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A NEW SOFT SET OPERATION: COMPLEMENTARY SOFT BINARY PIECEWISE INTERSECTION (∩) OPERATION

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Abstract: The soft set theory developed by Molodtsov has been applied both theoretically and practically in many fields. It is a useful piece of mathematics for handling uncertainty. Numerous variations of soft set operations have been described and used since its introduction. In this paper, we define a new soft set operation, called complementary soft binary piecewise intersection operation, a unique soft set operation, and examine its basic algebraic properties. Additionally, we aim to contribute to the literature on soft sets by illuminating the relationships between this new soft set operation and other kinds of soft set operations by researching the distribution of this new soft set operation over other soft set operations. Moreover, we prove that the set of all the soft sets with a fixed parameter set together with the complementary soft binary piecewise intersection operation and the soft binary piecewise union operation is a zero-symmetric near-semiring and also a hemiring.

Keywords: Soft sets, Soft set operations, Conditional complements, Near-semiring

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1. Introduction

Due to the existence of some types of uncertainty, we are unable to effectively employ traditional ways to address issues in many domains, including engineering, environmental and health sciences, and economics. Three well-known foundational theories that we could use as mathematical tools to deal with uncertainties are interval mathematics, fuzzy set theory, and probability theory. Molodtsov (1999) proposed Soft Set Theory as a mathematical method to deal with these uncertainties; however this method has limits as well because each of these theories has flaws of its own. Since then, this theory has been applied to a variety of fields, including information systems, decision-making as in Özlü (2022a, 202b), optimization theory, game theory, operations research, measurement theory, and some algebraic structures such as Özlü and Sezgin (2021). The initial contributions to soft set operations were released in Maji et al. (2003) and Pei and Miao (2005). Following this, Ali et al. (2009) introduced and discussed several soft set operations, including restricted and extended soft set operations. The basic traits of soft set operations were discussed by Sezgin and Atagün (2011), and the connections between them were shown. They also investigated and defined the idea of restricted symmetric difference of soft sets. Stojanovic (2021) defined the term "extended symmetric difference of soft sets" and its characteristics were investigated. A brand-new soft set operation called extended difference of soft sets was presented by Sezgin et al (2019). The two main categories into which the operations of soft set theory fall, according to the research, are restricted soft set operations and extended soft set operations.

The inclusive complement and exclusive complement of sets, a novel concept in set theory, were proposed and their relationships were investigatd by Çağman (2021). As a result of the inspiration from this study, certain novel complements of sets were developed in (Sezgin et al., 2023c). Additionally, Aybek (2023) constructed a number of additional restricted and extended soft set operations using these complements to soft set theory. Demirci (2023); Sarılalioğlu, 2023; Akbulut (2023) defined a new type of extended operation and in-depth examined their fundamental characteristics by changing the form of extended soft set operations using the complement at the first and second row of the piecewise function of extended soft set operations. Additionally, Eren and Çalışıcı (2019) created a brand-new class of soft difference operations. Soft binary piecewise operations were defined by Yavuz (2023), who also carefully analyzed their core characteristics. In addition, Sezgin and Demirci (2023), Sezgin and Aybek (2023), Sezgin et al. (2023a), Sezgin and Yavuz (2023) continued their work on soft set operations. They altered the soft binary operation's form by using the complement in the first row of piecewise operations.

This paper contributes to the literature on soft set theory by describing a novel soft set operation, which we call "complementary soft binary piecewise intersection operation". The organization of the paper is as follows: Section 3, definition and an example of the new operation are given. Also the full analysis of the algebraic properties of the new operation, including closure, associativity, unit, inverse element, and abelian property, is then made. In Section 4, to add to the body of knowledge on soft sets, the distributions of complementary soft binary piecewise intersection operation over extended soft set operations, complementary extended soft set operations, soft binary piecewise operations, complementary soft binary piecewise operations and restricted soft set operations are specifically targeted. In the conclusion section, we put into focus the meaning of the study's findings and its potential influence on the field.

