DOI: 10.38061/idunas.1318116

IDUNAS	NATURAL & APPLIED SCIENCES JOURNAL	2023 Vol. 6 No. 2 (24-30)

Some New Integral Inequalities via Caputo-Fabrizio Fractional Integral Operator

Research Article

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DOI: 10.38061/idunas.1318116

Received: 21.06.2023; Accepted: 31.08.2023

Abstract

On this note, some new integral inequalities were constructed for the product of two integrable functions by using Young's inequality with a well-known classical inequality via the Caputo-Fabrizio fractional integral operator. Some specific cases of the main findings are then given. The main results have the potential to be used in inequality theory.
Keywords: Caputo-Fabrizio fractional integral operator, Young inequality, Inequalities.

1. INTRODUCTION

Definition 1.1 [2] Let $f \in H^1(a,b), b > a, \alpha \in [0,1]$. The definitions of the left and right sides of the Caputo-Fabrizio fractional integral are given as:

$$\left({}^{CF}_{\ a}I^{\alpha}\right)(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}\int_a^t f(y)dy$$

and

$$\left({}^{CF}I_b^{\alpha}\right)(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}\int_t^b f(y)dy$$

where $B(\alpha) > 0$ is normalization function.

Subsequently in the paper, we will denote normalization function as $B(\alpha)$ with $B(0) = B(1) = 1$.

In [4], the authors provided an integral inequality of Hermite-Hadamard type for preinvex functions via Caputo-Fabrizio fractional integral operator as follows.

Theorem 1.2 Let $f: I = [k_1, k_1 + \mu(k_2, k_1)] \to (0, \infty)$ be a preinvex function on I° and $f \in L[k_1, k_1 + \mu(k_2, k_1)]$. If $\alpha \in [0,1]$, then the following inequality holds:

$$f\left(\frac{2k_1 + \mu(k_2, k_1)}{2}\right) \leq \frac{B(\alpha)}{\alpha\mu(k_2, k_1)}\left[{}_{k_1}^{CF}I^{\alpha}\{f(k)\} + {}^{CF}I^{\alpha}_{k_1+\mu(k_2,k_1)}\{f(k)\} - \frac{2(1-\alpha)}{B(\alpha)}\, f(k)\right]$$

$$\leq \frac{f(k_1) + f(k_2)}{2}$$

where $k \in [k_1, k_1 + \mu(k_2, k_1)]$.

For more information on various fractional operators and novel integral inequalities involving these operators, we recommend the following papers to readers: [1]-[21].

Suppose that

$$a, b, c, d > 0, \qquad 0 < \theta < 1,\;\; 0 < \beta < 1, \quad \theta + \beta = 1.$$

If $b < a, \quad d < c$, then

$$a^{\theta} b^{\beta} + c^{\theta} d^{\beta} \leq (a + c)^{\theta} + (b + d)^{\beta} \tag{1}$$

The fundamental inequality for integrable functions (1) can be given as:

Suppose f, g, h, r are integrable positive functions.

$$f(t), g(t), h(t), r(t) > 0, \qquad 0 < \theta < 1,\;\; 0 < \beta < 1, \quad \theta + \beta = 1.$$

If $g(t) < f(t),\; r(t) < h(t)$, then

$$f(t)^{\theta} g(t)^{\beta} + h(t)^{\theta} r(t)^{\beta} \leq \big(f(t) + h(t)\big)^{\theta} + \big(g(t) + r(t)\big)^{\beta}.$$

This inequality is a very basic inequality that holds for real numbers. We will use this inequality to prove our main findings.

The paper demonstrates some new integral inequalities for integrable functions using the Caputo-Fabrizio fractional integral operator. Young's inequality was used in some analysis methods.

