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A NONLINEAR PROGRAMMING APPROACH FOR THE SWING-UP CONTROL PROBLEM

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ABSTRACT: *A nonlinear programming approach for the inverted pendulum swing-up control is presented. Even though it is an energy-based method, it uses fundamentally different mathematical tools to achieve the swing-up goal. The control problem translated into nonlinear programming model with appropriate objective function and constraints. While the objective function provides energy increase in the system, physical restrictions of the system are handled in the constraints of the nonlinear programming model. It is also shown that this model is intrinsically suitable for embedding any nonlinear system constraints. Simulation results for illustrative cases are included to validate the design method.*
Keywords: *Swing-up control problem; Inverted pendulum; Underactuated systems; Nonlinear programming; Energy based methods.*

TERS SARKAÇ YUKARI KALDIRMA KONTROL PROBLEMİ İÇİN DOĞRUSAL OLMAYAN PROGRAMLAMA YAKLAŞIMI

ÖZET: *Ters sarkaç yukarı kaldırma kontrol problemi için doğrusal olmayan programlama yaklaşımı sunulmuştur. Önerilen yöntem her ne kadar enerji tabanlı olsa da, havaya kaldırma probleminde temelde farklı matematiksel araçlar kullanmaktadır. Kontrol problemi uygun amaç ve kısıt fonksiyonları kullanılarak doğrusal olmayan programlama modeline dönüştürülmüştür. Doğrusal olmayan programlama modelinde amaç fonksiyonu enerjinin artırımını hedeflerken, sisteme ait fiziksel sınırlamalar kısıtlar ile modellenmiştir. Önerilen modelin herhangi bir doğrusal olmayan kısıt tarifinde de kullanılabileceği gösterilmiştir. Tasarlanan modelin doğruluğunu göstermek üzere örnek durumlar için benzetim sonuçları eklenmiştir.*
Anahtar Kelimeler: *Havaya kaldırma kontrol problemi; Ters sarkaç; yetersiz uyarımlı sistemler; Doğrusal olmayan programlama; Enerji tabanlı yöntemler.*

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I. INTRODUCTION

In this paper, we introduce a nonlinear programming method for the swing-up control of an inverted pendulum system. It is a typical example for nonlinear underactuated systems in which pendulum angle and cart position are controlled by a single force input applied to the cart. The inverted pendulum's rich and nonlinear dynamics is useful in testing new control strategies. The control objective is to swing-up pendulum from the stable (hang-down) equilibrium point to the unstable (upright) equilibrium point and to stabilize it in its upright position. This control problem, in general, comes with various constraints such as with a limited track length and a limit on the size of the controlling force. These constraints are handled using nonlinear programming approach keeping the control objective. The method we propose is conceptually simple and can be generalizable to more complicated design requirements.

The swing-up problem has been studied extensively in the literature. A popular approach in designing swing-up controllers is based on controlling its energy. The logic behind this approach is injecting energy to the pendulum by applying appropriate control force to the cart. In [1], a bang-bang control is used to raise the energy of the pendulum towards a value equal to its steady state value at the upright position. In [2], a variable structure system version of energy-speed-gradient method is treated in a rigorous manner to show that attractivity of the upright equilibrium can be achieved by applying a control of arbitrary small magnitude. In [3], the sign condition in the derivative of the energy is exploited. In this paper a servo system having a low pass property is used for the swing-up. This servo system uses a sinusoidal reference input generated from the pendulum trajectory. In another significant energy-based work [4], the swing-up and stabilization of an inverted pendulum system with a restricted cart track length is achieved by using an energy-well built within the cart track. It is constructed in such a way that the cart experiences a repulsive force as it approaches the boundaries in the neighborhood of the limitations. They control the velocity similarly by using a velocity well. In the energy-based works, the stabilization phase is carried out, generally, by using controllers designed for the linearized model of the inverted pendulum. In [5], energy-based swinging strategies are compared with a fuzzy swing-up algorithm. In [6], a Lyapunov function is obtained by using the total energy of the system, and the convergence analysis carried out using the LaSalle's invariance principle. In fuzzy logic approaches, the states of the inverted pendulum system are used as inputs. For example, in [7], the fuzzy logic method is used in both swing-up and stabilization phases. Each state of the inverted pendulum is assigned with a single input rule module (SIRM) and a dynamic importance degree. Besides, a reader may also refer this paper for a very good review of other fuzzy logic works in the literature.

