-Liouville dynamic problem (23). Section 2 contains some basic concepts and notations regarding time scales and proportional derivatives on time scales. We demonstrate a number of spectral features for the problem (23) in Section 3 using various approaches, including self-adjointness, the Green Formula, Lagrange identity, the Abel formula, and orthogonality of eigenfunctions.

MATERIALS AND METHODS

We review the terms and ideas related to the time-scale proportional calculi that are required since they are utilized in the next section.

Definition 1 (Anderson and Ulness, 2015) Let $\alpha \in [0,1]$. The differential operatör $\overline{D}^\alpha$ is only referred to as a proportional derivative if $\overline{D}^0$ is the unit operator and $\overline{D}^1$ is the conventional differential operator. Particularly, the operator $\overline{D}^\alpha$ is referred to as being proportional for the derivative function $f{=}f(t)$, for which only

$$\overline{D}^0 f(t){=}f(t) \text{ and } \overline{D}^1 f(t){=}\frac{d}{dt}f(t){=}f'(t). \quad (1)$$

Remark 2 (Anderson and Ulness, 2015) Based on the employment of a proportional-derivative controller with an controller output u at time t, the fundamental idea of proportional derivative is established. This controller, $u(t)$, has an algorithm (Li et al., 2006)

$$u(t){=}\kappa_p E(t) + \kappa_d \frac{d}{dt}E(t).$$

In this case, E represents the error between the state variable and the process variable, while κ_p and κ_d represent the proportional and derivative gains, respectively.

Definition 3 (Anderson and Ulness, 2015) Assume that $\alpha \in [0,1]$, $\kappa_0, \kappa_1: [0,1] \times \mathbb{R} \to \mathbb{R}_0^+$ are continuous functions and that

$$\begin{cases} \lim_{\alpha\to 0^+} \kappa_0(\alpha,t) = 0, \quad \lim_{\alpha\to 0^+} \kappa_1(\alpha,t) = 1, \\ \lim_{\alpha\to 1^-} \kappa_0(\alpha,t) = 1, \lim_{\alpha\to 1^-} \kappa_1(\alpha,t) = 0, \\ \kappa_0(\alpha,t) \neq 0, \alpha \in (0,1], \quad \kappa_1(\alpha,t) \neq 0, \ \alpha \in [0,1), \end{cases} \tag{2}$$

are true. In this situation, if the function f is differentiable at t and $f'=\frac{d}{dt}f$, then the differential operator $\overline{D}^\alpha$ defined by

$$\overline{D}^\alpha f(t)=\kappa_1(\alpha,t)f(t) + \kappa_0(\alpha,t)f'(t), \tag{3}$$

is said to be proportional. Here, κ_1 is a type of proportional gain κ_p, κ_0 is a type of derivative gain κ_d, f is the error, and $u=\overline{D}^\alpha f$ is the controller output.

We must remember some basic concepts about time scales in order to get the basic findings for (23).The time scale $\mathbb{T}$ is a closed, non-empty subset of $\mathbb{R}$ in the standard topology of $\mathbb{R}$. The definitions of the forward and backward jump operators $\sigma, \rho: \mathbb{T} \to \mathbb{T}$, for $t \in \mathbb{T}$ are as follows:

$$\sigma(t)=\inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t)=\sup\{s \in \mathbb{T} : s < t\}.$$

This definition states that $\inf\emptyset = \sup\mathbb{T}$ and $\sup\emptyset=\inf\mathbb{T}$. If $\sigma(t) > t, \rho(t) < t, \rho(t) < t < \sigma(t), t$ is a right-scattered point, a left-scattered point, an isolated (discrete) point, respectively. On the other hand, if $t < \sup\mathbb{T}$ and $\sigma(t)=t$, t is referred to as right-dense, and if $t > \inf\mathbb{T}$ and $\rho(t)=t$, t is left-dense, and $\rho(t)=t=\sigma(t)$, then t is the dense point. The definition of the graininess function $\mu: \mathbb{T} \to [0,\infty)$ is $\mu(t)=\sigma(t) - t$. $\mathbb{T}^k=\mathbb{T} - \{m\}$ if there is a maximum point m of $\mathbb{T}$; else, $\mathbb{T}^k=\mathbb{T}$. The function $f : \mathbb{T} \to \mathbb{R}$ is called rd- continuously, provided that $\mathbb{T}$ has a left-sided limit at its right-dense points and at its left-scattered points and $C_{rd}(\mathbb{T})$ will be used to denote the collection of rd-continuous functions $f: \mathbb{T} \to \mathbb{R}$. Let $t \in \mathbb{T}^k$ and $f : \mathbb{T} \to \mathbb{R}$ be a function. $\forall\varepsilon > 0$, and for every s in a neighborhood U of point t, if there is a real number $f^\Delta(t)$, such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon\,|\sigma(t) - s|, \qquad \forall s \in U,$$

