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New Banach Sequence Spaces Defined by Jordan Totient Function

Uskan Devletli[1], Merve İlkhan Kara[2*]

Abstract
In this study, a special lower triangular matrix derived by combining Riesz matrix and Jordan totient matrix is used to construct new Banach spaces. $\alpha-,\beta-,\gamma-$duals of the resulting spaces are obtained and some matrix operators are characterized.

Keywords: Matrix mappings, Sequence space, $\alpha-,\beta-,\gamma-$duals
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[1] *Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey, ORCID: 0000-0001-9857-9513*
[2] *Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey, ORCID: 0000-0002-0831-1474*
*Corresponding author: merveilkhan@gmail.com
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1. Introduction and Background

A sequence space is a vector subspace of the space ω of all sequences with real entries. Well known classical sequence spaces are the space of p-absolutely summable sequences ℓ_p, the space of bounded sequences ℓ_∞, the space of null sequences c_0, the space of convergent sequences c. Throughout the study, the notion ℓ is used instead of ℓ_1. Also bs, cs_0 and cs are the most frequently encountered spaces consisting of sequences generating bounded, null and convergent series, respectively. A Banach sequence space having continuous coordinates is called a BK space. Examples of BK spaces are c_0 and c endowed with the supremum norm $\|u\|_\infty = \sup_i |u_i|$, where $\mathbb{N} = \{1,2,3,...\}$.

By virtue of the fact that the matrix mappings between BK-spaces are continuous, the theory of matrix mappings plays an important role in the study of sequence spaces. Let U and V be two sequence spaces, $\Lambda = (\lambda_{ij})$ be an infinite matrix with real entries and Λ_i indicate the i^{th} row of Λ. If each term of the sequence $\Lambda u = ((\Lambda u)_i) = (\sum_j \lambda_{ij} u_j)$ is convergent, this sequence is called Λ-transform of $u = (u_i)$. Further, if $\Lambda u \in V$ for every sequence $u \in U$, then the matrix Λ defines a matrix mapping from U into V. (U,V) represents the collection of all matrices defined from U into V. Additionally, $B(U,V)$ is the set of all bounded (continuous) linear operators from U to V. A matrix $\Lambda = (\lambda_{ij})$ is called a triangle if $\lambda_{ii} \neq 0$ and $\lambda_{ij} = 0$ for $j > i$.

The matrix domain U_Λ of the matrix Λ in the space U is defined by

$$U_\Lambda = \{u \in \omega : \Lambda u \in U\}.$$

Since this space is also a sequnce space, the matrix domain has a crucial role to construct new sequence spaces. Moreover given any triangle Λ and a BK-space U, the sequence space U_Λ gives a new BK-space equipped with the norm $\|u\|_{U_\Lambda} = \|\Lambda u\|_U$. Several authors applied this technique to construct new Banach spaces with the help of special triangles. For relevant literature, the papers [1–17] can be referred.

The spaces

$$U^{\alpha} = \left\{ t = (t_i) \in \omega : \sum_i |t_i u_i| < \infty \text{ for all } u = (u_i) \in U \right\},$$

$$U^{\beta} = \left\{ t = (t_i) \in \omega : \sum_i t_i u_i \text{ converges for all } u = (u_i) \in U \right\},$$

$$U^{\gamma} = \left\{ t = (t_i) \in \omega : \sup_i \left| \sum_i t_i u_i \right| < \infty \text{ for all } u = (u_i) \in U \right\},$$

are called the α-, β-, γ-duals of a sequence space U, respectively.

Note that $\frac{1}{p} + \frac{1}{q} = 1$ and $\sup_i$, $\sum_i$, $\lim_i$ mean $\sup_{i\in\mathbb{N}}$, $\sum_{i=1}^{\infty}$, $\lim_{i\to\infty}$, respectively.

The Euler totient matrix $\Phi = (\phi_{ij})$ is defined as in [18]

$$\phi_{ij} = \begin{cases} \frac{\varphi(j)}{i} & , \quad \text{if } j \mid i \\ 0 & , \quad \text{if } j \nmid i, \end{cases}$$

where φ is the Euler totient function. In the recent time, by using this matrix, many new sequence and series spaces are defined and studied in the papers [19–27].

For $i \in \mathbb{N}$ with $i \neq 1$, $\varphi(i)$ gives the number of positive integers less than i which are coprime with i and $\varphi(1) = 1$. Also, the equality

$$i = \sum_{j|i} \varphi(j)$$

holds for every $i \in \mathbb{N}$. For $i \in \mathbb{N}$ with $i \neq 1$, the Möbius function μ is defined as

$$\mu(i) = \begin{cases} (-1)^r & \text{if } i = p_1 p_2 ... p_r, \text{ where } p_1, p_2, ..., p_r \text{ are} \\ & \text{non-equivalent prime numbers} \\ 0 & \text{if } \tilde{p}^2 \mid i \text{ for some prime number } \tilde{p} \end{cases}$$

and $\mu(1) = 1$. The equality

$$\sum_{j|i} \mu(j) = 0 \tag{1.1}$$

holds except for $i = 1$.

The arithmetic function $J_r : \mathbb{N} \to \mathbb{N}$ with positive integer order r is called the Jordan totient function. This function generalizes the Euler totient function. If $r = 1$, it is reduced to the Euler totient function. The value $J_r(i)$ gives the number of r-tuples of positive integers all less than or equal to i that form a coprime $(r+1)$-tuples together with i.

The Jordan function J_r is multiplicative, i.e. for $n_1, n_2 \in \mathbb{N}$ with the greatest common divisor 1 the relation $J_r(n_1 n_2) = J_r(n_1) J_r(n_2)$ holds.