Our paper may be classified in energy-based methods. Even though we exploit the energy of the inverted pendulum system, we use fundamentally different mathematical tools in order to generate the control signal. Differing from the literature, we transform a problem that is mainly in the differential equation domain, into the domain of nonlinear programming: an algebraic domain. This also allows us to embed a variety of design specifications like system constraints in the problem in a conceptually simple way.

The outline of this paper is as follows. In the next section we present an equation of motion for the inverted pendulum system. Following this the energy of the pendulum is expressed in terms of its states. In the third section, problem specifications and its solution method using a nonlinear programming approach is presented. Prior to the concluding section we illustrate validity of our method by various simulation graphics.

II. PRELIMINARIES

Inverted pendulum attached to a moving cart (Figure 1) is widely used as benchmark for testing control algorithms. The pendulum and cart movements are restricted to a plane. The upside down equilibrium state of the pendulum is unstable, that is, the pendulum may fall over at any time in any direction within the plane of motion. Hence, an appropriate force input u has to be generated to drive and keep the pendulum up-straight.

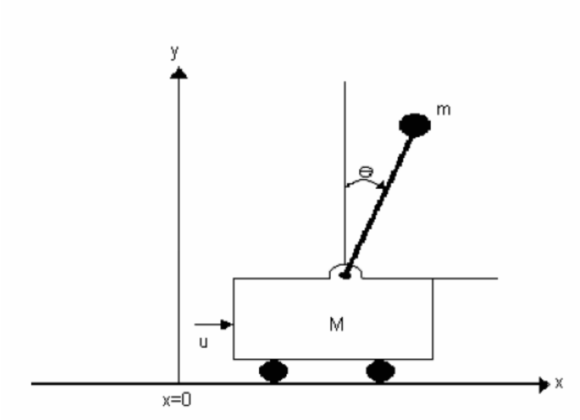

Figure 1. *The inverted pendulum system*

The nonlinear differential equation representing dynamics of an inverted pendulum shown in Figure 1 can be obtained using principles of the Newtonian mechanics [8]:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{-bx_2 + ml\sin(x_3)x_4^2 - mg\sin(x_3)\cos(x_3) + u}{M + m - m\cos(x_3)^2} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{(bx_2 - u - ml\sin(x_3)x_4^2)\cos(x_3) + (M+m)g\sin(x_3)}{l(M + m - m\cos(x_3)^2)}
\end{aligned} \tag{1}$$

where u denotes the control input and the components $x_1,\ldots,x_4$ of the system state vector X are defined by $x_1 := x$, $x_2 := \dot{x}$, $x_3 := \theta$, $x_4 := \dot{\theta}$. Table 1 shows definitions of the symbols used in (1) and typical values that we use in the simulations.

Table 1. *Definitions and typical values for the inverted pendulum*

Parameters	Symbol	Value	Unit
Mass of the Cart	M	3	kg
Mass of the inverted pendulum	m	0.5	kg
Length of the inverted pendulum	l	0.5	m
Friction Constant	b	2	kg/s
Gravitational force	g	9.81	m/s^2

The sum of the rotational kinetic energy of the pendulum and its potential energy, denoted by V, is [4]

$$V = \frac{1}{2} J x_4^2 + mgl(\cos x_3 + 1) \tag{2}$$

where J is the moment of inertia of the pendulum about the hinge. Expression (2) reveals that for zero initial velocity, the energy of the pendulum at the hanging down position is zero, and it is $V_{\max} (\approx 4.9 \text{ Joules})$ at the vertically upright position. If the energy of the pendulum is less than its energy at vertically upright position, the swinging up requires increasing the energy of the pendulum. Increasing the energy is equivalent to making the sign of dV/dt positive. In other words,

$$\dot{V} = -ml\dot{x}_2 x_4 \cos x_3 \geq 0 \tag{3}$$

must be satisfied. By using the pendulum system dynamics, expression (3) can be written as

$$\dot{V} = -mlx_4 \cos(x_3) \frac{-bx_2 + ml\sin(x_3)x_4^2 - mg\sin(x_3)\cos(x_3) + u}{M + m - m\cos(x_3)^2} \tag{4}$$

In the next section we design a nonlinear programming based controller that maximizes $\dot{V}$ during the swing-up phase, and satisfy the constraints on the track length and maximum input size.