$f^\Delta(t)$ is called the delta derivative of f at point t. If t is right-scattered and the function $f: \mathbb{T} \to \mathbb{R}$ is continuous at t, then

$$f^\Delta(t)=\frac{f(\sigma(t)) - f(t)}{\mu(t)}, \tag{4}$$

and if t is right-dense,

$$f^\Delta(t)= \lim_{s\to t}\frac{f(t) - f(s)}{t - s}. \tag{5}$$

Let's assume that $f, g: \mathbb{T} \to \mathbb{R}$ is differentiable in $t \in \mathbb{T}^k$. Each rd- continuous function has an inverse derivative F, i.e. $F^\Delta= f(t)$. For $s \in \mathbb{T}$,

$$F(t)= \int_s^t f(\tau)\,\Delta\tau, \qquad \forall t \in \mathbb{T},$$

is an antiderivative of f, i.e.

$$\left(\int_s^t f(\tau)\,\Delta\tau\right)^{\Delta} = f(t), \tag{6}$$

and also

$$\int_t^{\sigma(t)} f(t)\,\Delta t = \mu(t)\,f(t). \tag{7}$$

(Aulbach and Hilger, 1990; Agarwal et al., 2002; Bohner and Peterson, 2001, 2004; Bohner and Svetlin, 2016; Hilger, 1990) provide comprehensive information on the time scale.

The proportional delta derivative of the function $f:\mathbb{T}\to\mathbb{R}$ of order $\alpha\in[0,1]$ at point $t\in\mathbb{T}^k$ will now be defined.

Definition 4 (Segi Rahmat, 2019) Let $f:\mathbb{T}\to\mathbb{R}$ be a function, $t\in\mathbb{T}^k$, and κ_0 and κ_1 be continuous functions that fulfill the conditions (2). $\forall\varepsilon>0$, and for every s in a neighborhood U of point t, if there is a real number $D^\alpha f(t)$, $\alpha\in[0,1]$, such that

$$\left|\kappa_1(\alpha,t)f(t)[\sigma(t)-s]+\kappa_0(\alpha,t)\big[f\big(\sigma(t)\big)-f(s)\big]-(D^\alpha f)(t)[\sigma(t)-s]\right| \le \varepsilon\,[\sigma(t)-s], \tag{8}$$

that number is known as the α −th order proportional delta derivative of f at point t.

Let's define

$\mathfrak{J}(\mathbb{T})=\{f:\mathbb{T}\to\mathbb{R} : D^\alpha f(t)$ exists and is finite for all $t\in\mathbb{T}^k\}$,

as the collection of all proportional delta differentiable functions (Segi Rahmat, 2019).

Theorem 5 (Segi Rahmat, 2019) Assuming that $\mathbb{T}$ is a time scale, $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^k$.

(i) If $f\in\mathfrak{J}(\mathbb{T})$, then f is continuous at t.

(ii) If f is continuous at t, t is right-scattered, and

$$f^{\Delta}(t)=\frac{f\big(\sigma(t)\big)-f(t)}{\sigma(t)-t},$$

exists, then $f\in\mathfrak{J}(\mathbb{T})$. In this case

$$D^\alpha f(t)=\kappa_0(\alpha,t)\,f^{\Delta}(t)+\kappa_1(\alpha,t)\,f(t). \tag{9}$$

(iii) If t is right-dense, and

$$\lim_{t\to s}\frac{f(t)-f(s)}{t-s}=f'(t),$$

exists as a finite number, then $f\in\mathfrak{J}(\mathbb{T})$, and so

$$D^\alpha f(t)=\kappa_0(\alpha,t)f'(t)+\kappa_1(\alpha,t)f(t). \tag{10}$$

Lemma 6 (Segi Rahmat, 2019) If $f,g:\mathbb{T}\to\mathbb{R}$ are proportional delta differentiable at the point $t\in\mathbb{T}^k$ and κ_0 and κ_1 satisfy the conditions (2) and are continuous functions, then the following properties are provided:

(i) $D^\alpha[\rho f+\varsigma g]=\rho D^\alpha[f]+\varsigma D^\alpha[g]$, all $\rho,\varsigma\in\mathbb{R}$;

(ii) $D^\alpha[fg]=f^\sigma D^\alpha[g]+D^\alpha[f]g-f^\sigma g\kappa_1(\alpha,.)$;

(iii) $D^\alpha\left[\frac{1}{g}\right]=-\frac{D^\alpha[g]}{g.g^\sigma}+\left(\frac{1}{g}+\frac{1}{g^\sigma}\right)\kappa_1,\quad gg^\sigma\neq 0$;

(iv) $D^\alpha\left[\frac{f}{g}\right]=\frac{D^\alpha[f]g^\sigma-f.D^\alpha[g]}{g.g^\sigma}+\frac{f^\sigma}{g^\sigma}\kappa_1(\alpha,.),\quad gg^\sigma\neq 0.$

Definition 7 (Segi Rahmat, 2019) Let $\alpha \in [0,1]$ and $\kappa_0, \kappa_1 : [0,1] \times \mathbb{T} \to \mathbb{R}_0^+$ be continuous functions that fulfill (2). $p: \mathbb{T} \to \mathbb{R}$ is regarded as being α-regressive if the requirement

$$1 + \frac{p(\tau) - \kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)}\mu(\tau) \neq 0, \quad \text{all } \tau \in \mathbb{T}^k,$$

is hold. $\Re_\alpha = \Re_\alpha(\mathbb{T})$ is used to represent the collection of all rd-continuous and α-regressive functions on $\mathbb{T}$.