2. Preliminaries

Definition 2.1. Let U be the universal set, E be the parameter set, $P(U)$ be the power set of U and $A \subseteq E$. A pair (F, A) is called a soft set over U where F is a set-valued function such that $F: A \rightarrow P(U)$ (Molodtsov, 1999). Throughout this paper, the set of all the soft sets over U is designated by $S_E(U)$. Let A be a fixed subset of E and $S_A(U)$ be the collection of all soft sets over U with the fixed parameter set A. Clearly $S_A(U)$ is a subset of $S_E(U)$.

Definition 2.2. (K, D) is called a relative null soft set (with respect to the parameter set D), denoted by $\emptyset_D$, if $K(t) = \emptyset$ for all $t \in D$ and (K, D) is called a relative whole soft set (with respect to the parameter set D), denoted by U_D if $P(t) = U$ for all $t \in D$. The relative whole soft set U_E with respect to the universe set of parameters E is called the absolute soft set over U (Ali et al., 2009).

Definition 2.3. For two soft sets (K, D) and (R, J), we say that (K, D) is a soft subset of (R, J) and it is denoted by $(K, D) \tilde{\subseteq} (R, J)$, if $D \subseteq J$ and $K(t) \subseteq R(t)$, $\forall t \in D$. Two soft sets (K, D) and (R, J) are said to be soft equal if (K, D) is a soft subset of (R, J) and (R, J) is a soft subset of (K, D) (Pei and Miao, 2005).

Definition 2.4. The relative complement of a soft set (K, D), denoted by $(K, D)^r$, is defined by $(K, D)^r = (K^r, A)$, where $K^r: D \rightarrow P(U)$ is a mapping given by $(K, D)^r = U \backslash K(t)$ for all $t \in D$ (Ali et.al., 2009). From now on, $U \backslash K(t) = [K(t)]'$ will be designated by $K'(t)$ for the sake of designation.

Two conditional complements of sets, the inclusive complement and exclusive complement, were defined in (Çağman, 2021). For the ease of presentation, we represent their complements as + and θ, respectively. These complements, which are binary operations, are defined as follows: Assume that D and J are the two subsets of U. The formulas for the J-inclusive complement of D and J-exclusive complement are D+J= D'∪J and J-exlusive complement of D D⍰J = D'∩J', respectively. Here, U refers to a universe, D' is the complement of D over U. For more information, we refer to (Çağman, 2021). New complements were created as binary operations on sets:Let U have the two subsets D and J.D*J=D'∪J',D⍰J=D∩J, D⍰J=D∪J' are the results (Sezgin et al., 2023b). Aybek (2023) defined new restricted and extended soft set operations and looked at their characteristics by applying these set operations to soft sets.

Soft set operations can be grouped into the following categories in order to provide a summary: If " ∇ " is used to denote the set operations (i.e., here can be ∩, ∪, \, Δ, +,θ,*,λ,γ), then restricted operations, extended operations, complementary extended operations, soft binary piecewise operations, and complementary soft binary piecewise operations are as follows:

Definition 2.5. Let (K, D) and (R, J) be soft sets over U. The restricted ∇ operation of (K, D) and (R, J) is the soft set (S, F), denoted by $(K, D)\nabla_R(R, J) = (S, F)$, where $F = D \cap J \neq \emptyset$ and $\forall t \in F$, $S(t) = K(t)\nabla R(t)$ (Ali et al., 2009; Sezgin and Atagün, 2011; Aybek, 2023).