Let $p_1^{a_1} p_2^{a_2} ... p_k^{a_k}$ be the unique prime decomposition of $i \in \mathbb{N}$, then

$$J_r(i) = i^r (1 - \frac{1}{p_1^r})(1 - \frac{1}{p_2^r})...(1 - \frac{1}{p_k^r}).$$

Also, the following equations hold:

$$\sum_{j|i} J_r(j) = i^r$$

and

$$\sum_{j|i} \frac{\mu(j)}{j^r} = \frac{J_r(i)}{i^r}.$$

In [28], the authors have defined a new matrix $\Upsilon^r = (\upsilon^r_{ij})$ as

$$\upsilon^r_{ij} = \begin{cases} \frac{J_r(j)}{i^r} & , \quad \text{if } j \mid i \\ 0 & , \quad \text{if } j \nmid i \end{cases}$$

for each $r \in \mathbb{N}$. It is observed that this matrix is regular; that is a limit preserving mapping c into c. By using this matrix they introduce a space consisting of sequences whose Υ^r-transforms are in the space ℓ_p for $1 \le p < \infty$. Also, in [29], new Banach spaces are obtained by the aid of matrix domain of this matrix in the spaces ℓ_∞, c, c_0. In [30], the authors have studied the compact operators on the resulting spaces.

The Riesz matrix $E = (e_{ij})$ is defined as

$$e_{ij} = \begin{cases} \frac{q_j}{Q_i} & , \quad \text{if } 0 \le j \le i \\ 0 & , \quad \text{if } j > i, \end{cases}$$

where (q_j) is a sequence of positive numbers and $Q_i = \sum_{j=1}^{i} q_j$ for all $i \in \mathbb{N}$.

In a recent paper [31], the authors have constructed a new matrix called Riesz Euler totient matrix and study the domain of the matrix in the space ℓ_p. The Riesz Euler totient matrix $R_\Phi = (r_{ij})$ is defined as

$$r_{ij} = \begin{cases} \frac{q_j \varphi(j)}{Q_i} & , \quad \text{if } j \mid i \\ 0 & , \quad \text{if } j \nmid i. \end{cases}$$

The main purpose of this study is to construct new Banach spaces $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$. The matrix R_{Υ^r} is obtained by combining Jordan totient matrix and Riesz matrix. After studying certain properties of the resulting spaces, α-, β- and γ-duals are computed. Finally some matrix mappings from the resulting spaces to the classical spaces are characterized.

2. The Sequence Spaces $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$

In the present section, we introduce the sequence spaces $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$ by using the matrix R_{Υ^r}, where $1 < p < \infty$. Also, we present some theorems which give inclusion relations concerning these spaces.

The matrix $R_{\Upsilon^r} = (\nu_{ij})$ is defined as

$$\nu_{ij} = \begin{cases} \frac{q_j J_r(j)}{Q_i^r} & , \quad \text{if } j \mid i \\ 0 & , \quad \text{if } j \nmid i, \end{cases}$$

where $Q_i = q_1 + q_2 + ... + q_i$. We call this matrix as *Riesz Jordan totient matrix operator*.

Observe that in the special cases this matrix is reduced to the some matrices mentioned in the first section. If $r = 1$ and $q_j = 1$ for each j, it gives the Euler totient matrix. If $r = 1$, it gives the Riesz Euler totient matrix. If $q_j = 1$ for each j, it gives the Jorden totient matrix.

The inverse $R^{-1}_{\Upsilon^r} = (\nu^{-1}_{ij})$ of the matrix R_{Υ^r} is computed as

$$\nu^{-1}_{ij} = \begin{cases} \frac{\mu(\frac{i}{j})}{J_r(i)} \frac{Q_j^r}{q_i} & , \quad \text{if } j \mid i \\ 0 & , \quad \text{if } j \nmid i \end{cases}$$

for all $i, j \in \mathbb{N}$.

Now, we introduce the sequence spaces $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$ by

$$\ell_\infty(R_{\Upsilon^r}) = \left\{ u = (u_i) \in \omega : \sup_i \left| \frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) u_j \right| < \infty \right\},$$

$$\ell_p(R_{\Upsilon^r}) = \left\{ u = (u_i) \in \omega : \sum_i \left| \frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) u_j \right|^p < \infty \right\} \quad (1 < p < \infty),$$

$$\ell(R_{\Upsilon^r}) = \left\{ u = (u_i) \in \omega : \sum_i \left| \frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) u_j \right| < \infty \right\}.$$

Unless otherwise stated, $v = (v_i)$ will be the R_{Υ^r}-transform of a sequence $u = (u_i)$, that is, $v_i = (R_{\Upsilon^r} u)_i = \frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) u_j$ for all $i \in \mathbb{N}$.

Theorem 2.1. *The spaces $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$ are Banach spaces with the norms given by*

$$\|u\|_{\ell_\infty(R_{\Upsilon^r})} = \sup_i \left| \frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) u_j \right|,$$

$$\|u\|_{\ell_p(R_{\Upsilon^r})} = \left(\sum_i \left| \frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) u_j \right|^p \right)^{1/p} \quad (1 < p < \infty),$$

$$\|u\|_{\ell(R_{\Upsilon^r})} = \sum_i \left| \frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) u_j \right|.$$

Proof. We omit the proof which is straightforward. □

Corollary 2.2. *The spaces $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$ are BK-spaces, where $1 < p < \infty$.*

Theorem 2.3. *The space $U(R_{\Upsilon^r})$ is linearly isomorphic to U, where $U \in \{\ell_\infty, \ell_p, \ell\}$ and $1 < p < \infty$.*