2. MAIN RESULTS

Theorem 2.1. Let $I \subseteq R$. Suppose that $f, g, h, r : [a, b] \subseteq I \to R^{+}$ integrable positive functions for $0 < \theta < 1, \theta + \beta = 1$. Then, for the Caputo-Fabrizio fractional integral operator, the following inequality holds:

$$\frac{2(1-\alpha)}{B(\alpha)}(f + h)^{\theta}(k) + \frac{\alpha}{B(\alpha)}\int_a^b [f(t)]^{\theta}.[g(t)]^{\beta}\, dt + \frac{2(1-\alpha)}{B(\alpha)}(g + r)^{\beta}(k)$$

$$+ \frac{\alpha}{B(\alpha)}\int_a^b [h(t)]^{\theta}[r(t)]^{\beta} dt$$

$$\leq \left({}_{a}^{CF}I^{\alpha}(f + h)^{\theta}\right)(k) + \left({}^{CF}I_b^{\alpha}(f + h)^{\theta}\right)(k) + \left({}_{a}^{CF}I^{\alpha}(g + r)^{\beta}\right)(k) + \left({}^{CF}I_b^{\alpha}(g + r)^{\beta}\right)(k)$$

where $B(\alpha) > 0$ is normalization function, $k \in [a, b]$ and $\alpha \in [0,1]$.

Proof. We will start with

$$[f(t)]^{\theta}.[g(t)]^{\beta}+[h(t)]^{\theta}[r(t)]^{\beta}\leq[f(t)+h(t)]^{\theta}+[g(t)+r(t)]^{\beta}.$$

By multiplying both sides of the above inequality with $\frac{\alpha}{B(\alpha)}$, we have

$$\frac{\alpha}{B(\alpha)}[f(t)]^{\theta}.[g(t)]^{\beta}+\frac{\alpha}{B(\alpha)}[h(t)]^{\theta}[r(t)]^{\beta}\leq\frac{\alpha}{B(\alpha)}[f(t)+h(t)]^{\theta}+\frac{\alpha}{B(\alpha)}[g(t)+r(t)]^{\beta}.$$

By integrating both sides of the inequality over $[a,b]$ with respect to t, we get

$$\begin{aligned}&\frac{\alpha}{B(\alpha)}\int_a^b[f(t)]^{\theta}.[g(t)]^{\beta}dt+\frac{\alpha}{B(\alpha)}\int_a^b[h(t)]^{\theta}[r(t)]^{\beta}dt\\&\qquad\leq\frac{\alpha}{B(\alpha)}\int_a^b[f(t)+h(t)]^{\theta}dt+\frac{\alpha}{B(\alpha)}\int_a^b[g(t)+r(t)]^{\beta}dt\end{aligned}$$

and then we get

$$\begin{aligned}&\frac{\alpha}{B(\alpha)}\int_a^b[f(t)]^{\theta}.[g(t)]^{\beta}\,dt+\frac{\alpha}{B(\alpha)}\int_a^b[h(t)]^{\theta}[r(t)]^{\beta}dt\\&\qquad\leq\frac{\alpha}{B(\alpha)}\int_a^k[f(t)+h(t)]^{\theta}dt+\frac{\alpha}{B(\alpha)}\int_k^b[f(t)+h(t)]^{\theta}dt+\frac{\alpha}{B(\alpha)}\int_a^k[g(t)+r(t)]^{\beta}dt\\&\qquad+\frac{\alpha}{B(\alpha)}\int_k^b[g(t)+r(t)]^{\beta}dt.\end{aligned}$$

If we add $\frac{2(1-\alpha)}{B(\alpha)}(f+h)^{\theta}(k)$ and $\frac{2(1-\alpha)}{B(\alpha)}(g+r)^{\beta}(k)$ to both sides, we provide

$$\begin{aligned}&\frac{2(1-\alpha)}{B(\alpha)}(f+h)^{\theta}(k)+\frac{\alpha}{B(\alpha)}\int_a^b[f(t)]^{\theta}.[g(t)]^{\beta}\,dt+\frac{2(1-\alpha)}{B(\alpha)}(g+r)^{\beta}(k)\\&\qquad+\frac{\alpha}{B(\alpha)}\int_a^b[h(t)]^{\theta}[r(t)]^{\beta}dt\\&\qquad\leq\frac{(1-\alpha)}{B(\alpha)}(f+h)^{\theta}(k)+\frac{\alpha}{B(\alpha)}\int_a^k[f(t)+h(t)]^{\theta}dt+\frac{(1-\alpha)}{B(\alpha)}(f+h)^{\theta}(k)\\&\qquad+\frac{\alpha}{B(\alpha)}\int_k^b[f(t)+h(t)]^{\theta}dt+\frac{(1-\alpha)}{B(\alpha)}(g+r)^{\beta}(k)+\frac{\alpha}{B(\alpha)}\int_a^k[g(t)+r(t)]^{\beta}dt\\&\qquad+\frac{(1-\alpha)}{B(\alpha)}(g+r)^{\beta}(k)+\frac{\alpha}{B(\alpha)}\int_k^b[g(t)+r(t)]^{\beta}dt.\end{aligned}$$