III. THE NONLINEAR PROGRAMMING MODEL OF THE SWING-UP CONTROL PROBLEM

The control objective in inverted pendulum swing-up problem is to drive the pendulum to vertically upright position from its pendant initial conditions, and keep it in that position thereafter. This problem is solved in two phases: swing-up phase, stabilization phase.

III.1 The swing-up phase

In the swing-up phase pendulum is driven into a certain neighborhood of the vertically upright position. The swing-up phase of the control problem is modeled as a nonlinear programming problem. The nonlinear programming problem determines the control input u that maximizes the increase in pendulum energy while obeying certain constraints, such as limited track length and limited input size. In the swing-up phase we update the control input u at the beginning of every time interval $[k\Delta t, (k+1)\Delta t]$, $k = 0,1,\ldots,$ where the updating interval length Δt has a fixed value whose upper bound is determined by the smallest time constant of the inverted pendulum system. The updating is carried out by measuring the system state X and computing the control input u by solving a nonlinear programming problem at the beginning of every updating interval. The updated value of u is applied to the cart throughout the updating interval. For simplicity, we use linear feedback control inputs of the form $u = KX$ to form the control input, where $K = [k_1 \quad k_2 \quad k_3 \quad k_4] \in R^4$ denotes the feedback coefficient matrix. For the nonlinear programming problem, we next present the objective function and mathematical model of the constraints below:

a. The objective function: Our objective is to select a feedback coefficient matrix to form the control input which maximizes the energy of the pendulum. This can be done efficiently by maximizing the derivative of the pendulum energy (4). Therefore, the objective function below is chosen as derivative of (4) evaluated at the values given in Table 1. It is seen in the sequel that this maximization together with the design constraints generate a control input function that achieves the swing-up successfully. The objective function can be written as

$$\max_{K} \ -0.25x_4 \cos(x_3) \frac{-2x_2 + 0.25\sin(x_3)x_4^2 - 4.9\sin(x_3)\cos(x_3) + u}{3.5 - 0.5\cos(x_3)^2} \tag{5}$$

This maximization is subject to the constraints presented below.

b. Constraints*:* Inverted pendulum system has some restriction on the track length and input size. Beside, the energy level can also be restricted for control purpose. The details of the constraints are given below.

1. Limited Track Length Constraints

Let the track length be $2x_{\max}$, and let the position of the cart within this track be represented by the x values in $[-x_{\max}, x_{\max}]$. In order for the cart to stay within the track, corresponding to every cart position x we assign admissible cart velocities (Figure 2).

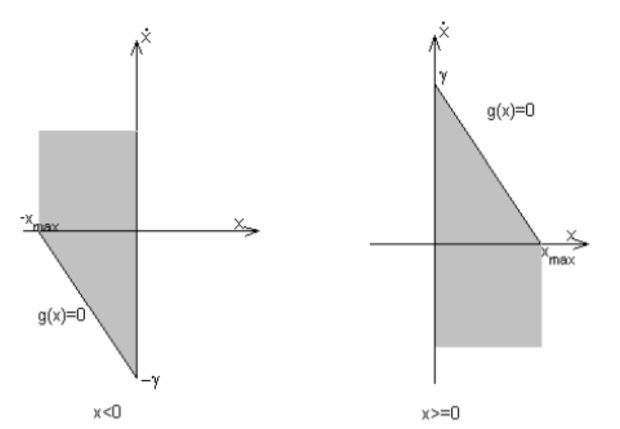

Figure 2. *Admissible velocities as a function of cart position*

In Figure 2, the admissible velocities are represented by the shaded regions. Notice that, for $x \geq 0$, the admissible velocities must be below the line $\mathrm{g(x)} = 0$. Likewise, for $\mathrm{x} < 0$, the admissible velocities must be above the line $\mathrm{g(x)} = 0$. The region of admissible cart velocities guarantees that the cart does not go beyond the ends of the track. For instance, when $\mathrm{x} = \mathrm{x}_{\max}$, positive cart velocities are not allowed. This means the cart cannot go in the positive x direction any more. The γ value is the maximum admissible velocity in the positive x direction at the centre of the track, and can be determined experimentally by using the maximum speed of the pendulum in its free fall from the vertically upright position. We show in the section of the simulations that any γ selected from a large range of γ values works successfully. The admissible region of velocities is represented by

