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Solutions to Differential-Differential Difference Equations with Variable Coefficients by Using Fourier Transform Method

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Abstract: In this paper, differential-differential difference equations with variable coefficients have been solved using the Fourier Transform Method (FTM). In addition, new definitions and theorems are introduced. Besides, the efficiency of the proposed method is verified by solving five important examples. Furthermore, we have noted that the Fourier transform method is a powerful technique for solving ordinary differential difference equations (ODDEs) with variable coefficients. It involves transforming the ODDEs into the frequency domain using the Fourier transform, solving the transformed equation, and then applying the inverse Fourier transform to obtain the solution in the time domain.

Key words: Linear differential equation, Variable coefficients, Fourier transform, Dirac delta function

1. Introduction

Differential equations are a fundamental concept in mathematics that describes how a function changes over time or in relation to other variables. They are widely used in various scientific fields, such as physics, engineering, economics, and biology, to model and understand natural phenomena, processes, and many other phenomena [7,9]. By studying the behavior and solutions of differential equations, we can gain insights into the underlying dynamics of these systems and make predictions about their future behavior. Differential difference equations combine both differential equations and difference equations, and they involve both continuous and discrete time variables. Solving such equations can be challenging, and various methods have been developed to address them. Here are some commonly used approaches: Laplace Transform, Elzaki transform [1], Taylor polynomial method [6], Mahgoub transform [2], differential transform method [8], and Generalized differential transform method [10].

In this article, we will solve ordinary differential- differential difference equations with variable coefficients given by the following formula :

$$\sum_{i=0}^{n}\sum_{k=0}^{m} C_{ik}(x)y^{(i)}(x-\mu_{ik}) = q(x)\,,\mu_{ik} \geq 0$$

The organization of the article is as follows: In the second section, new concepts and theories related to the proposed method were given. In the third section, five examples of differential equations and differential difference equations were solved, and finally the conclusion in the fourth section

2. Definitions and Theorems of Fourier Transform

Definition 2.1. [3] The Fourier transform of $f(t)$ is given by

$$\mathcal{F}[f(t)] = F(w) = \int_{-\infty}^{\infty} f(t).\, e^{-iwt} dt \qquad (1)$$

Definition 2.2. [3] The inverse Fourier transform of $F(w)$ is given by

$$f(t) = \mathcal{F}^{-1}[F(w)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)\, e^{iwt} dw \qquad (2)$$

Theorem2.1. If $a, b \in R$. Then

$$\mathcal{F}[af_1(t) + bf_2(t)] = a\mathcal{F}[f_1(t)] + b\mathcal{F}[f_2(t)]$$

That means: The Fourier Transform is a linear combination.

Theorem2.2. [4] Let $f(t)$ be continuous or partly continuous in $(-\infty, \infty)$ and $f(t), f'(t), f''(t), \dots, f^{(n-1)}(t) \to 0$ for $|t| \to \infty$.

Also, If $f(t), f'(t), f''(t), \dots, f^{(n-1)}(t)$ are absolutely integrable in $(-\infty, \infty)$, then

$$\mathcal{F}\left[f^{(n)}(t)\right] = (iw)^n \mathcal{F}[f(t)] \qquad (3)$$

Definition 2.3. [5] The Dirac delta distribution is limit for $\varepsilon \to 0$ function defined by

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}, & 0 < t < \varepsilon \\ 0, & t < 0 \\ 0, & t > \varepsilon \end{cases}$$

That is $\delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t)$.

And it's properties are given by

i. $\int_{-\infty}^{\infty} \delta(t) dt = 1$

ii. $\int_{-\infty}^{\infty} f(t)\delta(t - t_0) dt = f(t_0)$

iii. $\int_{-\infty}^{\infty} f(t)\delta^{(n)}(t - t_0) dt = (-1)^n f^{(n)}(t_0)$

iv. $\delta^{(n)}(w - w_0) = n!\,(-1)^n \frac{\delta(w-w_0)}{(w-w_0)^n}$

v. $\int_{-\infty}^{\infty} \frac{\delta(w-w_0) f(w)}{(w-w_0)^n} dw = \frac{1}{n!} \frac{d^n f(w)}{dw^n} |_{(w=w_0)}$,

where $\delta(w - w_0)$ is given by

$$\delta(w - w_0) = \begin{cases} 0, & w \neq w_0 \\ \infty, & w = w_0 \end{cases} \qquad (4)$$

