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STATIC ANALYSIS OF VISCOELASTIC BEAMS THROUGH FINITE ELEMENT METHOD

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ABSTRACT: *This study focuses on straight beams by taking viscoelastic behavior of material. Time-dependent behavior of the material is stated with the help of Prony series. A constant poisson ratio has been used. Constitution equations for beam are combined in one function with Hamilton Principle, and Laplace transformation is used to free it from time parameter. Finite element formulation is formed with linear shape functions. While integral operation of equations with a shear effect is executed with reduced integration method, integral operations of others are executed with full integration method. Following these analyses, results are obtained by using Reverse Laplace Transformation method developed by Honig and Hirdes.*
Keywords: *Viscoelastic Beam, Finite Element, Timoshenko Beam, Laplace Transform.*

VİSKOELASTİK KİRİŞLERİN SONLU ELEMANLAR YÖNTEMİYLE STATİK ANALİZİ

ÖZET: *Bu çalışmada malzeme için viskoelastik davranış kabulleri yapılarak, doğru eksenli kirişin analizi yapılmıştır. Malzemenin zamana bağlı davranışı Prony serisi yardımıyla ifade edilmiştir. Poisson oranı sabit olarak alınmıştır. Kiriş için elde edilen bünye denklemleri Hamilton Prensibi yardımıyla bir fonksiyonelde toplanmış, Laplace dönüşümü kullanılarak zaman parametresinden bağımsız hale getirilmiştir. Lineer olarak seçilen şekil fonksiyonları yardımıyla sonlu elemanlar formülasyonu oluşturulmuştur. Bu aşamada kayma etkisinin bulunduğu ifadelerin integral işlemi indirgenmiş integral (Reduce Integration) yöntemiyle, diğer ifadelerin integral işlemleri ise tam integral yöntemiyle yapılmıştır. Bu çözümlemelerden sonra Honig ve Hirdes'in geliştirdiği Ters Laplace Dönüşüm metodu kullanılmak suretiyle sonuçlar elde edilmiştir.*
Anahtar kelimeler: *Viskoelastik kiriş, Sonlu eleman, Timoshenko kirişi, Laplace Dönüşümü*

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I. INTRODUCTION

The importance of constructions (nuclear centrals, space structures…) is continuously increasing because of developing technology. Materials are generally accepted as elastic in engineering constructions due to calculation simplicity. But, used materials actually demonstrate a viscoelastic behavior. So, models which give the actual behavior of material with more time consuming computing capacity should be used for more precise determination of behavior of materials used in constructions. This requires viscoelastic material assumptions instead of the use of elastic material assumptions under normal conditions. The behavior of viscoelastic material under axial load can be explained with superposition of elastic and viscose elements. While elastic behavior is modeled by means of a simple spring (Hooke model), viscose behavior is modeled by means of a dashpot (Newton model). While constitution equations for elastic behavior are a function of stress and deformations, time function is included besides stress and deformations for viscoelastic behavior. In other words, deformation is now not dependent on only loading, but also loading speed and time. While stress-deformation is in a linear relationship, and independent from time for Hooke model, stress-deformation is a function of time for Newton model [1].

II. BEHAVIOR EQUATIONS OF VISCOELASTIC BEAMS

The cross section of a straight beam made from a viscoelastic material and positive direction of forces are given in Figure 1.

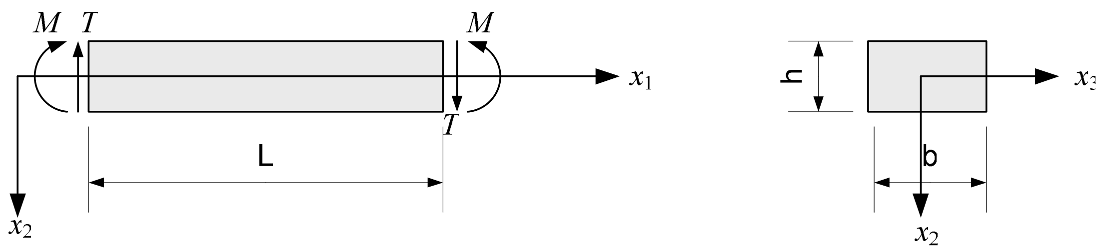

Figure 1. *Coordinate system and cross section of a beam element*

u_i being the displacement for vibrating objects, the acceleration is $\frac{\partial^2 u_i}{\partial t^2}$. Accordingly, motion equation in Euler coordinates;

$$t_{ij,j} + F_i = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{1}$$

Here ρ represents the density of material, and F_i body force (volume force). Let's consider δu_i virtual displacement, but take a vibrating body instead of a statistically balanced body [2]. Virtual work of volume and surface forces can be stated as;

$$W_e = \int_v F_i \delta u_i dv + \int_s \overset{v}{T_i} \delta u_i ds \tag{2}$$