We obtain

$$\frac{2(1-\alpha)}{B(\alpha)}(f+h)^{\theta}(k)+\frac{\alpha}{B(\alpha)}\int_a^b [f(t)]^{\theta}.[g(t)]^{\beta}\,dt+\frac{2(1-\alpha)}{B(\alpha)}(g+r)^{\beta}(k)$$
$$+\frac{\alpha}{B(\alpha)}\int_a^b [h(t)]^{\theta}[r(t)]^{\beta}dt$$
$$\leq \left({}_{a}^{CF}I^{\alpha}(f+h)^{\theta}\right)(k)+\left({}^{CF}I_b^{\alpha}(f+h)^{\theta}\right)(k)+\left({}_{a}^{CF}I^{\alpha}(g+r)^{\beta}\right)(k)+\left({}^{CF}I_b^{\alpha}(g+r)^{\beta}\right)(k).$$

which completes the proof.

Corollary 2.2. Under the assumptions of Theorem 2.1, if we choose $k=\frac{a+b}{2}$, then we have the following inequality:

$$\frac{2(1-\alpha)}{B(\alpha)}(f+h)^{\theta}\left(\frac{a+b}{2}\right)+\frac{\alpha}{B(\alpha)}\int_a^b [f(t)]^{\theta}.[g(t)]^{\beta}\,dt+\frac{2(1-\alpha)}{B(\alpha)}(g+r)^{\beta}\left(\frac{a+b}{2}\right)$$
$$+\frac{\alpha}{B(\alpha)}\int_a^b [h(t)]^{\theta}[r(t)]^{\beta}dt$$
$$\leq \left({}_{a}^{CF}I^{\alpha}(f+h)^{\theta}\right)\left(\frac{a+b}{2}\right)+\left({}^{CF}I_b^{\alpha}(f+h)^{\theta}\right)\left(\frac{a+b}{2}\right)+\left({}_{a}^{CF}I^{\alpha}(g+r)^{\beta}\right)\left(\frac{a+b}{2}\right)$$
$$+\left({}^{CF}I_b^{\alpha}(g+r)^{\beta}\right)\left(k\frac{a+b}{2}\right).$$

Corollary 2.3. Under the assumptions of Theorem 2.1, if we choose $\theta=\beta=\frac{1}{2}$, then we have the following inequality:

$$\frac{2(1-\alpha)}{B(\alpha)}(f+h)^{\frac{1}{2}}(k)+\frac{\alpha}{B(\alpha)}\int_a^b \sqrt{f(t)g(t)}\,dt+\frac{2(1-\alpha)}{B(\alpha)}(g+r)^{\frac{1}{2}}(k)+\frac{\alpha}{B(\alpha)}\int_a^b \sqrt{h(t)r(t)}dt$$
$$\leq \left({}_{a}^{CF}I^{\alpha}(f+h)^{\frac{1}{2}}\right)(k)+\left({}^{CF}I_b^{\alpha}(f+h)^{\frac{1}{2}}\right)(k)+\left({}_{a}^{CF}I^{\alpha}(g+r)^{\frac{1}{2}}\right)(k)$$
$$+\left({}^{CF}I_b^{\alpha}(g+r)^{\frac{1}{2}}\right)(k).$$