$$\mathrm{sign}(x)[\dot{x} - g(x)] \leq 0 \tag{6}$$

where

$$g(x) = \frac{-\gamma}{x_{\max}} x + \mathrm{sign}(x)\gamma \tag{7}$$

and

$$\operatorname{sign}(x)=\begin{cases}1 & \text{for } x\geq 0\\ -1 & \text{for x} < 0\end{cases}$$

We next associate $\dot{x}$ with the cart acceleration $\ddot{x}$, which is expressed in terms of the control input u in (1). Corresponding to every admissible velocity interval, we assign an admissible $\ddot{x}$ interval. Knowing the cart position, say $x = x_a$, corresponding admissible velocities satisfy the inequality (6), i.e., $\operatorname{sign}(x_a)[\dot{x} - g(x_a)] \leq 0$. Regarding this, for every x_a, we form the corresponding admissible $\ddot{x}$ values as in Figure 3. For a given x_a, we let the horizontal axis represent the admissible velocities, which are less than $g(x_a)$ when $x_a \geq 0$; and greater than $g(x_a)$ when $x_a < 0$. We then associate admissible velocities with the admissible accelerations. The functions representing the boundaries of the shaded region can be constructed by using a similar reasoning in the construction of g. One can notice the important feature of this region that the admissible $\ddot{x}$ values do not let $\dot{x}$ values occur outside $\operatorname{sign}(x_a)[\dot{x} - g(x_a)] \leq 0$ interval. Considering Figures 2 and 3 together, what we have done is associating every cart position x with corresponding admissible cart acceleration β. In the nonlinear programming formulation (that will appear in this section), this works as a constraint.

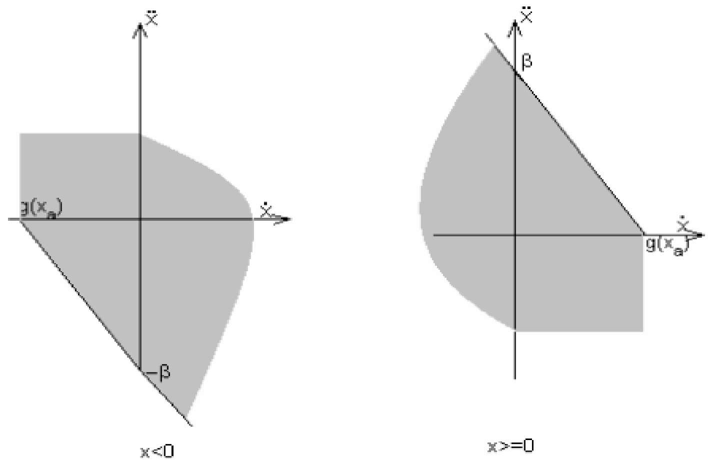

Figure 3. *Admissible acceleration values as a function of an admissible velocity interval*

We can write the track constraints to be satisfied by the input as

$$\operatorname{sign}(x_a)\left[\ddot{x} + \frac{\beta}{|g(x_a)|}\dot{x} - \operatorname{sign}(x_a)\beta\right] \le 0 \tag{8}$$

2. Energy Constraints

In the swing-up phase we select the control input to increase the energy of the pendulum to an energy level in the neighborhood of the vertically upright position. The stabilization phase starts, when the pendulum reaches to this neighborhood. In order to reach this neighborhood and transfer the control to the stabilization phase, we associate pendulum energy values with corresponding admissible pendulum energy derivatives. We represent this by the graphics depicted by Figure 4. In other words, the shaded region in the figure denotes the admissible $\dot{V}$ values versus V. For instance, when the pendulum is close to the vertically upright position, its energy is close to $V_{\max}$, the energy of the pendulum at the vertically upright position, and admissible rates of increase in energy is restricted. Particularly, at $V = V_{\max}$ no nonzero $\dot{V}$ is allowed. For simplicity in algebraic manipulation, we select the energy of the pendulum as a parabolic function of $\dot{V}$. Of course, there are other valid selections.