Definition 2.6. Let (K, D) and (R, J) be soft sets over U. The extended ∇ operation of (K, D) and (R, J) is the soft set (S,F), denoted by, $(K, D)\nabla_\varepsilon(R, J) = (S, F)$, where $F = D \cup J$ and $\forall t \in F$,

$$S(t) = \begin{cases} K(t), & t \in D\backslash J, \\ R(t), & t \in J\backslash D, \\ K(t)\nabla R(t), & t \in D \cap J. \end{cases}$$

(Maji et al., 2003; Ali et al., 2009; Sezgin et al., 2019; Stojanovic, 2021; Aybek, 2023).

Definition 2.7. Let (K, D) and (R, J) be soft sets over U. The complementary extended ∇ operation of (K, D) and (R, J) is the soft set (S,F), denoted by, $(K, D) \overset{*}{\nabla}_\varepsilon (R, J) = (S, F)$, where $F = D \cup J$ and $\forall t \in F$,

$$S(t) = \begin{cases} K'(t), & t \in D\backslash J, \\ R'(t), & t \in J\backslash D, \\ K(t)\nabla R(t), & t \in D \cap J. \end{cases}$$

(Sarıalioğlu, 2023; Demirci, 2023; Akbulut, 2023).

Definition 2.8. Let (K, D) and (R, J) be soft sets over U. The soft binary piecewise ∇ operation of (K, D) and (R, J) is the soft set (S,D), denoted by $(P, D)\underset{\nabla}{\sim}(R, J) = (S, D)$, where $\forall t \in D$,

$$S(t) = \begin{cases} K(t), & t \in D\backslash J \\ K(t)\nabla R(t), & t \in D \cap J \end{cases}$$

(Eren and Çalışıcı, 2019; Yavuz, 2023).

Definition 2.9. Let (K, D) and (R, J) be soft sets over U. The complementary soft binary piecewise ∇ operation of (K, D) and (R, J) is the soft set (S,D), denoted by $(K, D) \underset{\nabla}{\overset{*}{\sim}} (R, J) = (S, D)$, where $\forall t \in D$;

$$S(t)=\begin{cases} K'(t), & t\in D\setminus J \\ K(t)\ \nabla R(t), & t\in D\cap J \end{cases}$$

(Sezgin and Demirci, 2023; Sezgin and Aybek, 2023; Sezgin et al., 2023a; Sezgin and Yavuz, 2023)

Hoorn and Rootselaar (1967) discussed general theory of near-semirings. In mathematics, a near-semiring, also called a seminearring, is an algebraic structure more general than a near-ring or a semiring. Near-semirings arise naturally from functions on monoids. Near-semirings are a common abstraction of semirings and near-rings. An algebraic system (R, +, ·) is said to be a right (resp., left) near-semiring if R is a an additive monoid with identity 0 (not necessarily commutative) under addition, semigroup with respect to multiplication, satisfying right (resp., left) distributive law $(a+b)\cdot c = a\cdot c + b\cdot c$ (resp., $c\cdot(a + b) = c\cdot a + c\cdot b$), $\forall$ a, b, c $\in$ R and $0\cdot a = 0$, $\forall$ a $\in$ R (Accordingly 0 is a one-sided (right or left, respectively) absorbing element for the multitplication operation). In addition, if $0\cdot a = a\cdot 0 = 0$ for all a $\in$ R, then we call it a zero-symmetric near-semiring (or seminear-rings). For more about near-rings and ideals of near-ring and N-ideals, we refer to (Pilz, 1977; Taşdemir et al., 2013; Taşdemir and Taştekin, 2019)

The standard examples of near-semirings are typically of the form M(Γ), the set of all mappings on a monoid (Γ, +, 0), equipped with composition of mappings, pointwise addition of mappings, and the zero function. Subsets of M(Γ) closed under the operations provide further examples of near-semirings.