Proof. Let f be a mapping defined from $U(R_{\Upsilon^r})$ to U such that $f(u) = R_{\Upsilon^r} u$ for all $u \in U(R_{\Upsilon^r})$. It is clear that f is linear. Also it is injective since the kernel of f consists of only zero. To prove that f is surjective consider the sequence $u = (u_i)$ whose terms are

$$u_i = \sum_{j|i} \frac{\mu(\frac{i}{j})}{J_r(i)} \frac{Q_j^r}{q_i} v_j$$

for all $i \in \mathbb{N}$, where $v = (v_j)$ is any sequence in U. It follows from (1.1) that

$$\begin{aligned}(R_{\Upsilon^r} u)_i &= \frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) u_j = \frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) \sum_{k|j} \frac{\mu(\frac{j}{k})}{J_r(j)} \frac{Q_k^r}{q_j} v_k \\ &= \frac{1}{Q_i^r} \sum_{j|i} \sum_{k|j} \mu(\frac{j}{k}) Q_k^r v_k = \frac{1}{Q_i^r} \sum_{j|i} \left(\sum_{k|j} \mu(k) \right) Q_{\frac{i}{j}}^r v_{\frac{i}{j}} = \frac{1}{Q_i^r} \mu(1) Q_i^r v_i = v_i\end{aligned}$$

and so $u = (u_i) \in U(R_{\Upsilon^r})$. f preserves norms since the equality $\|u\|_{U(R_{\Upsilon^r})} = \|f(u)\|_U$ holds. □

Remark 2.4. *The space $\ell_2(R_{\Upsilon^r})$ is an inner product space with the inner product defined as $\langle u, \tilde{u} \rangle_{\ell_2(R_{\Upsilon^r})} = \langle R_{\Upsilon^r} u, R_{\Upsilon^r} \tilde{u} \rangle_{\ell_2}$, where $\langle ., . \rangle_{\ell_2}$ is the inner product on ℓ_2 which induces $\|.\|_{\ell_2}$.*

Theorem 2.5. *The space $\ell_p(R_{\Upsilon^r})$ is not an inner product space for $p \neq 2$.*

Proof. Consider the sequences $u = (u_i)$ and $\tilde{u} = (\tilde{u}_i)$, where

$$u_i = \begin{cases} \frac{\mu(i)}{J_r(i)} \frac{Q_1^r}{q_i} + \frac{\mu(\frac{i}{2})}{J_r(i)} \frac{Q_2^r}{q_i} & , \text{ if } i \text{ is even} \\ \frac{\mu(i)}{J_r(i)} \frac{Q_1^r}{q_i} & , \text{ if } i \text{ is odd} \end{cases}$$

and

$$\tilde{u}_i = \begin{cases} \frac{\mu(i)}{J_r(i)} \frac{Q_1^r}{q_i} - \frac{\mu(\frac{i}{2})}{J_r(i)} \frac{Q_2^r}{q_i} & , \text{ if } i \text{ is even} \\ \frac{\mu(i)}{J_r(i)} \frac{Q_1^r}{q_i} & , \text{ if } i \text{ is odd} \end{cases}$$

for all $i \in \mathbb{N}$. Then, we have $R_{\Upsilon^r} u = (1, 1, 0, ..., 0, ...) \in \ell_p$ and $R_{\Upsilon^r} \tilde{u} = (1, -1, 0, ..., 0, ...) \in \ell_p$. Hence, one can easily observe that

$$\|u + \tilde{u}\|^2_{\ell_p(R_{\Upsilon^r})} + \|u - \tilde{u}\|^2_{\ell_p(R_{\Upsilon^r})} \neq 2 \left(\|u\|^2_{\ell_p(R_{\Upsilon^r})} + \|\tilde{u}\|^2_{\ell_p(R_{\Upsilon^r})} \right).$$

□

Theorem 2.6. *The inclusion $\ell_p(R_{\Upsilon^r}) \subset \ell_q(R_{\Upsilon^r})$ strictly holds for $1 \le p < q < \infty$.*

Proof. It is clear that the inclusion $\ell_p(R_{\Upsilon^r}) \subset \ell_q(R_{\Upsilon^r})$ holds since $\ell_p \subset \ell_q$ for $1 \le p < q < \infty$. Also, $\ell_p \subset \ell_q$ is strict and so there exists a sequence $z = (z_i)$ in $\ell_q \backslash \ell_p$. By defining a sequence $u = (u_i)$ as

$$u_i = \sum_{j|i} \frac{\mu(\frac{i}{j})}{J_r(i)} \frac{Q_j^r}{q_i} z_j$$

for all $i \in \mathbb{N}$, we conclude that $u \in \ell_q(R_{\Upsilon^r}) \backslash \ell_p(R_{\Upsilon^r})$. Hence, the desired inclusion is strict. □

Theorem 2.7. *The inclusion $\ell_p(R_{\Upsilon^r}) \subset \ell_\infty(R_{\Upsilon^r})$ strictly holds for $1 \le p < \infty$.*

Proof. The inclusion is obvious since $\ell_p \subset \ell_\infty$ holds for $1 \le p < \infty$. Let $u = (u_i)$ be a sequence such that $u_i = \sum_{j|i} (-1)^j \frac{\mu(\frac{i}{j})}{J_r(i)} \frac{Q_j^r}{q_i}$ for all $i \in \mathbb{N}$. We obtain that $R_{\Upsilon^r} u = \left(\frac{1}{Q_i^r} \sum_{j|i} q_j J_r(j) \sum_{k|j} (-1)^k \frac{\mu(\frac{j}{k})}{J_r(j)} \frac{Q_k^r}{q_j} \right) = ((-1)^i) \in \ell_\infty \backslash \ell_p$ which implies that $u \in \ell_\infty(R_{\Upsilon^r}) \backslash \ell_p(R_{\Upsilon^r})$ for $1 \le p < \infty$. □

Lemma 2.8. *[32] The necessary and sufficient conditions for $\Lambda = (\lambda_{ij}) \in (U,V)$ with $U,V \in \{\ell_\infty, c, c_0, \ell_p, \ell\}$ and $p > 1$ can be read from Table 1. Here and in what follows, $\mathscr{N}$ denotes the family of all finite subsets of $\mathbb{N}$.*

To / From	ℓ_∞	c	c_0	ℓ_p	ℓ
ℓ_∞	**1.**	**4.**	**9.**	**14.**	**16.**
c	**1.**	**5.**	**10.**	**14.**	**16.**
c_0	**1.**	**6.**	**11.**	**14.**	**16.**
ℓ_p	**2.**	**7.**	**12.**	**-**	**17.**
ℓ	**3.**	**8.**	**13.**	**15**	**18.**

Table 1. The characterization of the class (U,V), where $U,V \in \{\ell_\infty, c, c_0, \ell_p, \ell\}$.