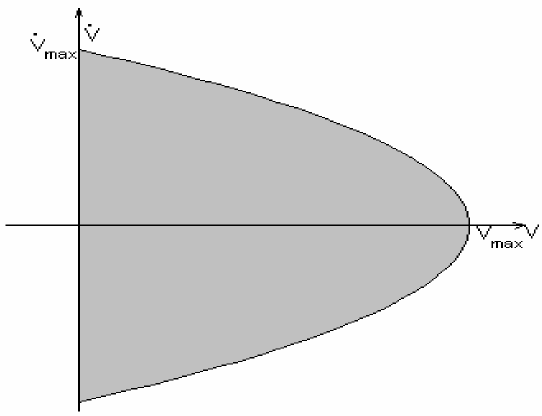

Figure 4. *Pendulum energy derivative as a function of pendulum*

We can write the energy constraint algebraically as $V - (-\dot{V}^2 + \dot{V}_{\max}^2)(\frac{V_{\max}}{\dot{V}_{\max}^2}) \le 0$ where $\dot{V}$ can be expressed in terms of the state variables by (4).

3. Input Size Constraints

Controlling a system by using a physical control input requires limited input sizes. Let the maximum input allowed be α . Then the input size constraint can be written as

$$|KX| \le \alpha$$

For use in the swing-up algorithm, we label the nonlinear programming problem by (P) and write it in terms of the state variables as:

Problem (P)

$$\max_{K} \; -0.25x_4\cos(x_3)\frac{-2x_2+0.25\sin(x_3)x_4^2-4.9\sin(x_3)\cos(x_3)+u}{3.5-0.5\cos(x_3)^2}-\lambda s$$

$$\text{subject to} \begin{cases} \operatorname{sign}(x_1)\left[\ddot{x}+\dfrac{\beta}{|g(x_1)|}\dot{x}-\operatorname{sign}(x_1)\beta\right]\le 0 \\ V-(-\dot{V}^2+\dot{V}_{\max}^2)(\dfrac{V_{\max}}{\dot{V}_{\max}^2})-s\le 0 \\ |KX|\le\alpha \\ s\ge 0 \end{cases} \tag{9}$$

where λ is a positive penalty parameter that is well-known in nonlinear programming theory for its weighting effect of the slack variable s. By the nature of such formulation, any large positive number, for instance, 10^7, works for an acceptable outcome of (9). The terms $\ddot{x}(=\dot{x}_2)$, V and $\dot{V}$ are given in terms of the state variables by (1), (2), and (3) respectively. The slack variable s is intended for avoiding infeasibility arising from this soft constraint. It has negative contribution to the objective function, therefore, its value is close to zero in vast majority of cases.

Of course, there is variety of ways in selecting the constraints in the swing up problem. In this paper we show only one which works and is simple in presentation. A reader may refer to [9] for a self contained descriptions and solutions of the nonlinear programming problems in the form of Problem (P).

3.2 Stabilization phase

The stabilization phase starts when the trajectory reaches a prespecified neighborhood of the vertically upright position (typically characterized by $|\theta| \leq 10$). In the stabilization phase the, the control strategy to keep the pendulum at the vertically upright position. In general, the stabilization phase is carried out by linear controllers by considering the model of inverted pendulum linearized at the vertically upright position. Even though our work reveals that the stabilization phase can be carried out by a nonlinear based controller by choosing appropriate objective function and constraints, for simplicity in presentation, we use a linear quadratic controller for the stabilization. Considering the typical pendulum system parameters given in Table 1, we use the linear quadratic controller coefficient matrix $K = [15.82 \quad 22.46 \quad 150.4 \quad 36.09]$ obtained by using lqr.m command of MATLAB.

3.3 The overall algorithm

Let t_0 and Δt denote the initial time and the control coefficient matrix updating time interval respectively. Then the overall swing-up algorithm can be written as follows:

Loop Start: For *i*=0,1,…, update *K* at time $t_0 + i\Delta t$ as follows:

$$K = \begin{cases} \text{Solution of problem (P),} & \text{if } |\theta| > 10 \\ [15.82 \quad 22.46 \quad 150.4 \quad 36.09] & \text{if } |\theta| \leq 10 \end{cases}$$

Use the K obtained above in the time interval $[t_0 + i\Delta t, t_0 + (i+1)\Delta t]$. Go to Loop *Start*.