As $\overset{v}{T_i} \delta u_i$ expression in the last integral, Gauss Theorem, t_{ij} tensor is symmetrical and using the Eq. 1 the following statement;

$$\int_s \overset{v}{T_i} \delta u_i ds = \int_v t_{ij,j} \delta u_i dv + \int_v t_{ij} \delta u_{i,j} dv = \int_v \left(\rho \frac{\partial^2 u_i}{\partial t^2} - F_i \right) dv + \int_v t_{ij} \delta e_{ij} dv \tag{3}$$

can be derived. Here the equation variation of motion is obtained as *W* deformation energy function.

$$\int_v W dv = \int_v t_{ij} \delta e_{ij} dv = \int_v \left(F_i - \rho \frac{\partial^2 u_i}{\partial t^2} \right) \delta u_i dv + \int_s \overset{v}{T_i} \delta u_i ds \tag{4}$$

If total deformation energy *U* is defined as;

$$U = \int_v W dv \tag{5}$$

the total works of deformation energy and internal forces are equal in a moving body. The work of internal forces for systems with only moment and shear forces can be stated as;

$$U = \int_v t_{ij} \delta e_{ij} dv = u_i = \int_R \left(\frac{M \delta M}{EI} + \frac{T \delta T}{\kappa AG} \right) dx_1 \tag{6}$$

Here if expression is written as;

$$\frac{M\delta M}{EI} = M\delta\left(\frac{d\varphi}{dx_1}\right); \quad \frac{T\delta T}{\kappa AG} = T\delta\left(\varphi + \frac{d\omega}{dx_1}\right) \tag{7}$$

Eq. 6 looks like;

$$\int_v t_{ij}\delta e_{ij} dv = \int_R \left(M\delta\left(\frac{d\varphi}{dx_1}\right) + T\delta\left(\varphi + \frac{d\omega}{dx_1}\right)\right) dx_1 \tag{8}$$

As a result the right side of equation (4) can be stated as;

$$U = \int_v F_i \delta u_i dv - \int_v \rho \frac{\partial^2 u_i}{\partial t^2} \rho u_i dv + \int_s \overset{v}{T_i} \rho u_i ds \tag{9}$$

If we convert volume integrals in the equation to linear integrals, and the expression Eq. 9 takes its place in the Eq. 3, it looks like;

$$\int_R \left[\rho A \frac{\partial^2 w}{\partial t^2} \delta w + \rho I \frac{\partial^2 \psi}{\partial t^2} \delta\psi + M\delta\left(\frac{d\psi}{dx_1}\right) + T\delta\left(\psi + \frac{dw}{dx_1}\right) - fA\delta w - p\delta w \right] dx_1 = 0 \tag{10}$$

Here, R represents defined area on the beam towards x_1, f volume forces, and p represents surface forces. Time variable t is eliminated by Laplace Transformation of the equation.

$$\begin{aligned} \int_R \Bigg\{ & \rho A\left[s^2 \overline{w} - s w(0, x_1) - \frac{\partial w}{\partial t}(0, x_1) \right] \delta \overline{w} \\ & + \rho I \left[s^2 \overline{\psi} - s\psi(0, x_1) - \frac{\partial \psi}{\partial t}(0, x_1) \right] \delta\overline{\psi} + \overline{M}\delta\left(\frac{\partial \overline{\psi}}{\partial x_1}\right) \\ & + \overline{T}\delta\left(\overline{\psi} + \frac{d\overline{w}}{dx_1}\right) - fA\delta\overline{w} - p\delta\overline{w} \Bigg\} dx_1 = 0 \end{aligned} \tag{11}$$

$\overline{M}$ and $\overline{T}$ values in the equation respectively;

$$\overline{M} = \left(IE_0 + \sum \frac{IE_n s}{s + a_n} \right) \frac{d\psi(s)}{dx_1} \tag{12}$$

$$\overline{T} = \left[\frac{\kappa AE_0}{2(1+\nu)} + \sum \frac{\kappa AE_n s}{2(1+\nu)} \frac{1}{s + a_n} \right] \left[\psi(s) + \frac{d\omega(s)}{dx_1} \right] \tag{13}$$

If we rewrite the Eq. 13 with matrix notation it becomes;

$$\begin{Bmatrix} \overline{M} \\ \overline{T} \end{Bmatrix} = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{Bmatrix} \dfrac{d\psi(s)}{dx_1} \\ \psi(s) + \dfrac{d\omega(s)}{dx_1} \end{Bmatrix} \tag{14}$$