Definition 8 (Segi Rahmat, 2019) Let $\alpha \in (0,1]$ and $p \in \Re_\alpha$. Assume that κ_0, κ_1 are continuous functions and $p/\kappa_0,\ \kappa_1/\kappa_0$ delta integrable functions on $\mathbb{T}$, and that (2) is satisfied.

$$\tilde{e}_p(t,s) = \exp\left[\int_s^t \frac{1}{\mu(\tau)} Log\left(1 + \frac{p(\tau) - \kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)}\mu(\tau)\right)\Delta\tau\right], \tag{11}$$

$$\tilde{e}_0(t,s) = \exp\left[\int_s^t \frac{1}{\mu(\tau)} Log\left(1 - \frac{\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)}\mu(\tau)\right)\Delta\tau\right],\ s,t \in \mathbb{T},$$

defines the proportional exponential function on $\mathbb{T}$ for operator D^α, where Log is the fundamental logarithm function. For $\mu(t) = 0$,

$$\tilde{e}_p(t,s) = \exp\left[\int_s^t \left(\frac{p(\tau) - \kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)}\right)\Delta\tau\right],\ \tilde{e}_0(t,s) = \exp\left[-\int_s^t \frac{\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)}\Delta\tau\right]. \tag{12}$$

Definition 9 (Segi Rahmat, 2019) Let $p : \mathbb{T} \to \mathbb{R}$ and $\alpha \in (0,1]$. Let's use $\Re_\alpha^+$ to define all positive α −regressive components of $\Re_\alpha$, that is,

$$\Re_\alpha^+ = \left\{p \in \Re_\alpha : 1 + \frac{p(\tau) - \kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)}\mu(\tau) > 0, \text{ all } t \in \mathbb{T}\right\}.$$

Lemma 10 (Segi Rahmat, 2019) Assume that $p \in \mathbb{R}$, $\alpha \in (0,1]$ and $t_0 \in \mathbb{T}$. If $p \in \Re_\alpha^+$, then $\tilde{e}_p(t,t_0) > 0,\ t \in \mathbb{T}$.

Theorem 11 (Segi Rahmat, 2019) If $p \in \Re_\alpha^+$ and $\alpha \in (0,1]$, the following properties are true:

(i) $\tilde{e}_p(\sigma(t),s) = \left(1 + \frac{p(t) - \kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}\mu(t)\right)\tilde{e}_p(t,s)$;

(ii) $\tilde{e}_p(t,s) = \frac{1}{\tilde{e}_p(s,t)}$;

(iii) $\tilde{e}_p(t,s)\tilde{e}_p(s,r) = \tilde{e}_p(t,r)$;

(iv) $\tilde{e}_p^\Delta(t,s) = \left(\frac{p(t) - \kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}\right)\tilde{e}_p(t,s)$;

(v) $\left(\frac{1}{\tilde{e}_p(t,s)}\right)^\Delta = -\left(\frac{p(t) - \kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}\right)\frac{1}{\tilde{e}_p(\sigma(t),s)}$.

Lemma 12 (Segi Rahmat, 2019) Let $\alpha \in (0,1]$ and $p \in \Re_\alpha$. For fixed $s \in \mathbb{T}$,

$$D^\alpha\left[\tilde{e}_p(.,s)\right] = p(t)\tilde{e}_p(.,s),$$

and for the proportional exponential function $\tilde{e}_0$,

$$D^\alpha\left[\int_a^t \frac{f(\tau)\tilde{e}_0(t,\sigma(\tau))}{\kappa_0(\alpha,\tau)}\Delta\tau\right] = f(t). \tag{13}$$

Definition 13 (Segi Rahmat, 2019) Assume that $f \in C_{rd}(\mathbb{R})$, $\alpha \in (0,1]$, and $t_0 \in \mathbb{T}$. The indefinite proportional integral (anti derivative) is defined as

$\int D^{\alpha}f(t)\Delta_{\alpha}\tau = f(t) + c\tilde{e}_0(t,t_0), \quad \forall t \in \mathbb{T}, c \in \mathbb{R},$

with respect to (12), Lemma 12.

$$\int_a^t f(\tau)\tilde{e}_0(t,\sigma(\tau))\,\Delta_{\alpha}\tau = \int_a^t \frac{f(\tau)\tilde{e}_0(t,\sigma(\tau))}{\kappa_0(\alpha,\tau)}\,\Delta\tau, \qquad \Delta_{\alpha}\tau = \frac{1}{\kappa_0(\alpha,\tau)}\,\Delta\tau, \tag{14}$$

describes the definite proportional integral of f on $[a,b]_{\mathbb{T}}$.