1.

$$\sup_i \sum_j |\lambda_{ij}| < \infty \tag{2.1}$$

2.

$$\sup_i \sum_j |\lambda_{ij}|^q < \infty \tag{2.2}$$

3.

$$\sup_{i,j} |\lambda_{ij}| < \infty \tag{2.3}$$

4.

$$\lim_i \lambda_{ij} \text{ exists for each } j \in \mathbb{N}, \tag{2.4}$$

$$\lim_i \sum_j |\lambda_{ij}| = \sum_j \left| \lim_i \lambda_{ij} \right|$$

5. (2.1), (2.4) *and*

$$\lim_i \sum_j \lambda_{ij} \text{ exists.}$$

6. (2.1) *and* (2.4)

7. (2.2) *and* (2.4)

8. (2.3) *and* (2.4)

9.

$$\lim_i \sum_j |\lambda_{ij}| = 0$$

10. (2.1) *and*

$$\lim_i \lambda_{ij} = 0 \text{ for each } j \in \mathbb{N}, \tag{2.5}$$

$$\lim_i \sum_j \lambda_{ij} = 0$$

11. (2.1) *and* (2.5)

12. (2.2) *and* (2.5)

13. (2.3) *and* (2.5)

14.

$$\sup_{K\in\mathscr{N}} \sum_i \left| \sum_{j\in K} \lambda_{ij} \right|^p < \infty$$

15.

$$\sup_j \sum_i |\lambda_{ij}|^p < \infty$$

16.

$$\sup_{N,K\in\mathscr{N}} \left| \sum_{i\in N} \sum_{j\in K} \lambda_{ij} \right| < \infty \Leftrightarrow \sup_{N\in\mathscr{N}} \sum_j \left| \sum_{i\in N} \lambda_{ij} \right| < \infty \Leftrightarrow \sup_{K\in\mathscr{N}} \sum_i \left| \sum_{j\in K} \lambda_{ij} \right| < \infty$$

17.

$$\sup_{N\in\mathscr{N}} \sum_j \left| \sum_{i\in N} \lambda_{ij} \right|^q < \infty$$

18.

$$\sup_j \sum_i |\lambda_{ij}| < \infty$$

3. The α-, β- and γ-duals

In this section, we determine the α-, β- and γ-duals of the sequence spaces $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$, where $1<p<\infty$. In the following theorem, we determine the α-duals.

Theorem 3.1. *The α-duals of the spaces $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$ are as follows:*

$$(\ell_\infty(R_{\Upsilon^r}))^\alpha=\left\{t=(t_i)\in\omega:\sup_{N\in\mathscr{N}}\sum_j\left|\sum_{i\in N,j|i}\frac{\mu(\frac{i}{j})}{J_r(i)}\frac{Q_j^r}{q_i}t_i\right|<\infty\right\},$$

$$(\ell_p(R_{\Upsilon^r}))^\alpha=\left\{t=(t_i)\in\omega:\sup_{N\in\mathscr{N}}\sum_j\left|\sum_{i\in N,j|i}\frac{\mu(\frac{i}{j})}{J_r(i)}\frac{Q_j^r}{q_i}t_i\right|^q<\infty\right\},$$

$$(\ell(R_{\Upsilon^r}))^\alpha=\left\{t=(t_i)\in\omega:\sup_j\sum_{i\in\mathbb{N},j|i}\left|\frac{\mu(\frac{i}{j})}{J_r(i)}\frac{Q_j^r}{q_i}t_i\right|<\infty\right\}.$$

Proof. Consider the matrix $C=(c_{ij})$ defined by

$$c_{ij}=\begin{cases}\frac{\mu(\frac{i}{j})}{J_r(i)}\frac{Q_j^r}{q_i}t_i & , \quad j\mid i\\ 0 & , \quad j\nmid i\end{cases}$$

for any sequence $t=(t_i)\in\omega$. Let $U\in\{\ell_\infty,\ell_p,\ell\}$. Given any $u=(u_i)\in U(R_{\Upsilon^r})$, we have $t_iu_i=(Cv)_i$ for all $i\in\mathbb{N}$. This implies that $tu\in\ell$ with $u\in U(R_{\Upsilon^r})$ if and only if $Cv\in\ell$ with $v\in U$. It follows that $t\in(U(R_{\Upsilon^r}))^\alpha$ if and only if $C\in(U,\ell)$ which completes the proof in view of Lemma 2.8. □

Lemma 3.2. *[33, Theorem 3.1] Let $B=(b_{ij})$ be defined via a sequence $t=(t_k)\in\omega$ and the inverse matrix $\tilde{\Delta}=(\tilde{\delta}_{ij})$ of the triangle matrix $\Delta=(\delta_{ij})$ by*

$$b_{ij}=\sum_{k=j}^{i}t_k\tilde{\delta}_{kj}$$

for all $i,j\in\mathbb{N}$. Then,

$$U_\Delta^\beta=\{t=(t_k)\in\omega:B\in(U,c)\}$$

and

$$U_\Delta^\gamma=\{t=(t_k)\in\omega:B\in(U,\ell_\infty)\}.$$

Consequently, we have the following theorem.