D_1 and terms in Eq. 14 are coefficients of viscoelastic materials which are independent from time parameter. D_1 stands for bending rigidity in elastic materials, and D_2 stands for shear rigidity. Terms in the Eq. 11 should be written in matrix notation to be able to use finite elements. For this reason $[N]$: As the Shape Function defined like;

$$\overline{w} = [N]\{\overline{w}\}; \quad \overline{\psi} = [N]\{\overline{\psi}\}; \quad [B] = {}^{d[N]}\!\big/_{dx}$$

If we rewrite and rearrange the Eq. 11 by using definitions;

$$\begin{aligned} \sum_{e=1}^{Ne} \Bigg\{ & \delta\{\overline{w}\}^T \int_{Re} \Bigg(\rho A[N]^T [N]\left[s^2\{\overline{w}\} - sw(0,x_1) - \frac{\partial w}{\partial t}(0,x_1) \right] + D_2[B]^T[B]\{\overline{w}\} \\ & + D_2[B]^T[N]\{\overline{\psi}\} - \delta\{\overline{w}\}^T[N]^T \overline{f}A - \delta\{\overline{w}\}^T[N]^T\overline{p} \Bigg) dx_1 \\ & + \delta\{\overline{\psi}\}^T \int_{Re} \Bigg(\rho I[N]^T[N]\left[s^2\{\overline{\psi}\} - s\psi(0,x_1) - \frac{\partial \psi}{\partial t}(0,x_1) \right] + D_2[N]^T[B]\{\overline{w}\} \\ & + \left[D_1[B]^T[B] + D_2[N]^T[N] \right]\{\overline{\psi}\} \Bigg) dx_1 \Bigg\} = 0 \end{aligned} \tag{15}$$

equation can be derived. *Re* in this equation represents the define area on beam elements towards x_1 axis, and *Ne* defines the number of elements. If this equation is written in matrix form for discretionarily selected $\delta\{\overline{w}\}$ and $\delta\{\overline{\psi}\}$ values, it looks like;

$$\begin{bmatrix} [M_1]+[K_{11}] & [K_{12}] \\ [K_{21}] & [M_2]+[K_{22}] \end{bmatrix} \begin{Bmatrix} \{\overline{w}\} \\ \{\overline{\psi}\} \end{Bmatrix} = \begin{Bmatrix} \{V_1\}+\{F\}+\{P\} \\ \{V_2\} \end{Bmatrix} \tag{16}$$

and here;

$$[M_1] = \int_{Re} \rho A[N]^T[N]s^2 dx_1 \qquad [M_2] = \int_{Re} \rho I[N]^T[N]s^2 dx_1$$

$$[K_{11}] = \int_{Re} D_2 [B]^T [B] dx_1 \qquad [K_{12}] = \int_{Re} D_2 [B]^T [N] dx_1$$

$$[K_{21}] = \int_{Re} D_2 [N]^T [B] dx_1 \qquad [K_{22}] = \int_{Re} \left\{ D_1 [B]^T [B] + D_2 [N]^T [N] \right\} dx_1$$

$$\{V_1\} = \int_{Re} \rho A [N]^T [N] \left[s w(0, x_1) - \frac{\partial w}{\partial t}(0, x_1) \right] dx_1$$

$$\{V_2\} = \int_{Re} \rho I [N]^T [N] \left[s \psi(0, x_1) - \frac{\partial \psi}{\partial t}(0, x_1) \right] dx_1 \qquad (17a\text{-}j)$$

$$\{F\} = \int_{Re} [N]^T \bar{f} A dx_1 \qquad \{P\} = \int_{Re} [N]^T \bar{p} dx_1$$

Finally, equation set can be written in matrix notation;

$$[K]\{\bar{q}\} = \{V\} + \{F\} + \{P\} \qquad (18)$$

Here, $[K]$ is defined as global stiffness matrix, $\{\bar{q}\}$ is defined as Laplace Transformation of point displacement vector, $\{V\}$ is defined as Laplace Transformation of global point load vector, $\{F\}$ is defined as Laplace transformation of volume forces, and $\{P\}$ is defined as Laplace Transformation of global outer forces [1, 3, 4].

Because the system is a beam in our problem, chosen elements are one dimensional and limit conditions change linearly.

II.1 Shape Functions

Coordinate transformation for shape functions is shown Figure 2.