Lemma 14 (Segi Rahmat, 2019) Let $\alpha \in (0,1]$, $f, g \in C_{rd}(\mathbb{R})$, and κ_0, κ_1 be continuous functions and satisfy (2). Then,

$$D^{\alpha}\left[\int_a^t f(\tau)\tilde{e}_0(t,\sigma(\tau))\,\Delta_{\alpha}\tau\right] = f(t). \tag{15}$$

Lemma 15 (Segi Rahmat, 2019) If $f, g \in \mathfrak{I}(\mathbb{T})$,

(i) $\int_a^t D^{\alpha}\,[g(\tau)]\;\tilde{e}_0(t,\sigma(\tau))\,\Delta_{\alpha}\tau = [g(\tau)\,\tilde{e}_0(t,\sigma(\tau))]_{\tau=a}^{t}$.

(ii) $\int_a^b f(t)\,D^{\alpha}[g(t)]\,\tilde{e}_0\big(b,\sigma(t)\big)\,\Delta_{\alpha}t = \big[f(t)g(t)\,\tilde{e}_0\big(b,\sigma(t)\big)\big]_{t=a}^{b}$

$$\int_a^b g^{\sigma}(t)\{D^{\alpha}\,[f(t)] - \kappa_1(\alpha,t)f(t)\}\tilde{e}_0(b,\sigma(t))\,\Delta_{\alpha}t.$$

Lemma 16 (Segi Rahmat, 2019) Suppose that $\alpha \in (0,1]$, the function $f : \mathbb{T}^2 \to \mathbb{R}$ is rd-continuous, κ_0 and κ_1 fulfill (2) and are continuous. In this instance,

$$D^{\alpha}\left[\int_a^t f(t,\tau)\,\tilde{e}_0(t,\sigma(\tau))\,\Delta_{\alpha}\tau\right] = \int_a^t [D_t^{\alpha}f(t,\tau) - \kappa_1(\alpha,t)f(t,\tau)]\tilde{e}_0(t,\sigma(\tau))\,\Delta_{\alpha}\tau + f(\sigma(t),t), \tag{16}$$

or

$$D^{\alpha}\left[\int_a^t f(t,\tau)\,\Delta_{\alpha}\tau\right] = \int_a^t D_t^{\alpha}\,f(t,\tau)\,\Delta_{\alpha}\tau + f(\sigma(t),t). \tag{17}$$

Take a look at equation

$$(D^{\alpha})^2 y + a(t)D^{\alpha}y + b(t)y = 0,\ t \in \mathbb{T}^{k^2}, \tag{18}$$

where $a, b \in C_{rd}(\mathbb{T})$.

Definition 17 (Anderson and Georgiev, 2020) The function $y \in C_{rd}^2(\mathbb{T})$ satisfying equation (18) is referred to as the solution of the equation.

Theorem 18 (Anderson and Georgiev, 2020) Make y_1 and y_2 the results of solving equation (18). Then, $py_1 + qy_2$ is a solution to equation (18) for $p, q \in \mathbb{R}$.

Definition 19 (Anderson and Georgiev, 2020) Any two functions $y_1, y_2 \in C_{rd}^1(\mathbb{T})$ have a proportional Wronskian defined as

$$W(y_1,y_2) = \det\begin{pmatrix} y_1 & y_2 \\ D^{\alpha}y_1 & D^{\alpha}y_2 \end{pmatrix}.$$

Definition 20 (Anderson and Georgiev, 2020) If for any $t \in \mathbb{T}^k$ the condition

$$W(y_1, y_2)(t) \neq 0,$$

is hold, the solutions y_1 and y_2 of (18) are said that forms the basic solution set for (18).

Remark 21 (Anderson and Georgiev, 2020) With $\wp_c^+$, the collection of functions $f: \mathbb{T} \to \mathbb{R}$ that provide for the condition

$$\kappa_0 + \mu(f - \kappa_1) > 0, \quad \kappa_0 - \mu\kappa_1 \neq 0,$$

will be shown.

Definition 22 (Bohner and Peterson, 2001) Let $t \in \mathbb{T}^k$ and $f: \mathbb{T} \to \mathbb{R}$ be a function. $\forall \varepsilon > 0$, and for every s in a neighborhood U of point t, if there is a real number $f^{\nabla}(t)$, such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)[\rho(t) - s]| \leq \varepsilon\,|\rho(t) - s|, \quad \forall s \in U,$$

$f^{\nabla}(t)$ is referred to as the nabla derivative of f at point t.