Definition 2.4. [4] The Heaviside function is defined by

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{5}$$

Lemma 2.1. $H'(t) = \delta(t)$

Theorem 2.3. The Fourier transform of $\delta(t)$ is 1. That means $\mathcal{F}[\delta(t)] = 1$

Theorem 2.4.[3,4] The Fourier transforms for some functions are following

I. $\mathcal{F}[1] = 2\pi\delta(w)$
II. $\mathcal{F}[t^n] = 2\pi i^n \delta^{(n)}(w)$
III. $\mathcal{F}[t^n f(t)] = i^n \frac{d^n \mathcal{F}[f(t)]}{dw^n}$
IV. $\mathcal{F}\left[e^{iw_0 t}\right] = 2\pi\delta(w - w_0)$.
V. If $\mathcal{F}[f(t)] = F(w)$, then $\mathcal{F}\left[e^{iw_0 t} f(t)\right] = F(w - w_0)$
VI. $\mathcal{F}[e^{at}] = 2\pi\delta(w + ia)$
VII. If $\mathcal{F}[f(t)] = F(w)$, then $\mathcal{F}[e^{at} f(t)] = F(w + ia)$
VIII. If $\mathcal{F}[f(t)] = F(w)$, then $\mathcal{F}[f(t - t_0)] = e^{-iwt_0} F(w)$

Theorem 2.5. If Fourier transform of $f(t)$ is $F(w)$ then

a. $\int_{-\infty}^{t} f(u)du = -i\mathcal{F}^{-1}\left[\frac{F(w)}{w}\right]$

b. $\int_{-\infty}^{u} F(u)du = i\mathcal{F}\left[\frac{f(t)}{t}\right]$

To prove a. we use the definition of the inverse Fourier transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{iwt}dw$$

By taking integral on both sides from $-\infty$ to t, we have

$$\int_{-\infty}^{t} f(u)\, du = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)\frac{e^{iwt}}{iw}dw = -i\mathcal{F}^{-1}\left[\frac{F(w)}{w}\right]$$

To prove b. we use the definition of the Fourier transform

$$F(w) = \int_{-\infty}^{\infty} f(t)e^{-iwt}dt$$

By taking integral on both sides from $-\infty$ to w, we obtain

$$\int_{-\infty}^{w} F(s)\, ds = \int_{-\infty}^{\infty} f(t)\left(\int_{-\infty}^{w} e^{-iut}\, du\right)dt = i\int_{-\infty}^{\infty}\frac{f(t)}{t}.e^{-iwt}dt = i\mathcal{F}\left[\frac{f(t)}{t}\right]$$

Theorem2.6. Let $w > 0$, then

a. $\mathcal{F}\left[\frac{1}{t}\right] = -2\pi i H(w)$

b. $\mathcal{F}\left[\frac{1}{t^2}\right] = -2\pi w$

c. $\mathcal{F}\left[\frac{1}{t^3}\right] = \pi i w^2$

d. $\mathcal{F}\left[\frac{1}{t^4}\right] = \pi\frac{w^3}{3}$

e. $\mathcal{F}\left[\frac{1}{t^5}\right] = -\pi i\frac{w^4}{12}$

f. $\mathcal{F}[lnt] = -2\pi\frac{H(w)}{w}$

Proof a: $\mathcal{F}\left[\frac{1}{t}\right] = \frac{1}{i}\int_{-\infty}^{w}\mathcal{F}(1)du = -i\int_{-\infty}^{w}2\pi\delta(u)du = -2\pi iH(w)$

Proof b: $\mathcal{F}\left[\frac{1}{t^2}\right] = -i\int_{-\infty}^{w}\mathcal{F}\left(\frac{1}{t}\right)du = -i\int_{-\infty}^{w}-2\pi iH(u)du \quad = -2\pi w$

Proof c: $\mathcal{F}\left[\frac{1}{t^3}\right] = -i\int_{-\infty}^{w}\mathcal{F}\left(\frac{1}{t^2}\right)du = -i\int_{-\infty}^{w}-2\pi u du = \pi i w^2$

Proof f: We know that $\mathcal{F}[f'(t)] = iw\mathcal{F}[f(t)]$, and $(lnt)' = \frac{1}{t}$

We have $\mathcal{F}\left[\frac{1}{t}\right] = iw\mathcal{F}[lnt]$

Then we have $\mathcal{F}[lnt] = \frac{1}{iw}\mathcal{F}\left[\frac{1}{t}\right] = \frac{-2\pi H(w)}{w}$

Moreover this theorem can be seen with other way.