Theorem 2.4. Let $I\subseteq R$. Suppose that $f,g,h,r:[a,b]\subseteq I\to R^{+}$ integrable positive functions for $0<\theta<1, \theta+\beta=1$. Then, we have following inequality for Caputo-Fabrizio fractional integral operator:

$$\frac{2(1-\alpha)}{B(\alpha)}(f+h)^{p\theta}(k)+\frac{p\alpha}{B(\alpha)}\int_a^b [f(t)]^{\theta}.[g(t)]^{\beta}\,dt$$
$$+\frac{2(1-\alpha)}{B(\alpha)}(g+r)^{p\beta}(k)+\frac{p\alpha}{B(\alpha)}\int_a^b [h(t)]^{\theta}[r(t)]^{\beta}dt-\frac{2(b-a)}{q}\frac{p\alpha}{B(\alpha)}$$
$$\leq \left({}_{a}^{CF}I^{\alpha}(f+h)^{p\theta}\right)(k)+\left({}^{CF}I_b^{\alpha}(f+h)^{p\theta}\right)(k)+\left({}_{a}^{CF}I^{\alpha}(g+r)^{p\beta}\right)(k)$$
$$+\left({}^{CF}I_b^{\alpha}(g+r)^{p\beta}\right)(k)$$

where $B(\alpha) > 0$ is normalization function, $q > 1, \frac{1}{q} + \frac{1}{p} = 1, k \in [a, b]$ and $\alpha \in [0,1]$.

Proof. We can write

$$[f(t)]^{\theta}.[g(t)]^{\beta} + [h(t)]^{\theta}[r(t)]^{\beta} \leq [f(t) + h(t)]^{\theta} + [g(t) + r(t)]^{\beta}.$$

By multiplying both sides of the above inequality with $\frac{p\alpha}{B(\alpha)}$, we get

$$\frac{p\alpha}{B(\alpha)}[f(t)]^{\theta}.[g(t)]^{\beta} + \frac{p\alpha}{B(\alpha)}[h(t)]^{\theta}[r(t)]^{\beta} \leq \frac{p\alpha}{B(\alpha)}[f(t) + h(t)]^{\theta} + \frac{p\alpha}{B(\alpha)}[g(t) + r(t)]^{\beta}.$$

By integrating both sides of the inequality over $[a, b]$ with respect to t, we obtain

$$\frac{p\alpha}{B(\alpha)}\int_a^b [f(t)]^{\theta}.[g(t)]^{\beta} dt + \frac{p\alpha}{B(\alpha)}\int_a^b [h(t)]^{\theta}[r(t)]^{\beta} dt$$
$$\leq \frac{p\alpha}{B(\alpha)}\int_a^b [f(t) + h(t)]^{\theta} dt + \frac{p\alpha}{B(\alpha)}\int_a^b [g(t) + r(t)]^{\beta} dt.$$

If we apply the Young's inequality to the right -hand side of the inequality, we get

$$\frac{p\alpha}{B(\alpha)}\int_a^b [f(t)]^{\theta}.[g(t)]^{\beta}\, dt + \frac{p\alpha}{B(\alpha)}\int_a^b [h(t)]^{\theta}[r(t)]^{\beta} dt$$
$$\leq \frac{p\alpha}{B(\alpha)}\left(\frac{1}{p}\int_a^k [f(t) + h(t)]^{p\theta} dt + \frac{1}{q}\int_a^k 1^q dt + \frac{1}{p}\int_k^b [f(t) + h(t)]^{p\theta} dt + \frac{1}{q}\int_k^b 1^q dt\right)$$
$$+ \frac{p\alpha}{B(\alpha)}\left(\frac{1}{p}\int_a^k [g(t) + r(t)]^{p\beta} dt + \frac{1}{q}\int_a^k 1^q dt + \frac{1}{p}\int_k^b [g(t) + r(t)]^{p\beta} dt + \frac{1}{q}\int_k^b 1^q dt\right)$$
$$= \frac{p\alpha}{B(\alpha)}\left(\frac{1}{p}\int_a^k [f(t) + h(t)]^{p\theta} dt + \frac{k-a}{q} + \frac{1}{p}\int_k^b [f(t) + h(t)]^{p\theta} dt + \frac{b-k}{q}\right)$$
$$+ \frac{p\alpha}{B(\alpha)}\left(\frac{1}{p}\int_a^k [g(t) + r(t)]^{p\beta} dt + \frac{k-a}{q} + \frac{1}{p}\int_k^b [g(t) + r(t)]^{p\beta} dt + \frac{b-k}{q}\right).$$