Definition 23 (Anderson and Georgiev, 2020) Assume that κ_0 and κ_1 provide for (2). On the time scale $\mathbb{T}$, the derivative $\widehat{D}^{\alpha}$ defined by

$$\widehat{D}^{\alpha} f(t) = \kappa_1(\alpha, t) f(t) + \kappa_0(\alpha, t) f^{\nabla}(t), \quad t \in \mathbb{T}_k, \tag{19}$$

is known as the proportional nabla derivative.

Definition 24 (Anderson and Georgiev, 2020) Suppose that κ_0 and κ_1 provide for (2). When

$$\kappa_0(\alpha, t) - \nu(t)(f(t) - \kappa_1(\alpha, t)) \neq 0, \tag{20}$$

where $\nu(t) = t - \rho(t)$ is the graininess function, the function $f: \mathbb{T} \to \mathbb{R}$ is said to be proportional ν −regressive for any $\alpha \in (0, 1]$ and any $t \in \mathbb{T}$. $\widehat{\mathfrak{R}}_c$ will stand for the collection of every proportional ν −regressive function on $\mathbb{T}$.

Definition 25 (Anderson and Georgiev, 2020) If $\alpha \in (0, 1]$, $s, t \in \mathbb{T}$ ve $p \in \widehat{\mathfrak{R}}_c$, then

$$\hat{e}_p(t, s) = \check{e}_{\frac{p - \kappa_1}{\kappa_0}}(t, s), \tag{21}$$

defines the proportional nabla exponential function with regard to $\widehat{D}^{\alpha}$. In this situation, the relationship

$$\widehat{D}^{\alpha} \hat{e}_p(t, s) = p(t) \hat{e}_p(t, s), \quad t \in \mathbb{T}_k, \ s \in \mathbb{T}, \tag{22}$$

is true and $\check{e}$ represents the exponential function of nabla derivative on the time scale.

RESULTS AND DISCUSSION

Think about the Sturm-Liouville problem for

$$\begin{cases} L^{\alpha} y \equiv D^{\alpha} D^{\alpha} y(t) + q(t) y(t) = \lambda y(t), \quad \alpha \in (0,1], \quad t \in [a, b] \cap \mathbb{T}^k, \\ \eta y(a) + \beta D^{\alpha} y(a) = 0, \\ \delta y(b) + \gamma D^{\alpha} y(b) = 0, \end{cases} \tag{23}$$

where λ is a spectral parameter, q is a continuous function, $\eta, \beta, \delta, \gamma$ are constant values, and

$$\eta^2 + \beta^2 \neq 0, \ \delta^2 + \gamma^2 \neq 0.$$

Theorem 26 For L^{α} and $\alpha \in (0,1]$ given in (23), suppose that

$\kappa_0^{\sigma}(\alpha,t)+\mu(t)\kappa_1^{\sigma}(\alpha,t)\neq 0, \qquad t\in\mathbb{T}^k.$

If $x,y\in\mathbb{D}$, we get

$$x(L^{\alpha}y)-y(L^{\alpha}x)=\left(\frac{\kappa_0^{\sigma}+\mu\kappa_1^{\sigma}}{\kappa_0^{\sigma}}\right)D^{\alpha}[W(x,y)]+\frac{\kappa_1^{\sigma}(\kappa_0-\mu\kappa_1)}{\kappa_0^{\sigma}}W(x,y),\ t\in\mathbb{T}^k. \tag{24}$$

Consequently, we obtain Lagrange identity

$$\hat{\tilde{e}}_0(t,b)D^{\alpha}\left[\frac{W(x,y)(t)}{\tilde{e}_0(t,b)}\right]=x(L^{\alpha}y)-y(L^{\alpha}x) \tag{25}$$

for $t,b\in\mathbb{T}^k$.

Proof Assume that $x,y\in\mathbb{D}$ that indicating that $D^{\alpha}x$, and especially x^{Δ} is continuous. The proportional delta derivative's product rule on $\mathbb{T}^k$ allows us to determine that

$$D^{\alpha}\big(W(x,y)\big)=xD^{\alpha}(D^{\alpha}y)+D^{\alpha}x(D^{\alpha}y)^{\sigma}-\kappa_1x(D^{\alpha}y)^{\sigma}-yD^{\alpha}(D^{\alpha}x)-D^{\alpha}y(D^{\alpha}x)^{\sigma}+\kappa_1y(D^{\alpha}x)^{\sigma}$$

$$=\kappa_0[x^{\Delta}(D^{\alpha}y)^{\sigma}-y^{\Delta}(D^{\alpha}x)^{\sigma}]+y(L^{\alpha}x)-x(L^{\alpha}y)$$

$$=\kappa_0{\kappa_1}^{\sigma}(x^{\Delta}y-y^{\Delta}x)^{\sigma}+y(L^{\alpha}x)-x(L^{\alpha}y)$$