By using Theorems 2.5. a, 2.6. a, we have

$$\mathcal{F}^{-1}[H(w)] = \frac{1}{-2\pi it}$$

$$\mathcal{F}^{-1}\left[\frac{H(w)}{w}\right] = \frac{i}{-2\pi i}\int_{-\infty}^{t}\frac{1}{u}du = \frac{lnt}{-2\pi}$$

Thus

$$\mathcal{F}[lnt] = -2\pi\frac{H(w)}{w}$$

Theorem2.7. $\mathcal{F}^{-1}\left[\frac{H(w)}{w^2}\right] = \frac{1}{2\pi i}(tlnt - t)$

Proof: $\mathcal{F}^{-1}\left[\frac{H(w)}{w^2}\right] = \frac{1}{-i}\int_{-\infty}^{t}\mathcal{F}\left[\frac{H(w)}{w}\right]du = \frac{1}{-i}\int_{-\infty}^{t}\frac{lnu}{-2\pi}du = \frac{1}{2\pi i}(tlnt - t)$

3. Examples of Applying The Fourier Transform on Differential Equations:

In this section five examples are given and exact solution is found using FTM.

Example 3.1: Let's consider the following ordinary differential difference equation

$$y'(x) + y(x-1) = x^2 + 1. \tag{6}$$

Solution. By taking Fourier transform to Eq (6), we get

$$\mathcal{F}\big(y'(x) + y(x-1)\big) = \mathcal{F}(x^2+1)$$

$$\big(iw + e^{-iw}\big)Y = 2\pi i^2 \delta''(w) + 2\pi\delta(w)$$

$$Y = \frac{-2\pi\delta''(w) + 2\pi\delta(w)}{iw + e^{-iw}}.$$

By taking the inverse Fourier Transform of the above equation, we obtain the solution $y(x)$.

$$y = \mathcal{F}^{-1}\left(\frac{-2\pi\delta''(w) + 2\pi\delta(w)}{iw + e^{-iw}}\right)$$

$$= \int_{-\infty}^{\infty} \frac{-\delta''(w)}{iw + e^{-iw}} e^{iwx}\, dw + \int_{-\infty}^{\infty} \frac{\delta(w)}{iw + e^{-iw}} e^{iwx}\, dw$$

$$= -2\int_{-\infty}^{\infty} \frac{\delta(w)}{w^2(iw + e^{-iw})} e^{iwx}\, dw + 1$$

$$= -2\ \frac{1}{2!}\frac{d^2}{dw^2}\left(\frac{e^{iwx}}{iw + e^{-iw}}\right)|_{w=0} + 1 = x^2.$$

Example 3.2: Let find a special solution of the following ordinary differential equation with variable coefficients

$$xy'' - (2x+1)y' + (x+1)y = x^2 e^x. \qquad (7)$$

Solution. By taking Fourier transform to Eq (7), we get

$$\mathcal{F}(xy'' - (2x+1)y' + (x+1)y) = \mathcal{F}(x^2 e^x).$$

Therefore

$$\mathcal{F}(xy'') - 2\mathcal{F}(xy') - \mathcal{F}(y') + \mathcal{F}(xy) + \mathcal{F}(y) = \mathcal{F}(x^2 e^x).$$

$$i\frac{d}{dw}(-w^2Y) - 2i\frac{d}{dw}(iwY) - iwY + i\frac{dY}{dw} + Y = -2\pi\delta''(w+i)$$

$$-2iwY - iw^2Y' + 2Y + 2wY' - iwY + iY' + Y = -2\pi\delta''(w+i)$$

$$(-iw^2 + 2w + i)Y' + (3 - 3iw)Y = -2\pi\delta''(w+i)$$

$$Y' + \frac{3(1-iw)}{-i(w+i)^2}Y = \frac{2\pi\delta''(w+i)}{i(w+i)^2}.$$

The previous equation is a linear differential equation of first order.