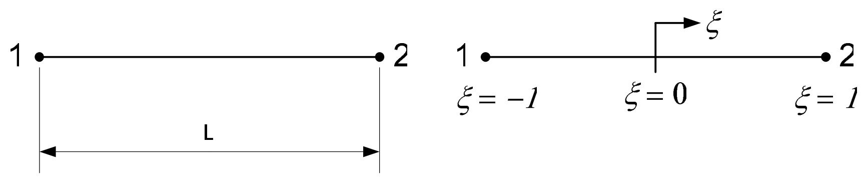

Figure 2. Coordinate transformation

Remembering $dx_1 = \frac{L}{2} d\xi$, and depending on ξ, shape functions can be defined as:

$$N_1 = \frac{1}{2}(1-\xi); \qquad N_2 = \frac{1}{2}(1+\xi) \tag{19}$$

According to these shape functions, displacement function can be written in matrix notation as;

$$\overline{w}(\xi) = \begin{bmatrix} N_1(\xi) & N_2(\xi) \end{bmatrix} \begin{bmatrix} \overline{w_1} \\ \overline{w_2} \end{bmatrix} = [N]\{\overline{w}\} \tag{20}$$

Here, $\overline{w}$ expression is Laplace Transformed state of displacement function [5, 6]. Similarly, spin functions are:

$$\overline{\psi}(\xi) = \begin{bmatrix} N_1(\xi) & N_2(\xi) \end{bmatrix} \begin{bmatrix} \overline{\psi_1} \\ \overline{\psi_2} \end{bmatrix} = [N]\{\overline{\psi}\} \tag{21}$$

Here, $\overline{\psi}$ expression is Laplace Transformed state of spin function. If Eq. 19 expression is written in its place in the Eq. 17a-j before integral operation, and necessary arrangements it looks like;

$$[M_1] = \rho A s^2 \frac{L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} [M_2] = \rho I s^2 \frac{L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[B] = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} [K_{11}] = \frac{D_2}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K_{12}] = \frac{D_2}{L} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} [K_{21}] = \frac{D_2}{L} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$[K_{22}] = D_1 \frac{1}{L} + D_2 \frac{L}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{22}$$

And, if we write these given matrix terms to their places in $[K]$ matrix given inEq. 18;

$$[K] = \begin{bmatrix} \rho A s^2 \frac{L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{D_2}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \frac{D_2}{L} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \\ \frac{D_2}{L} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} & \rho I s^2 \frac{L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + D_1 \frac{1}{L} + D_2 \frac{L}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{bmatrix} \tag{23}$$

can be derived [7].

Solved problem examples in the literature are used to compare this developed solution. Presented solutions are compared to solve problems with the same material specifications, loading and geometric conditions. Material properties are formulated with the help of Prony Series. Integral expressions in the calculation of stiffness matrix elements are integrated by the means of two different methods for Timoshenko beam. Integral operations of terms which express bending rigidity are executed with full integration method, and reduced integration method is used for terms of shear rigidity. The purpose is the elimination of shear rigidity dominancy in the solution, and avoid from divergence called locking during the solution. If the behavior of the beam is investigated according to different L/h ratio, the aforementioned locking can occur. Reduced integration is used with Gauss-Legendre Rule method to overcome that problem.

Matrix manipulation operations are executed with Mathematica 4.0 program, mentioned above, and numerical method developed by Honig and Hirdes is used for Reverse Laplace Transformation [8-11].

III. NUMERICAL EXAMPLE

A simple support beam which is 10 m long, 2 m wide and has a 0.5 m thickness has been loaded with a uniform load of q = 10 N/m as shown in Figure 3. Material specifications are exactly the same as in the literature. All three material models represented by a spring-dashpot model are illustrated in Figure 4. Linear shape function has been chosen, and the beam has been divided into eight pieces for coherence with executed studies. Obtained solutions are compared to final solutions on graphs. Because Poisson ratio of the material varies within a small space, the function of time is assumed as constant [10, 12].

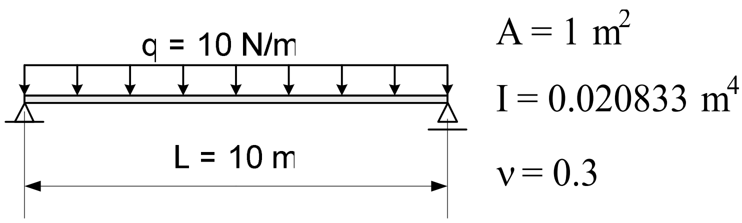

Figure 3. *Loading condition of simple support beam*

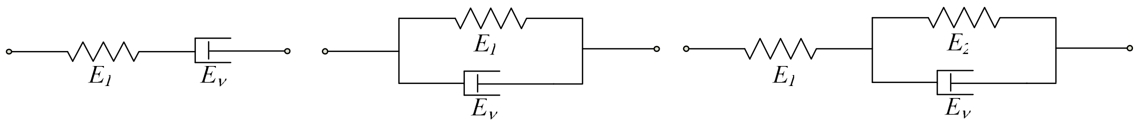

Figure 4. *Viscoelastic models:(a) Maxwell model (b) Kelvin model (c) Three-parameter model*