$$=\kappa_0{\kappa_1}^{\sigma}\left[\left(\frac{D^{\alpha}x-\kappa_1x}{\kappa_0}\right)y-\left(\frac{D^{\alpha}y-\kappa_1y}{\kappa_0}\right)x\right]^{\sigma}+y(L^{\alpha}x)-x(L^{\alpha}y)$$

$$=\frac{\kappa_0{\kappa_1}^{\sigma}}{{\kappa_0}^{\sigma}}[(D^{\alpha}x)y-(D^{\alpha}y)x]^{\sigma}+y(L^{\alpha}x)-x(L^{\alpha}y)$$

$$\Rightarrow D^{\alpha}\big(W(x,y)\big)=y(L^{\alpha}x)-x(L^{\alpha}y)-\left(\frac{\kappa_1}{\kappa_0}\right)^{\sigma}\kappa_0W(x,y)^{\sigma}.$$

$$\kappa_0f^{\sigma}=(\kappa_0-\kappa_1\mu)f+\mu D^{\alpha}f,$$

is discovered for any delta differentiable function f. Indeed,

$$\kappa_0f^{\sigma}=\kappa_0(\mu f^{\Delta}+f)$$

$$=\kappa_0\mu\left(\frac{D^{\alpha}f-\kappa_1f}{\kappa_0}\right)+\kappa_0f$$

$$=(\kappa_0-\kappa_1\mu)f+\mu D^{\alpha}f.$$

Therefore, we follow

$$D^{\alpha}\big(W(x,y)\big)=y(L^{\alpha}x)-x(L^{\alpha}y)-\left(\frac{\kappa_1}{\kappa_0}\right)^{\sigma}\left[(\kappa_0-\kappa_1\mu)W(x,y)+\mu D^{\alpha}\big(W(x,y)\big)\right]$$

$$\Rightarrow D^{\alpha}\big(W(x,y)\big)=\frac{\kappa_0^{\sigma}}{\kappa_0^{\sigma}+\mu\kappa_1^{\sigma}}[y(L^{\alpha}x)-x(L^{\alpha}y)]-\frac{\kappa_1^{\sigma}(\kappa_0-\kappa_1\mu)}{\kappa_0^{\sigma}+\mu\kappa_1^{\sigma}}W(x,y). \tag{26}$$

Let

$$\xi(t)=\kappa_1-\frac{\kappa_1^{\sigma}\kappa_0}{\kappa_0^{\sigma}+\mu\kappa_1^{\sigma}}. \tag{27}$$

Knowing that $\tilde{e}_0 = \hat{\tilde{e}}_\xi$

$$\tilde{e}_\xi^\sigma = \left(1 + \frac{(\xi - \kappa_1)}{\kappa_0}\mu\right)\tilde{e}_\xi$$

$$= \left[1 - \frac{\mu\kappa_1^\sigma}{\kappa_0^\sigma + \mu\kappa_1^\sigma}\right]\tilde{e}_\xi$$

$$\Rightarrow \quad \frac{\tilde{e}_\xi}{\tilde{e}_\xi^\sigma} = \frac{\kappa_0^\sigma + \mu\kappa_1^\sigma}{\kappa_0^\sigma}.$$

Additionally, based on the quotient rule

$$\tilde{e}_\xi D^\alpha\left(\frac{W}{\tilde{e}_\xi}\right) = \tilde{e}_\xi\left[\frac{(D^\alpha W)\tilde{e}_\xi - W\left(D^\alpha \tilde{e}_\xi\right)}{\tilde{e}_\xi \tilde{e}_\xi^\sigma} + \kappa_1 \frac{W}{\tilde{e}_\xi}\right]$$

$$\Rightarrow \tilde{e}_\xi D^\alpha\left(\frac{W}{\tilde{e}_\xi}\right) = \frac{\tilde{e}_\xi}{\tilde{e}_\xi^\sigma}(D^\alpha W - W\xi) + \kappa_1 W. \tag{28}$$

(26), replacing it with (28), in our case,

$$\tilde{e}_\xi D^\alpha\left(\frac{W}{\tilde{e}_\xi}\right) = \frac{\tilde{e}_\xi}{\tilde{e}_\xi^\sigma}\left[\frac{\kappa_0^\sigma}{\kappa_0^\sigma + \mu\kappa_1^\sigma}(y(L^\alpha x) - x(L^\alpha y)) - \left(\frac{\kappa_1^\sigma(\kappa_0 - \kappa_1\mu)}{\kappa_0^\sigma + \mu\kappa_1^\sigma}\right)W - \left(\kappa_1 - \frac{\kappa_1^\sigma\kappa_0}{\kappa_0^\sigma + \mu\kappa_1^\sigma}\right)W\right] + \kappa_1 W$$

$$= y(L^\alpha x) - x(L^\alpha y) + \left(\frac{\kappa_0^\sigma + \mu\kappa_1^\sigma}{\kappa_0^\sigma}\right)W\left[-\frac{\kappa_1^\sigma(\kappa_0 - \kappa_1\mu)}{\kappa_0^\sigma + \mu\kappa_1^\sigma} - \left(\kappa_1 - \frac{\kappa_1^\sigma\kappa_0}{\kappa_0^\sigma + \mu\kappa_1^\sigma}\right)\right] + \kappa_1 W,$$

and, from here

$$\tilde{e}_\xi D^\alpha\left(\frac{W}{\tilde{e}_\xi}\right) = y(L^\alpha x) - x(L^\alpha y).$$