This implies,

$$\frac{p\alpha}{B(\alpha)}\int_a^b [f(t)]^{\theta}[g(t)]^{\beta}\, dt + \frac{p\alpha}{B(\alpha)}\int_a^b [h(t)]^{\theta}[r(t)]^{\beta} dt - \frac{2(b-a)}{q}\frac{p\alpha}{B(\alpha)}$$
$$\leq \frac{p\alpha}{B(\alpha)}\left(\frac{1}{p}\int_a^k [f(t) + h(t)]^{p\theta} dt + \frac{1}{p}\int_k^b [f(t) + h(t)]^{p\theta} dt\right)$$
$$+ \frac{p\alpha}{B(\alpha)}\left(\frac{1}{p}\int_a^k [g(t) + r(t)]^{p\beta} dt + \frac{1}{p}\int_k^b [g(t) + r(t)]^{p\beta} dt\right).$$

If we add $\frac{2(1-\alpha)}{B(\alpha)}(f + h)^{p\theta}(k)$ and $\frac{2(1-\alpha)}{B(\alpha)}(g + r)^{p\beta}(k)$ to both sides of the inequality,

$$
\begin{aligned}
&\frac{p\alpha}{B(\alpha)}\int_a^b [f(t)]^{\theta}[g(t)]^{\beta}\,dt+\frac{p\alpha}{B(\alpha)}\int_a^b [h(t)]^{\theta}[r(t)]^{\beta}dt-\frac{2(b-a)}{q}\frac{p\alpha}{B(\alpha)}+\frac{2(1-\alpha)}{B(\alpha)}(f+h)^{p\theta}(k)\\
&\quad+\frac{2(1-\alpha)}{B(\alpha)}(g+r)^{p\beta}(k)\\
&\quad\le\left(\frac{(1-\alpha)}{B(\alpha)}(f+h)^{p\theta}(k)+\frac{\alpha}{B(\alpha)}\int_a^k [f(t)+h(t)]^{p\theta}dt\right)\\
&\quad+\left(\frac{(1-\alpha)}{B(\alpha)}(f+h)^{p\theta}(k)+\frac{\alpha}{B(\alpha)}\int_k^b [f(t)+h(t)]^{p\theta}dt\right)\\
&\quad+\left(\frac{(1-\alpha)}{B(\alpha)}(g+r)^{p\beta}(k)+\frac{\alpha}{B(\alpha)}\int_a^k [g(t)+r(t)]^{p\beta}dt\right)\\
&\quad+\left(\frac{(1-\alpha)}{B(\alpha)}(g+r)^{p\beta}(k)+\frac{\alpha}{B(\alpha)}\int_k^b [g(t)+r(t)]^{p\beta}dt\right).
\end{aligned}
$$

Therefore, we conclude

$$
\begin{aligned}
&\frac{p\alpha}{B(\alpha)}\int_a^b [f(t)]^{\theta}[g(t)]^{\beta}\,dt+\frac{p\alpha}{B(\alpha)}\int_a^b [h(t)]^{\theta}[r(t)]^{\beta}dt-\frac{2(b-a)}{q}\frac{p\alpha}{B(\alpha)}+\frac{2(1-\alpha)}{B(\alpha)}(f+h)^{p\theta}(k)\\
&\quad+\frac{2(1-\alpha)}{B(\alpha)}(g+r)^{p\beta}(k)\\
&\quad\le\left({}_{a}^{CF}I^{\alpha}(f+h)^{p\theta}\right)(k)+\left({}^{CF}I_b^{\alpha}(f+h)^{p\theta}\right)(k)+\left({}_{a}^{CF}I^{\alpha}(g+r)^{p\beta}\right)(k)\\
&\quad+\left({}^{CF}I_b^{\alpha}(g+r)^{p\beta}\right)(k)
\end{aligned}
$$

This completes the proof.

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