Theorem 3.3. *Let define the following sets:*

$$A_1=\left\{t=(t_k)\in\omega:\lim_i\sum_{k=j,j|k}^{i}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}t_k \text{ exists for each } j\in\mathbb{N}\right\},$$

$$A_2=\left\{t=(t_k)\in\omega:\sup_i\sum_j\left|\sum_{k=j,j|k}^{i}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}t_k\right|^q<\infty\right\},$$

$$A_3=\left\{t=(t_k)\in\omega:\lim_i\sum_j\left|\sum_{k=j,j|k}^{i}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}t_k\right|=\sum_j\left|\sum_{k=j,j|k}^{\infty}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}t_k\right|\right\},$$

$$A_4=\left\{t=(t_k)\in\omega:\sup_{i,j}\left|\sum_{k=j,j|k}^{i}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}t_k\right|<\infty\right\}.$$

The β- and γ-duals of the spaces $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$ are as follows:
$(\ell_\infty(R_{\Upsilon^r}))^\beta=A_1\cap A_3$, $(\ell_p(R_{\Upsilon^r}))^\beta=A_1\cap A_2$, $(\ell(R_{\Upsilon^r}))^\beta=A_1\cap A_4$.
$(\ell_\infty(R_{\Upsilon^r}))^\gamma=A_2$ with $q=1$, $(\ell_p(R_{\Upsilon^r}))^\gamma=A_2$, $(\ell(R_{\Upsilon^r}))^\gamma=A_4$.

Proof. Let $t=(t_k)\in\omega$, $U\in\{\ell_\infty,\ell_p,\ell\}$ and $B=(b_{ij})$ be an infinite matrix with terms

$$b_{ij}=\begin{cases}\sum_{k=j,j|k}^{i} t_k\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k} & , \quad \text{if } 1\le j\le i\\ 0 & , \quad \text{if } j>i.\end{cases}$$

Hence it follows that

$$\sum_{j=1}^{i} t_j u_j=\sum_{j=1}^{j} t_j\left(\sum_{k|j}\frac{\mu(\frac{j}{k})}{J_r(j)}\frac{Q_k^r}{q_j}v_k\right)=\sum_{j=1}^{i}\left(\sum_{k=j,j|k}^{i} t_k\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}\right)v_j=(Bv)_i$$

for any $u=(u_i)\in U(R_{\Upsilon^r})$. This equality yields that $tu\in cs$ for $u\in U(R_{\Upsilon^r})$ if and only if $Bv\in c$ for $v\in U$. That is, $t\in(U(R_{\Upsilon^r}))^\beta$ if and only if $B\in(U,c)$. Hence, by Lemma 2.8, it is concluded that $(\ell_\infty(R_{\Upsilon^r}))^\beta=A_1\cap A_3$, $(\ell_p(R_{\Upsilon^r}))^\beta=A_1\cap A_2$, $(\ell(R_{\Upsilon^r}))^\beta=A_1\cap A_4$.

This equality also yields that $tu\in bs$ for $u\in U(R_{\Upsilon^r})$ if and only if $Bv\in\ell_\infty$ for $v\in U$. That is, $t\in(U(R_{\Upsilon^r}))^\gamma$ if and only if $B\in(U,\ell_\infty)$. Hence, by Lemma 2.8, it is concluded that $(\ell_\infty(R_{\Upsilon^r}))^\gamma=A_2$ with $q=1$, $(\ell_p(R_{\Upsilon^r}))^\gamma=A_2$, $(\ell(R_{\Upsilon^r}))^\gamma=A_4$. □

4. Certain Matrix Transformations

In this section, characterization of certain classes of matrices is given. The following result is obtained from Theorem 4.1 in [34] and this result is required to characterize the classes of matrices from $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$ into ℓ_∞,c,c_0,ℓ.

Theorem 4.1. *Let $1<p<\infty$, $U\in\{\ell_\infty,\ell_p,\ell\}$ and $V\subset\omega$. Then, $\Lambda=(\lambda_{ij})\in(U_{R_{\Upsilon^r}},V)$ if and only if $\Theta^{(i)}=\left(\theta_{lj}^{(i)}\right)\in(U,c)$ for each fixed $i\in\mathbb{N}$ and $\Theta=(\theta_{ij})\in(U,V)$, where*

$$\theta_{lj}^{(i)}=\begin{cases}\sum_{k=j,j|k}^{l}\lambda_{ik}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k} & , \quad 1\le j\le l\\ 0 & , \quad j>l\end{cases}$$

and

$$\theta_{ij}=\sum_{k=j,j|k}^{\infty}\lambda_{ik}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}.$$

Proof. Let $\Lambda\in(U_{R_{\Upsilon^r}},V)$ and $u\in U_{R_{\Upsilon^r}}$. Then, the equality

$$\begin{aligned}\sum_{j=1}^{l}\lambda_{ij}u_j &= \sum_{j=1}^{l}\lambda_{ij}\left(\sum_{k|j}\frac{\mu(\frac{j}{k})}{J_r(j)}\frac{Q_k^r}{q_j}v_k\right)\\ &= \sum_{j=1}^{l}\left(\sum_{k=j,j|k}^{l}\lambda_{ik}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}\right)v_j=\sum_{j=1}^{l}\theta_{lj}^{(i)}v_j\end{aligned} \tag{4.1}$$

holds. Since Λu exists, it follows that $\Theta^{(i)}\in(U,c)$ for each fixed $i\in\mathbb{N}$. It is deduced that $\Lambda u=\Theta v$ as $l\to\infty$ in (4.1). Hence, $\Lambda u\in V$ implies that $\Theta v\in V$; that is $\Theta\in(U,V)$.

Conversely, suppose that $\Theta^{(i)}=\left(\theta_{lj}^{(i)}\right)\in(U,c)$ *for each fixed* $i\in\mathbb{N}$ and $\Theta=(\theta_{ij})\in(U,V)$. Let $u\in U_{R_{\Upsilon^r}}$. Then, $(\theta_{ij})\in U^\beta$ for each fixed $i\in\mathbb{N}$ implies that $(\lambda_{ij})\in U^\beta_{R_{\Upsilon^r}}$ for each fixed $i\in\mathbb{N}$. Hence, Λu exists. From equality (4.1), it follows that $\Lambda u=\Theta v$ as $l\to\infty$. This proves that $\Lambda\in(U_{R_{\Upsilon^r}},V)$. □

Theorem 4.2. *Let $\Lambda=(\lambda_{ij})$ be an infinite matrix. Then, the following statements hold:*