Definition 27 Assume that $\alpha \in (0, 1]$ and the condition $\kappa_0^\sigma(\alpha, t) + \mu(t)\kappa_1^\sigma(\alpha, t) \neq 0$ are satisfied where ξ is defined with (27).The formula for the inner product of $f, g \in C_{rd}(\mathbb{T}^k)$ on $[a, b]_\mathbb{T} \subseteq \mathbb{T}^k$

$$\langle f, g\rangle = \int_a^b \frac{f(t)g(t)\tilde{e}_0(b, \sigma(t))}{\tilde{e}_\xi(t, b)\kappa_0(\alpha, t)}\Delta t = \int_a^b \frac{f(t)g(t)}{\tilde{e}_\xi(t, b)}\Delta_{a,b}t, \qquad \Delta_{a,b}t = \frac{\tilde{e}_0(b, \sigma(t))\Delta t}{\kappa_0(\alpha, t)}. \tag{29}$$

Lemma 28 Assuming $\alpha \in (0, 1]$ and

$$\kappa_0^\sigma(\alpha, t) + \mu(t)\kappa_1^\sigma(\alpha, t) \neq 0, \quad t \in \mathbb{T}^k,$$

let L^α be supplied as in (23). For $x, y \in \mathbb{D}$, Green's formula

$$\langle L^\alpha x, y\rangle - \langle x, L^\alpha y\rangle = W(x, y)(b) - \frac{W(x, y)(a)}{\tilde{e}_\xi(a, b)}\tilde{e}_0(b, a), \tag{30}$$

is provided. Additionally, if $x, y \in \mathbb{D}$, and x, y fulfill the self-adjoint boundary conditions

$$W(x, y)(b) = \frac{W(x, y)(a)}{\tilde{e}_0(a, b)\hat{\tilde{e}}_0(a, b)} \tag{31}$$

Operator L^α is only self-adjoint via inner product (29), i.e.

$$\langle x, L^\alpha y\rangle = \langle L^\alpha x, y\rangle\,. \tag{32}$$

Proof From Theorem 26 the Lagrangian identity (25) given by

$$\tilde{e}_\xi D^\alpha\left(\frac{W}{\tilde{e}_\xi}\right)=y(L^\alpha x)-x(L^\alpha y),$$

is true.

$$\int_a^b D^\alpha\left(\frac{W}{\tilde{e}_\xi}\right)\Delta_{a,b}t=\int_a^b \frac{[y(L^\alpha x)\ -\ x(L^\alpha y)]}{\tilde{e}_\xi(t,b)}\Delta_{a,b}t,$$

is determined by multiplying both sides of this identity by ξ, then by $\frac{\tilde{e}_0(b,\sigma(t))}{\tilde{e}_\xi(t,b)\kappa_0(\alpha,t)}$, and then by integrating from a to b. Based

$$\frac{W(x,y)(b)}{\tilde{e}_\xi(b,b)}-\frac{W(x,y)(a)}{\tilde{e}_\xi(a,b)}\tilde{e}_0(b,a)=\langle L^\alpha x,y\rangle\ -\ \langle x,L^\alpha y\rangle,$$

from Lemma 15, we arrive to Green's formula (30). From this, if $x,y\in\mathbb{D}$ coincide the criteria (31), it could easily deduced the self-adjointness, i.e. $\langle x,L^\alpha y\rangle=\langle L^\alpha x,y\rangle$.

Lemma 29 (Abel Formula) Assume that $\alpha\in(0,1]$,

$$\kappa_0^\sigma(\alpha,t)+\mu(t)\kappa_1^\sigma(\alpha,t)\neq 0,\quad t\in\mathbb{T}^k,$$

and L^α is supplied by (23). If $x,y\in\mathbb{D}$ are solution of $L^\alpha x=0$, then the Wronskian is

$$W(x,y)(t)=\frac{W(x,y)(b)}{\tilde{e}_0(b,t)\hat{\tilde{e}}_0(b,t)}=\frac{W(x,y)(a)}{\tilde{e}_0(a,t)\hat{\tilde{e}}_0(a,t)},\quad t\in\mathbb{T}^k, \tag{33}$$

for the constant $a\in\mathbb{T}^k$.