$$\lambda = e^{\int \frac{3(1-iw)}{-i(w+i)^2}dw} = e^{\int \frac{3}{w+i}dw} = (w+i)^3$$

$$((w+i)^3 Y)' = \frac{2\pi}{i}(w+i)\delta''(w+i)$$

$$((w+i)^3 Y)' = \frac{4\pi}{i}(w+i)\frac{\delta(w+i)}{(w+i)^2} = -4\pi i\frac{\delta(w+i)}{(w+i)} = 4\pi i\delta'(w+i).$$

Therefore

$$((w+i)^3 Y)' = 4\pi i\delta'(w+i).$$

By taking integral to the above equation, we have

$$(w+i)^3 Y = 4\pi i\delta(w+i)$$

$$Y = \frac{4\pi i\delta(w+i)}{(w+i)^3}.$$

By taking the inverse Fourier Transform of the above equation, we obtain the solution $y(x)$.

$$y = \mathcal{F}^{-1}(Y) = \mathcal{F}^{-1}\left(\frac{-4\pi i\delta(w+i)}{(w+i)^3}\right) = 2i\int_{-\infty}^{\infty}\frac{\delta(w+i)}{(w+i)^3}e^{iwx}dw = 2i\frac{(ix)^3}{6}e^x$$
$$= \frac{x^3}{3}e^x.$$

Example 3.3: Let's consider the following ordinary differential equation

$$y'' - 2y' + y = \frac{e^x}{x^5}. \tag{7}$$

Solution. By taking Fourier transform to Eq (7), we get

$$\mathcal{F}(y'' - 2y' + y) = \mathcal{F}\left(\frac{e^x}{x^5}\right).$$

Therefore

$$\mathcal{F}(y'') - 2\mathcal{F}(y') + \mathcal{F}(y) = \mathcal{F}\left(\frac{e^x}{x^5}\right)$$

$$(-w^2 - 2iw + 1)Y = -\pi i\frac{(w+i)^4}{12}$$

$$Y = \pi i\frac{(w+i)^2}{12}.$$

By taking the inverse Fourier Transform of the above equation, we obtain the solution $y(x)$.

$$y=\mathcal{F}^{-1}(Y)=\mathcal{F}^{-1}\left(\pi i\frac{(w+i)^2}{12}\right)=\frac{i}{2}\int_{-\infty}^{\infty}\frac{(w+i)^2}{12}e^{iwx}dw=\frac{2\pi i}{2}\frac{1}{\pi i}\frac{1}{12}\frac{e^x}{x^3}=\frac{e^x}{12x^3}.$$

Example 3.4: Let find a special solution of the following ordinary differential equation with variable coefficients

$$(1-x)y''+xy'-y=2(x-1)^2e^{-x}. \tag{8}$$

Solution. By taking Fourier transform to Eq (8), we get

$$\mathcal{F}(y''-xy''+xy'-y)=2\mathcal{F}((x-1)^2e^{-x})$$

$$(iw)^2Y-i(-w^2Y)'+i(iwY)'-Y=-4\pi e^{-i(w-i)}\delta''(w-i)$$

$$-w^2Y+iw^2Y'+2iwY-Y-wY'-Y=-4\pi e^{-i(w-i)}\delta''(w-i)$$

$$(iw^2-w)Y'+(-w^2+2iw-2)Y=-4\pi e^{-i(w-i)}\delta''(w-i)$$

$$Y'+\frac{-w^2+2iw-2}{iw^2-w}Y=\frac{-4\pi e^{-i(w-i)}\delta''(w-i)}{iw^2-w}$$

$$\lambda=e^{\int\frac{-w^2+2iw-2}{iw^2-w}dw}=e^{\int\left(i+\frac{2}{w}+\frac{1}{w+i}\right)dw}=w^2(w+i)e^{iw}$$

$$\left(w^2(w+i)e^{iw}Y\right)'=\frac{-4\pi e^{-i(w-i)}\delta''(w-i)}{iw^2-w}w^2(w+i)e^{iw}=4\pi ie^{-1}w\delta''(w-i)$$

$$w^2(w+i)e^{iw}Y=4\pi ie^{-1}\int w\delta''(w-i)dw=4\pi ie^{-1}\left(w\delta'(w-i)-\delta(w-i)\right)$$

$$Y=\frac{4\pi ie^{-1}\left(w\delta'(w-i)-\delta(w-i)\right)}{w^2(w+i)e^{iw}}.$$

By taking the inverse Fourier Transform of the above equation, we obtain the solution $y(x)$.