1. $\Lambda\in(\ell_\infty(R_{\Upsilon^r}),\ell_\infty)$ if and only if

$$\lim_{l\to\infty}\sum_{k=j,j|k}^{l}\lambda_{ik}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k} \text{ exists for each fixed } i,j\in\mathbb{N}, \tag{4.2}$$

$$\lim_{l}\sum_{j}\left|\sum_{k=j,j|k}^{l}\lambda_{ik}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}\right|=\sum_{j}\left|\lim_{l}\sum_{k=j,j|k}^{l}\lambda_{ik}\frac{\mu(\frac{k}{j})}{J_r(k)}\frac{Q_j^r}{q_k}\right| \tag{4.3}$$

and

$$\sup_i \sum_j \left| \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| < \infty. \tag{4.4}$$

2. $\Lambda \in (\ell_\infty(R_{\Upsilon^r}), c)$ *if and only if* (4.2), (4.3),

$$\lim_i \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \text{ exists for each } j \in \mathbb{N}, \tag{4.5}$$

$$\lim_i \sum_j \left| \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| = \sum_j \left| \lim_i \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right|.$$

3. $\Lambda \in (\ell_\infty(R_{\Upsilon^r}), c_0)$ *if and only if* (4.2), (4.3),

$$\lim_i \sum_j \left| \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| = 0.$$

4. $\Lambda \in (\ell_\infty(R_{\Upsilon^r}), \ell)$ *if and only if* (4.2), (4.3) *and*

$$\sup_{N,K \in \mathscr{N}} \left| \sum_{i \in N} \sum_{j \in K} \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| < \infty. \tag{4.6}$$

Proof. The proof follows from Lemma 2.8 and Theorem 4.1. □

Theorem 4.3. *Let $\Lambda = (\lambda_{ij})$ be an infinite matrix and $p > 1$. Then, the following statements hold:*

1. $\Lambda \in (\ell_p(R_{\Upsilon^r}), \ell_\infty)$ *if and only if* (4.2),

$$\sup_{l \in \mathbb{N}} \sum_{j=1}^{l} \left| \sum_{k=j,j|k}^{l} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right|^q < \infty \text{ for each fixed } i \in \mathbb{N}, \tag{4.7}$$

$$\sup_i \sum_j \left| \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right|^q < \infty. \tag{4.8}$$

2. $\Lambda \in (\ell_p(R_{\Upsilon^r}), c)$ *if and only if* (4.2), (4.7), (4.5), (4.8).

3. $\Lambda \in (\ell_p(R_{\Upsilon^r}), c_0)$ *if and only if* (4.2), (4.7), (4.8),

$$\lim_i \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} = 0 \text{ for each } j \in \mathbb{N}. \tag{4.9}$$

4. $\Lambda \in (\ell_p(R_{\Upsilon^r}), \ell)$ *if and only if* (4.2), (4.7),

$$\sup_{N \in \mathscr{N}} \sum_j \left| \sum_{i \in N} \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right|^q < \infty.$$

Proof. The proof follows from Lemma 2.8 and Theorem 4.1. □

Theorem 4.4. *Let $\Lambda = (\lambda_{ij})$ be an infinite matrix. Then, the following statements hold:*

1. $\Lambda \in (\ell(R_{\Upsilon^r}), \ell_\infty)$ *if and only if* (4.2),

$$\sup_{l,j} \left| \sum_{k=j,j|k}^{l} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| < \infty \text{ for each fixed } i \in \mathbb{N}, \tag{4.10}$$

$$\sup_{i,j} \left| \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| < \infty. \tag{4.11}$$

2. $\Lambda \in (\ell(R_{\Upsilon^r}), c)$ *if and only if* (4.2), (4.10), (4.5), (4.11).
3. $\Lambda \in (\ell(R_{\Upsilon^r}), c_0)$ *if and only if* (4.2), (4.10), (4.9), (4.11).
4. $\Lambda \in (\ell(R_{\Upsilon^r}), \ell)$ *if and only if* (4.2), (4.10),

$$\sup_{j} \sum_{i} \left| \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| < \infty.$$

Proof. The proof follows from Lemma 2.8 and Theorem 4.1. □

Corollary 4.5. *Let $\Lambda = (\lambda_{ij})$ be an infinite matrix. Then, the following statements hold:*

1. $\Lambda \in (\ell_\infty(R_{\Upsilon^r}), bs)$ *if and only if* (4.2), (4.3),

$$\sup_{i} \sum_{j} \left| \sum_{l=1}^{i} \sum_{k=j,j|k}^{\infty} \lambda_{lk} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| < \infty. \tag{4.12}$$

2. $\Lambda \in (\ell_\infty(R_{\Upsilon^r}), cs)$ *if and only if* (4.2), (4.3),

$$\lim_{i} \sum_{l=1}^{i} \sum_{k=j,j|k}^{\infty} \lambda_{lk} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \text{ exists for each } j \in \mathbb{N}, \tag{4.13}$$

$$\lim_{i} \sum_{j} \left| \sum_{l=1}^{i} \sum_{k=j,j|k}^{\infty} \lambda_{lk} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| = \sum_{j} \left| \lim_{i} \sum_{l=1}^{i} \sum_{k=j,j|k}^{\infty} \lambda_{lk} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right|.$$

3. $\Lambda \in (\ell_\infty(R_{\Upsilon^r}), cs_0)$ *if and only if* (4.2), (4.3)

$$\lim_{i} \sum_{j} \left| \sum_{l=1}^{i} \sum_{k=j,j|k}^{\infty} \lambda_{lk} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| = 0.$$

Corollary 4.6. *Let $\Lambda = (\lambda_{ij})$ be an infinite matrix. Then, the following statements hold:*

1. $\Lambda \in (\ell_p(R_{\Upsilon^r}), bs)$ *if and only if* (4.2), (4.7),

$$\sup_{i} \sum_{j} \left| \sum_{l=1}^{i} \sum_{k=j,j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right|^q < \infty. \tag{4.14}$$

2. $\Lambda \in (\ell_p(R_{\Upsilon^r}), cs)$ *if and only if* (4.2), (4.7), (4.13), (4.14).