Data for Maxwell model:

$E_1 = 9.8\times10^7 N/m^2$, $E_v = 2.744\times10^9 Ns/m^2$, $\nu = 0.3$ and relaxation module obtained by Prony series; $E(t) = 9.8\times10^7 e^{-t/t_0} N/m^2$; $t_0 = 28s$

Data for Kelvin model:

$E_1 = 9.8\times10^7 N/m^2$, $E_v = 2.744\times10^9 Ns/m^2$, $\nu = 0.3$

Data for three parameter model:

$E_1 = 9.8\times10^7 N/m^2, E_2 = 2.45\times10^7 N/m^2$, $E_v = 2.744\times10^9 Ns/m^2$, $\nu = 0.3$ and relaxation module obtained by Prony series;

$E(t) = 1.96\times10^7 + 7.84\times10^7 e^{-t/t_0} N/m^2$; $t_0 = 2.24s$

Analytic solution of the system according to this information is as follows.
Displacement value at the middle point for Timoshenko beam:

$$w(t) = \frac{5qL^4}{384I}\left[1 + 1.6\frac{1+\nu}{\kappa}\left(\frac{h}{L}\right)^2\right]J(t) \tag{24}$$

Displacement value at the middle point for traditional (Bernoulli) beam:

$$w(t) = \frac{5qL^4}{384I}J(t) \tag{25}$$

$J(t)$ expression in equations (24) and (25) is defined as the creep function of materials, and $J(t)$ values for given materials are presented Table 1[13].

Table 1. *Viscoelastic material models used in our study and their mechanical properties.*

	Maxwell Model	Kelvin Model	Three Parameter Model
Creep relation	$J(t)=\frac{1}{E_1}+\frac{1}{E_v}t$	$J(t)=\frac{1}{E_1}\left(1-\exp\left[-\frac{E_1}{E_v}t\right]\right)$	$J(t)=\frac{1}{E_2}\exp\left[-\frac{E_1}{E_v}t\right]+ \frac{1}{E_1}\left(1-\exp\left[-\frac{E_1}{E_v}t\right]\right)$
Relaxation relation	$Y(t)=E_1\exp\left[-\frac{E_1}{E_v}t\right]$	$Y(t)=E_1\left(1-\exp\left[-\frac{E_1}{E_v}t\right]\right)$	$Y(t)=E_2\exp\left[-\frac{E_1}{E_v}t\right]+ E_1\exp\left(1-\exp\left[-\frac{E_1}{E_v}t\right]\right)$

Obtained solution for Maxwell model as shown in Figure 5 is very close to exact solution. A sudden displacement occurred on Maxwell model with load application, and increased linearly depending on time. Displacement on Kelvin model with load application started from zero, and took its final position in a very short time period as shown in Figure 6. A sudden displacement occurred on three parameter model with load application, and then displacement continued with down scaling as shown Figure 7. This model maintains good properties of both Maxwell and Kelvin models. The comparisons of the Maxwell model, Kelvin model and tree parameter model are listed in Figure 8 shows that the time dependent displacements at the center of beam for all material models. Figure 9 illustrate time-dependent displacement change on Maxwell model. As Figure 9, displacement increases on Maxwell model as the time increases. It does no converge to any value. As Figure 10 on, the displacement demonstrated a convergence on Kelvin model, and the displacement did not increase although time had increased. Three parameter model also demonstrated a graph which is similar to Kelvin model Figure 11 Displacement increase scaled down, and converged to a value. As a result, three parameter model, within these three models, is the closest model to the actual behavior of materials. Other models will fail to express actual behavior of materials, or even will be mistaken. These models are studied to understand the theory easily, but different models which can express actual behavior can be applied, too.