Noting that the more interesting distinct features of the control strategies appear in the swing-up phase, we model the swing-up phase as a nonlinear programming problem. Using this model, it is shown that the system configuration constraints and the desired performance constraints on the swing-up problem can easily be embedded to the problem. Also, using the formulation of this paper, initial conditions for the problem can be extended to a larger set. Particularly, releasing the pendulum from any angular position, the nonlinear programming formulation achieves the swing-up phase successfully.

3.4 The convergence

In this subsection we show existence of control input u that increases the energy of the inverted pendulum monotonically at every updating interval provided that the α is not less than some threshold value. Expression (3) reveals that the derivative of the energy, $\dot{V}$ depends on the states x_2, x_3 and x_4. In order to illustrate convergence of the pendulum to the upright equilibrium state, we show x_2, x_3, x_4 values for which it is possible to find energy-increasing u. For simplicity in presentation, we superpose x_2 vs. x_4 graphics for sufficiently many sampled values of x_3 such that $|x_3| \geq 10^0$. For $\alpha = 15$ and $\alpha = 20$, Figures 5 and 6 respectively show superposed x_2 vs. x_4 graphics. The nonshaded region in each graphics represents the states for which energy increase is possible for the corresponding α value. It will be illustrated in the simulations section that when $\alpha \geq 15$, the nonlinear programming problems of the overall algorithm yield u values that monotonically increase the pendulum energy. It will appear that this threshold value of α (or higher) is sufficient for a convergent trajectory.

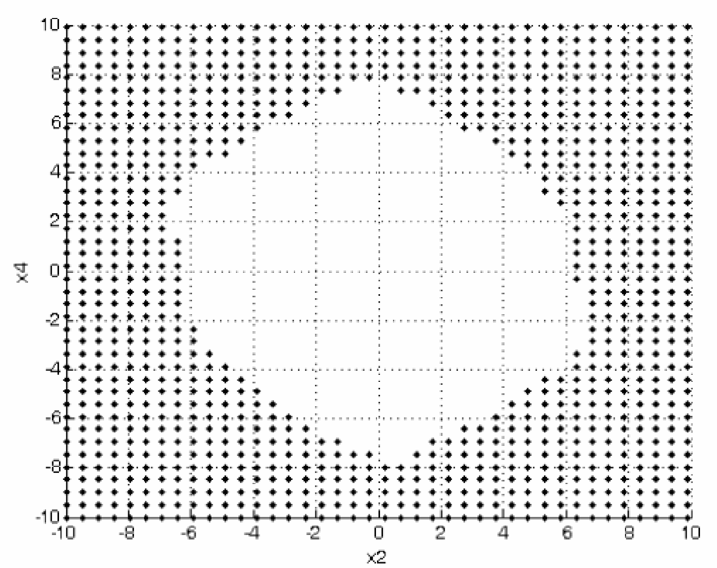

Figure 5. *Region where energy increasing control inputs exist for* $\alpha = 15$

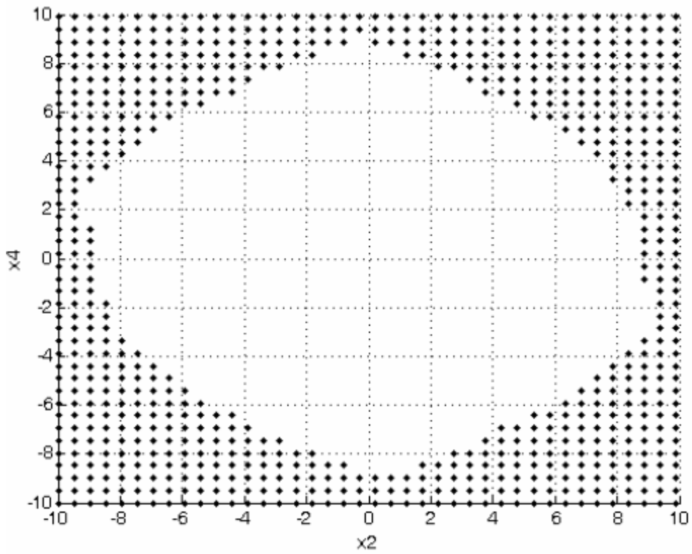

Figure 6. *Region where energy increasing control inputs exist for* $\alpha = 20$