3. $\Lambda \in (\ell_p(R_{\Upsilon^r}), cs_0)$ *if and only if* (4.2), (4.7), (4.14),

$$\lim_i \sum_{l=1}^{i} \sum_{k=j, j|k}^{\infty} \lambda_{lk} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} = 0 \text{ for each } j \in \mathbb{N}. \tag{4.15}$$

Corollary 4.7. *Let $\Lambda = (\lambda_{ij})$ be an infinite matrix. Then, the following statements hold:*

1. $\Lambda \in (\ell(R_{\Upsilon^r}), bs)$ *if and only if* (4.2), (4.10),

$$\sup_{i,j} \left| \sum_{l=1}^{i} \sum_{k=j, j|k}^{\infty} \lambda_{ik} \frac{\mu(\frac{k}{j})}{J_r(k)} \frac{Q_j^r}{q_k} \right| < \infty. \tag{4.16}$$

2. $\Lambda \in (\ell(R_{\Upsilon^r}), cs)$ *if and only if* (4.2), (4.10), (4.13), (4.16).

3. $\Lambda \in (\ell(R_{\Upsilon^r}), cs_0)$ *if and only if* (4.2), (4.10), (4.15), (4.16).

Theorem 4.8. *Let $\Lambda = (\lambda_{ij})$ be an infinite matrix and $p > 1$. Then, the following statements hold:*
(a) $\Lambda \in (\ell_\infty, \ell_p(R_{\Upsilon^r})) = (c, \ell_p(R_{\Upsilon^r})) = (c_0, \ell_p(R_{\Upsilon^r}))$ *if and only if*

$$\sup_{K \in \mathscr{N}} \sum_i \left| \sum_{j \in K} \sum_{l|i} \frac{q_l J_r(l)}{Q_i^r} \lambda_{lj} \right|^p < \infty.$$

(b) $\Lambda \in (\ell, \ell_p(R_{\Upsilon^r}))$ *if and only if*

$$\sup_j \sum_i \left| \sum_{l|i} \frac{q_l J_r(l)}{Q_i^r} \lambda_{lj} \right|^p < \infty.$$

Proof. The proof is given only for the matrix in $(\ell_\infty, \ell_p(R_{\Upsilon^r}))$ since the other case can be proven similarly. Given any infinite matrix $\Lambda = (\lambda_{ij}) \in (\ell_\infty, \ell_p(R_{\Upsilon^r}))$, define a new matrix $\hat{\Lambda} = (\hat{\lambda}_{ij})$ by

$$\hat{\lambda}_{ij} = \sum_{l|i} \frac{q_l J_r(l)}{Q_i^r} \lambda_{lj}$$

for all $i, j \in \mathbb{N}$. Then, for any $u = (u_j) \in \ell_\infty$, the equality

$$\sum_j \hat{\lambda}_{ij} u_j = \sum_{l|i} \frac{q_l J_r(l)}{Q_i^r} \sum_j \lambda_{lj} u_j$$

means that $(\hat{\Lambda}u)_i = (R_{\Upsilon^r}(\Lambda u))_i$ for all $i \in \mathbb{N}$. This implies that $\Lambda u \in \ell_p(R_{\Upsilon^r})$ for $u = (u_j) \in \ell_\infty$ if and only if $\hat{\Lambda}u \in \ell_p$ for $u = (u_j) \in \ell_\infty$. Hence, we conclude from Lemma 2.8 that

$$\sup_{K \in \mathscr{N}} \sum_i \left| \sum_{j \in K} \sum_{l|i} \frac{q_l J_r(l)}{Q_i^r} \lambda_{lj} \right|^p < \infty.$$

□

Theorem 4.9. *Let $\Lambda = (\lambda_{ij})$ be an infinite matrix. Then, the following statements hold:*
(a) $\Lambda \in (\ell_\infty, \ell_\infty(R_{\Upsilon^r})) = (c, \ell_\infty(R_{\Upsilon^r})) = (c_0, \ell_\infty(R_{\Upsilon^r}))$ *if and only if*

$$\sup_i \sum_j \left| \sum_{l|i} \frac{q_l J_r(l)}{Q_i^r} \lambda_{lj} \right| < \infty.$$

(b) $\Lambda \in (\ell, \ell_\infty(R_{\Upsilon^r}))$ *if and only if*

$$\sup_{i,j} \left| \sum_{l|i} \frac{q_l J_r(l)}{Q_i^r} \lambda_{lj} \right| < \infty.$$

Proof. The proof follows with the same way in the proof of Theorem 4.8. □

Theorem 4.10. *Let $\Lambda = (\lambda_{ij})$ be an infinite matrix. Then, the following statements hold:*
(a) $\Lambda \in (\ell_\infty, \ell(R_{\Upsilon^r})) = (c, \ell(R_{\Upsilon^r})) = (c_0, \ell(R_{\Upsilon^r}))$ if and only if

$$\sup_{K\in\mathscr{N}} \sum_i \left| \sum_{j\in K} \sum_{l|i} \frac{q_l J_r(l)}{Q_i^r} \lambda_{lj} \right| < \infty.$$

(b) $\Lambda \in (\ell, \ell(R_{\Upsilon^r}))$ if and only if

$$\sup_j \sum_i \left| \sum_{l|i} \frac{q_l J_r(l)}{Q_i^r} \lambda_{lj} \right| < \infty.$$

Proof. The proof follows with the same way in the proof of Theorem 4.8. □

Now, we investigate the norm of the bounded linear matrix operators from $\ell_\infty(R_{\Upsilon^r})$, $\ell_p(R_{\Upsilon^r})$, $\ell(R_{\Upsilon^r})$ into $\ell_\infty(R_{\Upsilon^r})$ and $\ell(R_{\Upsilon^r})$. Firstly, we have a lemma which is essential for our investigation.