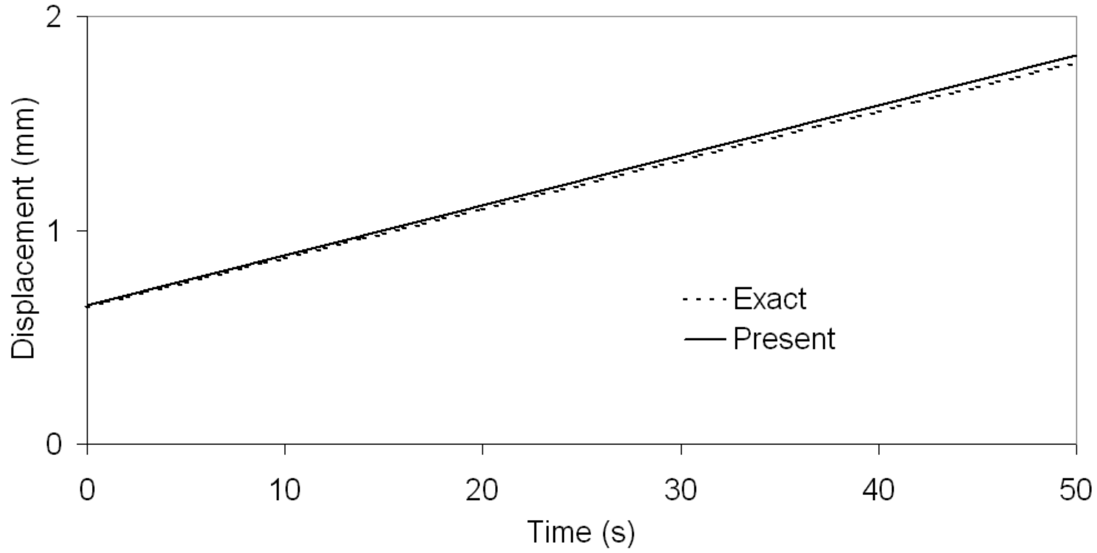

Figure 5. *The time-dependent displacement at the mid-span of the beam for Maxwell model*

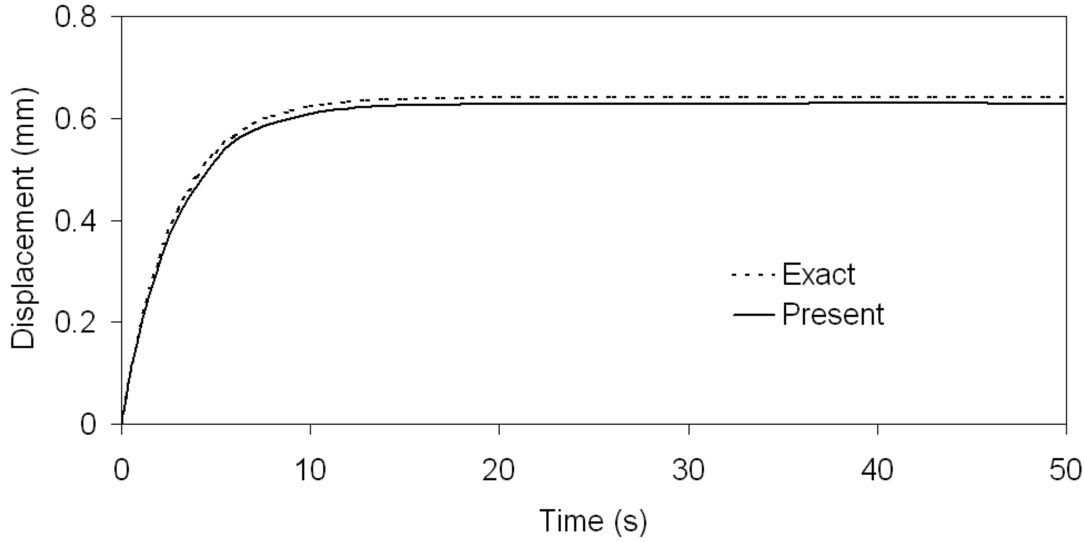

Figure 6. *The time-dependent displacement at the mid-span of the beam for Kelvin model*