IV. THE SIMULATION

In this section, we illustrate performance of the nonlinear programming approach by presenting the simulation outputs corresponding to various input constraints and design parameters. The simulations are performed by using the ordinary differential equation solver ode23.m and nonlinear programming solver fmincon.m of MATLAB. For all subsequent simulations we use control input updating time interval $\Delta t = 0.01$ sec., and the penalty parameter value $\lambda = 10^5$. Figure 7 presents graphical outputs corresponding to the hang-down initial condition with the design parameters $\alpha = 15$, $\gamma = 1$, $\beta = 5$. It can be observed that the swing-up is achieved successfully in three swings. It also shows projection of the corresponding trajectory on the x_2- x_4 plane. This illustrates that the algorithm yields a trajectory whose states at each updating instant allows monotonic increase in the pendulum energy. In Figure 8 the simulations are repeated for $\alpha = 20$.

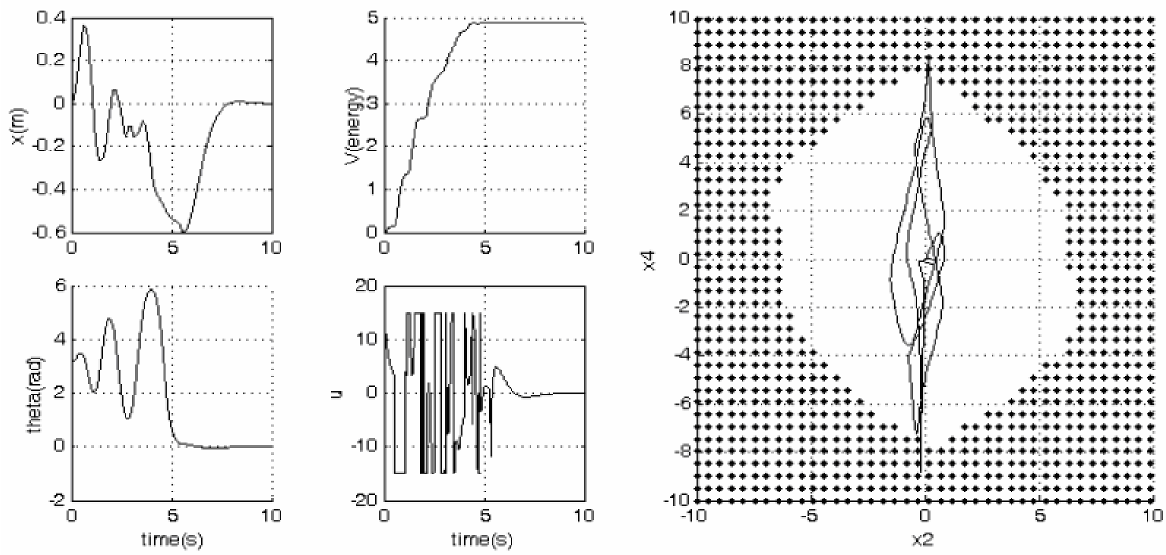

Figure 7. *State vs. time graphics and* x_2 *vs.* x_4 *trajectory when* $\alpha = 15$