$$y=2ie^{-1}\int_{-\infty}^{\infty}\frac{\left(w\delta'(w-i)-\delta(w-i)\right)}{w^2(w+i)e^{iw}}e^{iwx}dw$$

$$=2ie^{-1}\int_{-\infty}^{\infty}\frac{w\delta'(w-i)}{w^2(w+i)}e^{iw(x-1)}dw-2ie^{-1}\int_{-\infty}^{\infty}\frac{\delta(w-i)}{w^2(w+i)}e^{iw(x-1)}dw$$

$$=-2ie^{-1}\frac{d}{dw}\left(\frac{e^{iw(x-1)}}{w(w+i)}\right)|_{w=i}-2ie^{-1}\frac{e^{1-x}}{i^2 2i}$$

$$y=-\left(x+\frac{1}{2}\right)e^{-x}+e^{-x}=\left(\frac{1}{2}-x\right)e^{-x}.$$

Example 3.5: Let's consider the following ordinary differential difference equation

$$(1-x)y''+xy'-y=2(x-1)^2e^{-x}. \tag{9}$$

Solution. By taking Fourier transform to Eq (9), we get

$$\mathcal{F}\big(y''(x) + y(x-2)\big) = \mathcal{F}(-x^2 + 2\ )$$

$$\left(e^{-2iw} - w^2\right)Y = -2\pi i^2 \delta''(w) + 4\pi\delta(w)$$

$$Y = \frac{2\pi\delta''(w) + 4\pi\delta(w)}{e^{-2iw} - w^2}.$$

By taking the inverse Fourier Transform of the above equation, we obtain the solution $y(x)$.

$$y = \mathcal{F}^{-1}\left(\frac{2\pi\delta''(w) + 4\pi\delta(w)}{e^{-2iw} - w^2}\right)$$

$$= \int_{-\infty}^{\infty} \frac{\delta''(w)}{e^{-2iw} - w^2} e^{iwx}\, dw + 2\int_{-\infty}^{\infty} \frac{\delta(w)}{e^{-2iw} - w^2} e^{iwx}\, dw$$

$$= 2\int_{-\infty}^{\infty} \frac{\delta(w)}{w^2(e^{-2iw} - w^2)} e^{iwx}\, dw + 2$$

$$= 2\ \frac{1}{2!}\frac{d^2}{dw^2}\left(\frac{e^{iwx}}{e^{-2iw} - w^2}\right)|_{w=0} + 2 = -4x - x^2.$$

4. Conclusion

In conclusion, delving into the realm of solutions for differential-differential difference equations with variable coefficients through the application of the Fourier Transform Method unveils a powerful and versatile approach. The Fourier Transform's ability to seamlessly navigate between the time and frequency domains provides a unique lens through which these complex equations can be unraveled. By transforming the differential-differential difference equations into simpler algebraic expressions in the frequency domain, we gain valuable insights into the system's behavior and characteristics. This method not only simplifies the mathematical complexities but also opens doors to a wide array of analytical tools that facilitate the exploration of solutions. Moreover, the Fourier Transform method shines particularly bright when faced with problems featuring variable coefficients. Its adaptability to changes in coefficients allows for dynamic and nuanced analysis of systems that may exhibit variations over time. This adaptability is crucial in capturing the intricate dynamics of real-world phenomena where coefficients are seldom constant. Also in future studies, we can solve partial differential difference equations and integro differential difference equations using this method.

Authorship contribution statement

M. Düz: Methodology, Investigation, Review and Editing, Formal Analysis; **S. Avezov**: Validation, Formal Analysis, Original Draft Writing; **A. Issa**: Review and Editing, Formal Analysis, Methodology, Original Draft Writing, Validation.

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Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Ethics Committee Approval and Informed Consent

As the authors of this study, we declare that we do not have any ethics committee approval and/or informed consent statement.

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