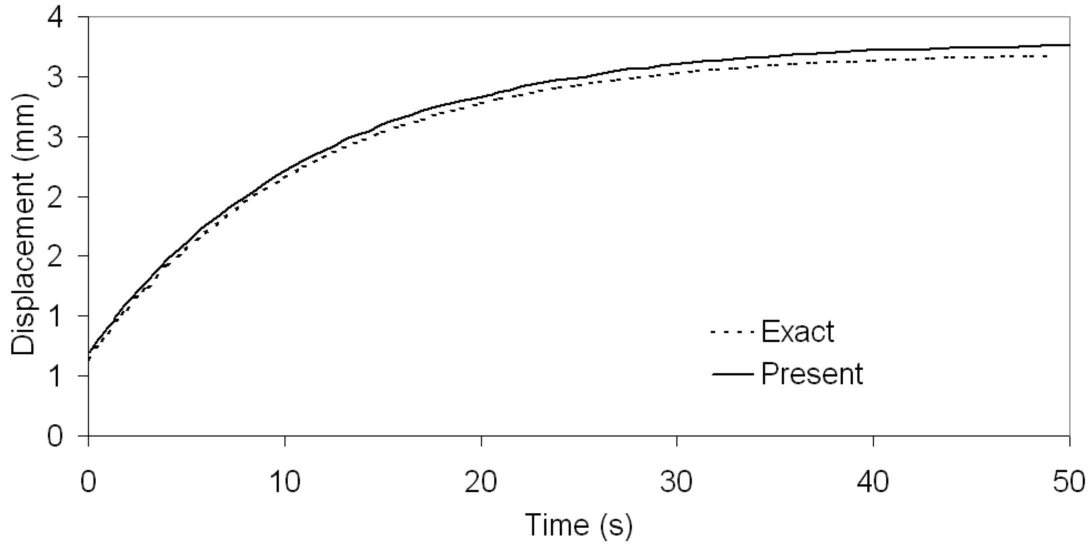

Figure 7. *The time-dependent displacement at the mid-span of the beam for three-parameter model*

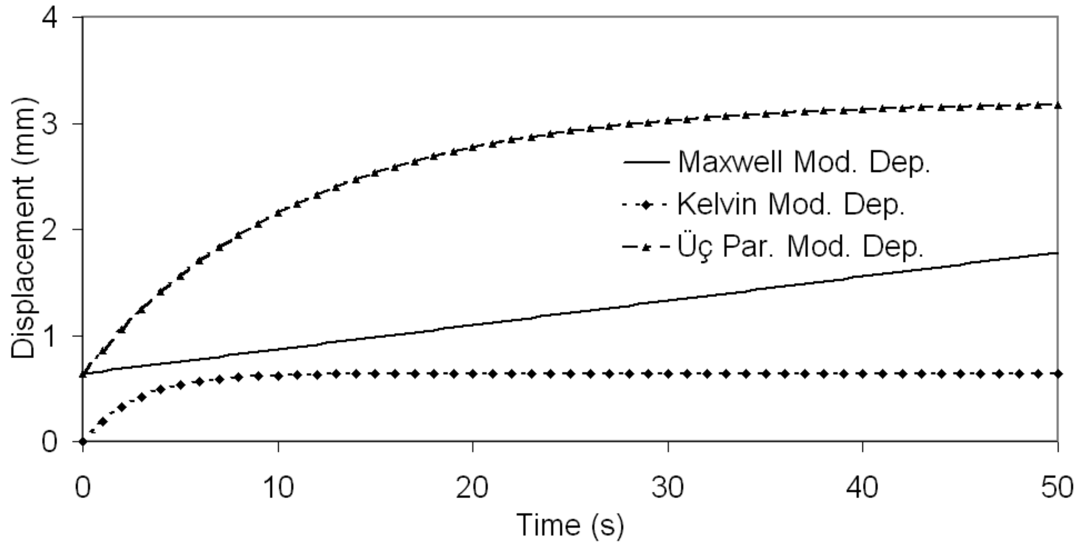

Figure 8. *The time-dependent displacement at the center of the beam for all three material models*