3. Complementary Soft Binary Piecewise Intersection Operation and Its Properties

Definition 3.1. Let (F, A) and (G, B) be soft sets over U. The complementary soft binary piecewise intersection (∩) operation of (F, A) and (G, B) is the soft set (H,A), denoted by, $(F, A)\overset{*}{\underset{\cap}{\sim}}(G, B) = (H, A)$, where $\forall\omega\in A$,

$$H(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus B \\ F(\omega)\cap G(\omega), & \omega\in A\cap B \end{cases}$$

Example 3.2. Let $E=\{e_1,e_2,e_3,e_4\}$ be the parameter set $A=\{e_1, e_3\}$ and $B=\{e_2, e_3, e_4\}$ be the subsets of E and $U=\{h_1,h_2,h_3,h_4,h_5\}$ be the initial universe set. Assume that (F,A) and (G,B) are the soft sets over U defined as following:

$(F,A)=\{(e_1,\{h_2,h_5\}),(e_3,\{h_1,h_2,h_5\})\}$,

$(G,B)=\{(e_2,\{h_1,h_4,h_5\}), (e_3,\{h_2,h_3,h_4\}),(e_4,\{h_3,h_5\})\}$.

Let $(F,A)\overset{*}{\underset{\cap}{\sim}}(G,B)=(H,A)$. Then,

$$H(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus B \\ F(\omega)\cap G(\omega), & \omega\in A\cap B \end{cases}$$

Since $A\setminus B=\{e_1\}$ and $A\cap B=\{e_3\}$, $H(e_1)=F'(e_1)=\{h_1,h_3,h_4\}$, $H(e_3)=F(e_3)\cap G(e_3)=\{h_1, h_2,h_5\}\cap\{h_2,h_3,h_4\}=\{h_2\}$. Thus, $(F,A)\overset{*}{\underset{\cap}{\sim}}(G,B)=\{(e_1,\{h_1,h_3,h_4\}), (e_3, \{h_2\})\}$.

Theorem 3.3. (Algebraic properties of the operation)

1) The set $S_E(U)$ is closed under the operation $\overset{*}{\underset{\cap}{\sim}}$. That is, when (F,A) and (G,X) are two soft sets over U, then so is $(F,A)\overset{*}{\underset{\cap}{\sim}}(G,X)$.

Proof: It is clear that $\overset{*}{\underset{\cap}{\sim}}$ is a binary operation in $S_E(U)$.

That is,

$$\overset{*}{\underset{\cap}{\sim}}: S_E(U)\times S_E(U)\rightarrow S_E(U)$$

$$((F,A), (G,X)) \rightarrow (F,A)\overset{*}{\underset{\cap}{\sim}}(G,X)=(H,A)$$

Hence, the set $S_E(U)$ is closed under the operation $\overset{*}{\underset{\cap}{\sim}}$.

2) $[(F,A)\overset{*}{\underset{\cap}{\sim}}(G,A)]\overset{*}{\underset{\cap}{\sim}}(H,A)=(F,A)\overset{*}{\underset{\cap}{\sim}}[(G,A)\overset{*}{\underset{\cap}{\sim}}(H,A)]$.

Proof: Let $(F,A)\overset{*}{\underset{\cap}{\sim}}(G,A)=(T,A)$, where $\forall\omega\in A$;

$$T(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus A=\emptyset \\ F(\omega)\cap G(\omega), & \omega\in A\cap A=A \end{cases}$$

Let $(T,A)\overset{*}{\underset{\cap}{\sim}}(H,A)=(M,A)$, where $\forall\omega\in A$;

$$M(\omega)=\begin{cases} T'(\omega), & \omega\in A\setminus A=\emptyset \\ T(\omega)\cap H(\omega), & \omega\in A\cap A=A \end{cases}$$

Thus,

$$M(\omega)=\begin{cases} T'(\omega), & \omega\in A\setminus A=\emptyset \\ [F(\omega)\cap G(\omega)]\cap H(\omega), & \omega\in A\cap A=A \end{cases}$$