Proof Similar to (25) and the demonstration of Lemma 26, for $x,y\in\mathbb{D}$

$$\tilde{e}_0(t,b)D^\alpha\left[\frac{W(x,y)(t)}{\tilde{e}_0(t,b)}\right]=x(L^\alpha y)-y(L^\alpha x).$$

If x,y are solutions of (23) on $\mathbb{T}^k$, then $L^\alpha x=0=L^\alpha y$ and

$$\tilde{e}_\xi D^\alpha\left(\frac{W}{\tilde{e}_\xi}\right)=0\Rightarrow D^\alpha\left(\frac{W}{\tilde{e}_\xi}\right)=0\quad(\tilde{e}_\xi\neq 0),$$

$$\Rightarrow \tilde{e}_0(t,b)D^\alpha\left[\frac{W(x,y)(t)}{\tilde{e}_0(t,b)}\right]=0,$$

thus,

$$D^\alpha\left[\frac{W(x,y)(t)}{\tilde{e}_0(t,b)}\right]=0,$$

and

$$\frac{W(x,y)(t)}{\tilde{e}_0(t,a)}=c\,\hat{\tilde{e}}_0(t,b)\hat{\tilde{e}}_0(a,t),$$

where $c=W(x,y)(b)$. According to (33)

$W(x,y)(t)=\tilde{e}_0(t,a)\hat{\tilde{e}}_0(t,a)W(x,y)(a).$

Theorem 30 Self-adjointness exists in the proportional Sturm-Liouville problem (23).

Proof According to Green formula (30)

$$\langle L^\alpha x, y\rangle = \frac{W(x,y)(b)}{\tilde{e}_\xi(b,b)} - \frac{W(x,y)(a)}{\tilde{e}_\xi(a,b)}\tilde{e}_0(b,a) + \langle x, L^\alpha y\rangle$$

$$= x(b)D^\alpha y(b) - y(b)D^\alpha x(b) - \frac{x(a)D^\alpha y(a) - y(a)D^\alpha x(a)}{\tilde{e}_0(a,b)\hat{\tilde{e}}_0(a,b)} + \langle x, L^\alpha y\rangle$$

$$= \langle x, L^\alpha y\rangle,$$

thereby concluding the proof.

Theorem 31 Eigenfunctions $x(t)$ and $y(t)$ corresponding to different eigenvalues λ_1, λ_2 of the proportional Sturm-Liouville problem (23) are orthogonal, i.e.

$$\int_a^b \frac{x(t)y(t)}{\tilde{e}_\xi(t,b)}\Delta_{a,b}t = 0. \tag{34}$$

Proof From the Green formula (30)

$$W(x,y)(b) - \frac{W(x,y)(a)}{\tilde{e}_\xi(a,b)}\tilde{e}_0(b,a) = \langle L^\alpha x, y\rangle - \langle x, L^\alpha y\rangle,$$

and the conditions $W(x,y)(a)=0$ and $W(x,y)(b)=0$ are considered

$$\langle L^\alpha x, y\rangle - \langle x, L^\alpha y\rangle = 0,$$

$$\langle \lambda_1 x, y\rangle - \langle x, \lambda_2 y\rangle = 0,$$

$$\int_a^b \frac{\lambda_1 x(t)y(t)}{\tilde{e}_\xi(t,b)}\Delta_{a,b}t - \int_a^b \frac{\lambda_2 x(t)y(t)}{\tilde{e}_\xi(t,b)}\Delta_{a,b}t = 0,$$

$$(\lambda_1 - \lambda_2)\int_a^b \frac{x(t)y(t)}{\tilde{e}_\xi(t,b)}\Delta_{a,b}t = 0,$$

since $\lambda_1 \neq \lambda_2$, (34) is found.

Theorem 32 Any two solutions to the proportional Sturm-Liouville problem (23) are linearly dependent if and only if $W=0$.

Proof If any two solutions of the proportional Sturm-Liouville problem (23) are linearly dependent, then $y(x)=cz(x)$, and from here

$$W(y,z)(x) = \det\begin{pmatrix} y(x) & z(x) \\ D^\alpha y(x) & D^\alpha z(x)\end{pmatrix} = \det\begin{pmatrix} cz(x) & z(x) \\ cD^\alpha z(x) & D^\alpha z(x)\end{pmatrix} = 0,$$

and then

$$D^\alpha y = D^\alpha(cz) = \kappa_1(cz) + \kappa_0(cz)^\Delta = (\kappa_1 + \kappa_0 z^\Delta)c \Rightarrow D^\alpha y = cD^\alpha z.$$

CONCLUSION

As a generic instance of a conformable derivative, the proportional derivative was used to analyze the Sturm-Liouville dynamic problem. Several spectrum properties were proven for this problem utilizing a variety of techniques, such as self-adjointness, the Green Formula, Lagrange identity, the Abel formula, and orthogonality of eigenfunctions.

ACKNOWLEDGEMENTS

This research is a part of second author's master's thesis, which is carried out at Firat University, Türkiye.

Conflict of Interest

The article authors declare that there is no conflict of interest between them.

Author's Contributions

The authors declare that they have contributed equally to the article.

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