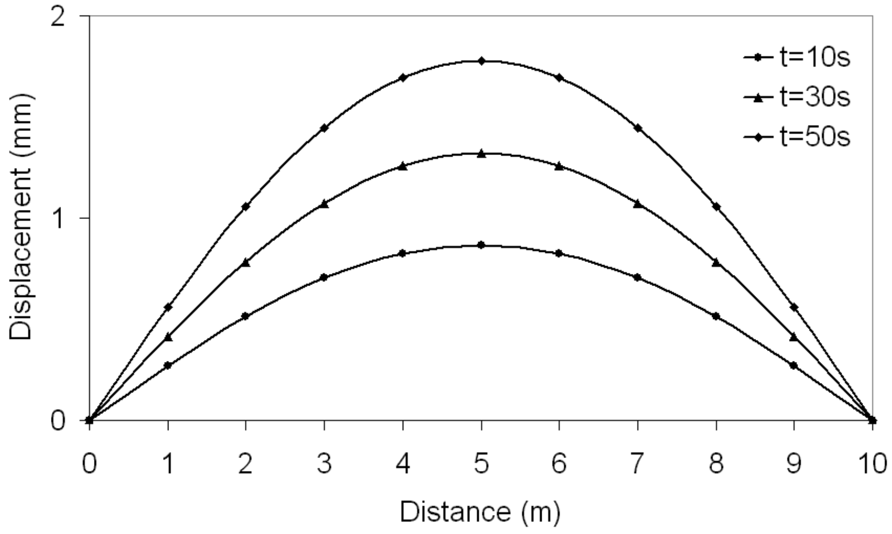

Figure 9. *Displacement variation on Maxwell model along beam axis*

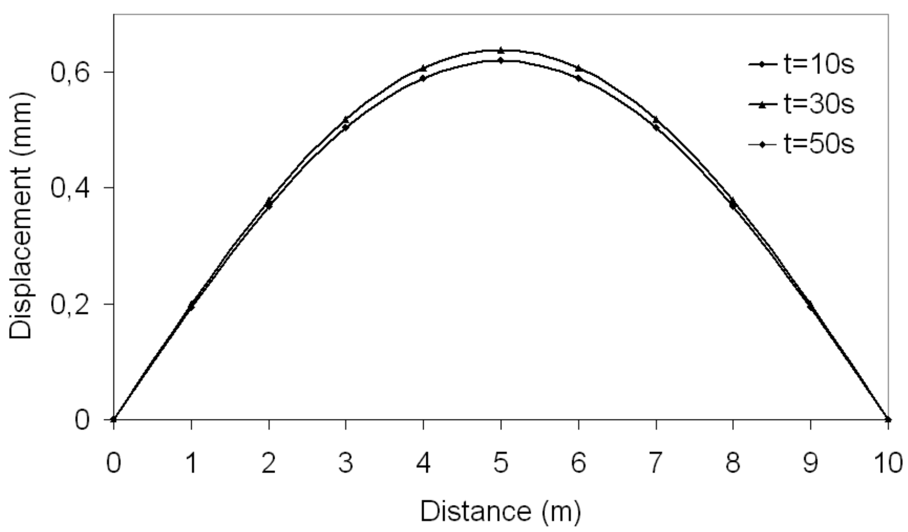

Figure 10. *Displacement variation on Kelvin model along beam axis*

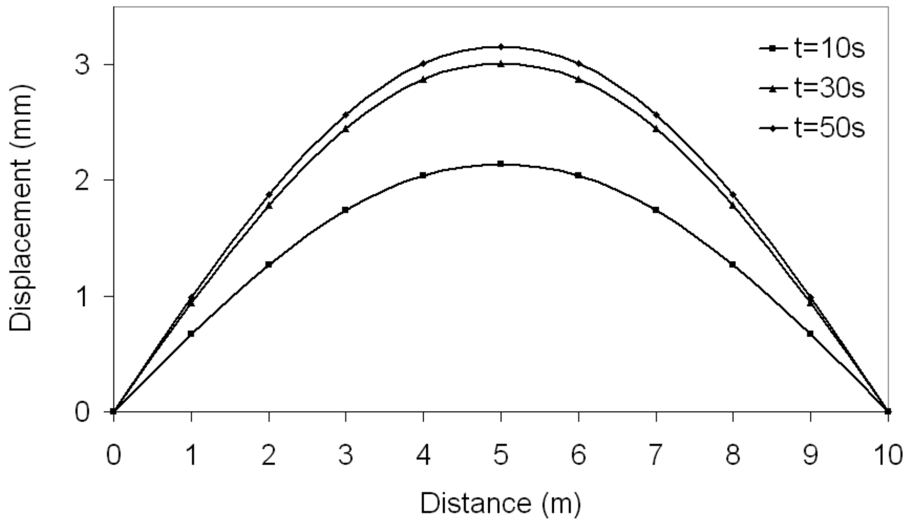

Figure 11. *Displacement variation on three-parameter model along beam axis*

IV. CONCLUSION

The actual behavior of engineering materials is determined more realistically with the use of viscoelastic materials. Some solutions have been reached in this study by using some basic viscoelastic materials through a developed theory. The relaxation function of the material has been expressed with the help of a specially chosen Prony series. This simplified the Reverse Laplace Transformation of emerging equations during the course. Finite element operations and consequent Reverse Transformation operations are executed with Mathematica 4.0 program. Obtained results are compared to the literature, and targeted approximation has been achieved.

When the deformation of Maxwell and Kelvin models from the time of force application is considered, these models are apparently far from representing the actual behavior of the material. The most appropriate one within given models is the three parameter model. A deformation which is decreasing in time following a sudden deformation at the beginning has been revealed. More complex models than ones mentioned above can be chosen practically. Because of the simplicity of operations in theory and comprehensibility, the simplest models of viscoelasticity have been used. Viscoelastic models can be improved with experiments on materials.

Shape functions have been chosen linearly in finite elements method. While high level shape functions do not change the sensitivity a lot, they reduce the computing capability. Instead, choosing shape function linearly, and the execution of operations on divided elements, as happened in that study, is more convenient.

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