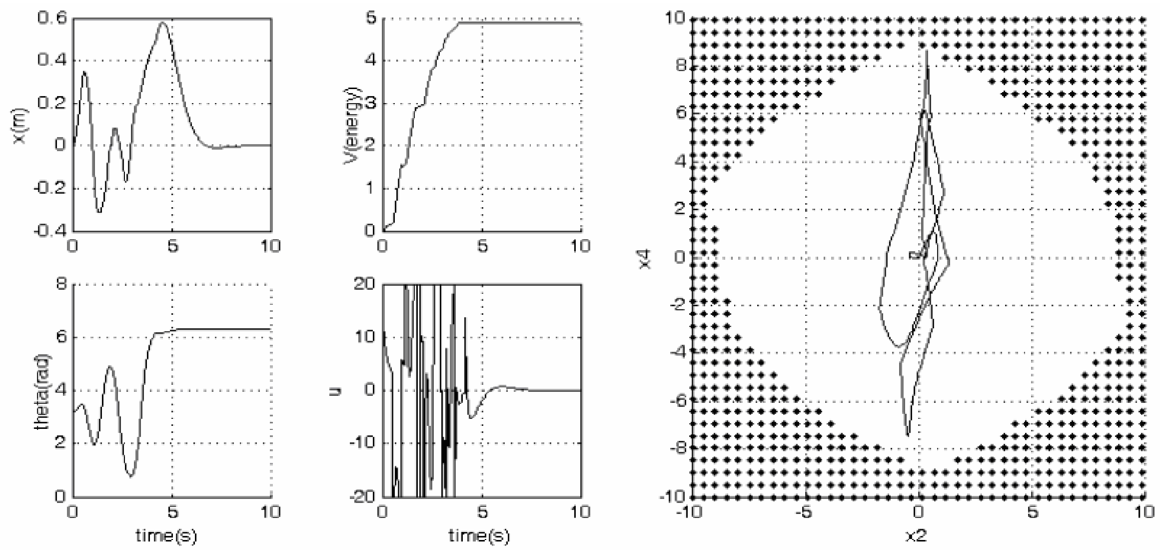

Figure 8. *State vs. time graphics and* x_2 *vs.* x_4 *trajectory when* $\alpha = 20$

In Figure 9 we present performance of the controller for the $\alpha = 15$ constraint with the doubled design parameters, $\gamma = 2,\ \beta = 10$. The swing-up is achieved successfully with the only significant change in the x-trajectory. This illustrates that achieving the swing-up goal is not sensitive to a significant amount of changes in γ and β.

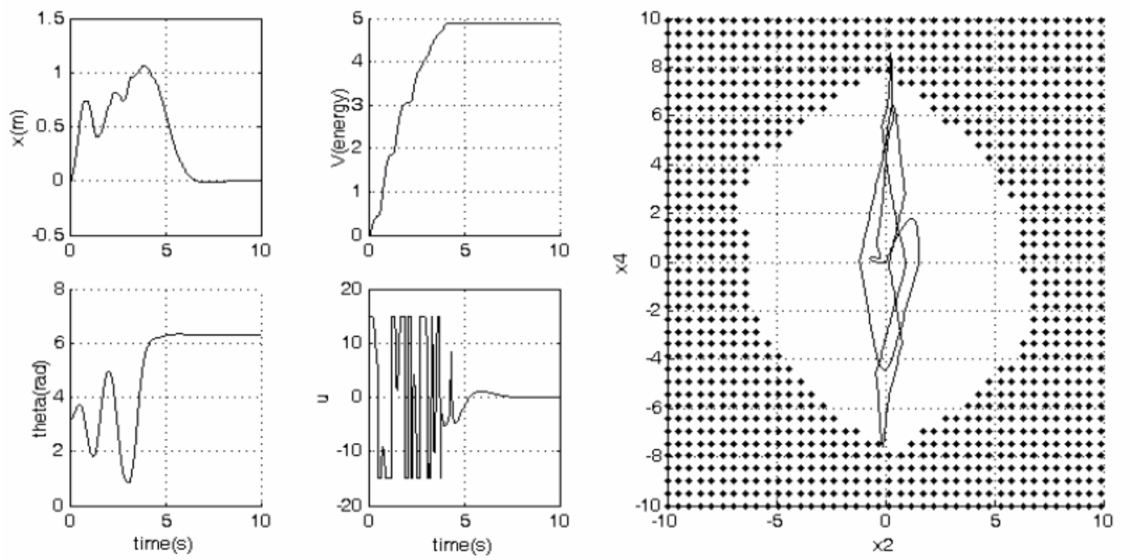

Figure 9. *State vs. time graphics and x_2 vs. x_4 trajectory with $\alpha = 15$, doubled γ and β.*

V. CONCLUSIONS

We presented a nonlinear programming based controller for the swing-up control of an inverted pendulum. We showed that the design is conceptually simple and capable of incorporating constraints such as a limited track length and a limited input size into the control problem. Although our approach is energy-based approach, due to the types of the constraints it handles, it differs from the literature. Also, the nonlinear programming modeling used in this work is suitable to be generalized to a larger class of dynamic systems including various control benchmark problems. This is an open research field which we consider to resolve in our emerging studies. In the simulation section we validated our method for various input bounds and design parameters.

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