Lemma 4.11. *Given any infinite matrix $\Lambda = (\lambda_{ij})$, the norm of bounded linear operators is defined by*

$$\|\Lambda\|_{(\ell_\infty,\ell_\infty)} = \|\Lambda\|_{(\ell_p,\ell_\infty)} = \sup_i \sum_j |\lambda_{ij}|^q$$

$$\|\Lambda\|_{(\ell,\ell_\infty)} = \sup_{i,j} |\lambda_{ij}|$$

$$\|\Lambda\|_{(\ell_\infty,\ell)} = \|\Lambda\|_{(\ell_p,\ell)} = \sup_{K\in\mathscr{N}} \sum_j \left| \sum_{i\in K} \lambda_{ij} \right|^q$$

$$\|\Lambda\|_{(\ell,\ell)} = \sup_j \sum_i |\lambda_{ij}|.$$

Theorem 4.12. *Let $\Lambda = (\lambda_{ij})$ be an infinite matrix.*
(a) If $\Lambda \in B(\ell_\infty(R_{\Upsilon^r}), \ell_\infty(R_{\Upsilon^r}))$ or $\Lambda \in B(\ell_p(R_{\Upsilon^r}), \ell_\infty(R_{\Upsilon^r}))$, then

$$\sup_i \sum_j \left| \sum_{j|l} \frac{\mu(\frac{l}{j})}{J_r(l)} \frac{Q_j^r}{q_l} \sum_{k|i} \frac{q_k J_r(k)}{Q_i^r} \lambda_{kl} \right|^q < \infty$$

and

$$\|\Lambda\|_{(\ell_\infty(R_{\Upsilon^r}),\ell_\infty(R_{\Upsilon^r}))} = \|\Lambda\|_{(\ell_p(R_{\Upsilon^r}),\ell_\infty(R_{\Upsilon^r}))} = \sup_i \sum_j \left| \sum_{j|l} \frac{\mu(\frac{l}{j})}{J_r(l)} \frac{Q_j^r}{q_l} \sum_{k|i} \frac{q_k J_r(k)}{Q_i^r} \lambda_{kl} \right|^q.$$

(b) If $\Lambda \in B(\ell(R_{\Upsilon^r}), \ell_\infty(R_{\Upsilon^r}))$, then

$$\sup_{i,j} \left| \sum_{j|l} \frac{\mu(\frac{l}{j})}{J_r(l)} \frac{Q_j^r}{q_l} \sum_{k|i} \frac{q_k J_r(k)}{Q_i^r} \lambda_{kl} \right| < \infty$$

and

$$\|\Lambda\|_{(\ell(R_{\Upsilon^r}),\ell_\infty(R_{\Upsilon^r}))} = \sup_{i,j} \left| \sum_{j|l} \frac{\mu(\frac{l}{j})}{J_r(l)} \frac{Q_j^r}{q_l} \sum_{k|i} \frac{q_k J_r(k)}{Q_i^r} \lambda_{kl} \right|.$$

(c) If $\Lambda \in B(\ell_\infty(R_{\Upsilon^r}), \ell(R_{\Upsilon^r}))$ or $\Lambda \in B(\ell_p(R_{\Upsilon^r}), \ell(R_{\Upsilon^r}))$, then

$$\sup_{K\in\mathscr{N}} \sum_j \left| \sum_{i\in K} \sum_{j|l} \frac{\mu(\frac{l}{j})}{J_r(l)} \frac{Q_j^r}{q_l} \sum_{k|i} \frac{q_k J_r(k)}{Q_i^r} \lambda_{kl} \right|^q < \infty$$

and

$$\|\Lambda\|_{(\ell_\infty(R_{\Upsilon^r}),\ell(R_{\Upsilon^r}))} = \|\Lambda\|_{(\ell_p(R_{\Upsilon^r}),\ell(R_{\Upsilon^r}))} = \sup_{K\in\mathscr{N}} \sum_j \left| \sum_{i\in K} \sum_{j|l} \frac{\mu(\frac{l}{j})}{J_r(l)} \frac{Q_j^r}{q_l} \sum_{k|i} \frac{q_k J_r(k)}{Q_i^r} \lambda_{kl} \right|^q.$$

(d) If $\Lambda \in B(\ell(R_{\Upsilon^r}), \ell(R_{\Upsilon^r}))$, then

$$\sup_j \sum_i \left| \sum_{j|l} \frac{\mu(\frac{l}{j})}{J_r(l)} \frac{Q_j^r}{q_l} \sum_{k|i} \frac{q_k J_r(k)}{Q_i^r} \lambda_{kl} \right| < \infty$$

and

$$\|\Lambda\|_{(\ell(R_{\Upsilon^r}),\ell(R_{\Upsilon^r}))} = \sup_j \sum_i \left| \sum_{j|l} \frac{\mu(\frac{l}{j})}{J_r(l)} \frac{Q_j^r}{q_l} \sum_{k|i} \frac{q_k J_r(k)}{Q_i^r} \lambda_{kl} \right|.$$

Proof. Let $\tilde{\Lambda} = R_{\Upsilon^r} \Lambda R_{\Upsilon^r}^{-1}$. From Theorem 2.3, it is known that the spaces $U(R_{\Upsilon^r})$ and U are linearly isomorphic. Hence, we deduce from the following diagram

$$\begin{array}{ccc} U(R_{\Upsilon^r}) & \xrightarrow{\Lambda} & V(R_{\Upsilon^r}) \\ R_{\Upsilon^r}^{-1} \uparrow & & \downarrow R_{\Upsilon^r} \\ U & \xrightarrow[\tilde{\Lambda} = R_{\Upsilon^r} \Lambda R_{\Upsilon^r}^{-1}]{} & V \end{array}$$

that $\|\Lambda\|_{(U(R_{\Upsilon^r}),V(R_{\Upsilon^r}))} = \|\tilde{\Lambda}\|_{(U,V)}$, where $U \in \{\ell_\infty, \ell_p, \ell\}$ and $V \in \{\ell_\infty, \ell\}$. Thus, the desired results follows from Lemma 4.11. □

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