Let $(G,A)\overset{*}{\underset{\cap}{\sim}}(H,A)=(L,A)$, where $\forall\omega\in A$;

$$L(\omega)=\begin{cases} G'(\omega), & \omega\in A\setminus A=\emptyset \\ G(\omega)\cap H(\omega), & \omega\in A\cap A=A \end{cases}$$

Let (F,A) $\overset{*}{\underset{\cap}{\sim}}$ (L,A) =(N,A), where $\forall\omega\in A$;

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus A=\emptyset \\ F(\omega)\cap L(\omega), & \omega\in A\cap A=A \end{cases}$$

Thus,

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus A=\emptyset \\ F(\omega)\cap[G(\omega)\cap H(\omega)], & \omega\in A\cap A=A \end{cases}$$

It is seen that M=N.

That is, for the soft sets whose parameter sets are the same, the operation $\overset{*}{\underset{\cap}{\sim}}$ has associativity property. However we have the following:

3) [(F,A) $\overset{*}{\underset{\cap}{\sim}}$ (G,X)] $\overset{*}{\underset{\cap}{\sim}}$ (H,L) ≠ (F,A) $\overset{*}{\underset{\cap}{\sim}}$ [(G,X) $\overset{*}{\underset{\cap}{\sim}}$ (H,L)]

Proof: Let (F,A) $\overset{*}{\underset{\cap}{\sim}}$ (G,X)=(T,A), where $\forall\omega\in A$;

$$T(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus X \\ F(\omega)\cap G(\omega), & \omega\in A\cap X \end{cases}$$

Let (T,A) $\overset{*}{\underset{\cap}{\sim}}$ (H,L) =(M,A), where $\forall\omega\in A$;

$$M(\omega)=\begin{cases} T'(\omega), & \omega\in A\setminus L \\ T(\omega)\cap H(\omega), & \omega\in A\cap L \end{cases}$$

Thus,

$$M(\omega)=\begin{cases} F(\omega), & \omega\in(A\setminus X)\setminus L=A\cap X'\cap L' \\ F'(\omega)\cup G'(\omega), & \omega\in(A\cap X)\setminus L=A\cap X\cap L' \\ F'(\omega)\cap H(\omega), & \omega\in(A\setminus X)\cap L=A\cap X'\cap L \\ [F(\omega)\cap G(\omega)]\cap H(\omega), & \omega\in(A\cap X)\cap L=A\cap X\cap L \end{cases}$$

Let (G,X) $\overset{*}{\underset{\cap}{\sim}}$ (H,L)=(K,X), where $\forall\omega\in X$;

$$K(\omega)=\begin{cases} G'(\omega), & \omega\in X\setminus L \\ G(\omega)\cap H(\omega), & \omega\in X\cap L \end{cases}$$

Let (F,A) $\overset{*}{\underset{\cap}{\sim}}$ (K,X) =(S,A), where $\forall\omega\in A$;

$$S(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus X \\ F(\omega)\cap K(\omega), & \omega\in A\cap X \end{cases}$$

Thus,

$$S(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus X \\ F(\omega)\cap G'(\omega), & \omega\in A\cap(X\setminus L)=A\cap X\cap L' \\ F(\omega)\cap[G(\omega)\cap H(\omega)], & \omega\in A\cap(X\cap L)=A\cap X\cap L \end{cases}$$

Here let handle $\omega\in A\setminus X$ in the second equation of the first line. Since A\X= A∩X', if k∈X', then $\omega\in L\setminus X$ or $\omega\in(X\cup L)'$. Hence, if $\omega\in A\setminus X$, then $\omega\in A\cap X'\cap L'$ or $\omega\in A\cap X'\cap L$. Thus, it is seen that M≠S. That is, the operation $\overset{*}{\underset{\cap}{\sim}}$ has not associativity property on the set $S_E(U)$.

4) (F,A) $\overset{*}{\underset{\cap}{\sim}}$ (G,X)≠(G,X) $\overset{*}{\underset{\cap}{\sim}}$ (F,A)

Proof: Let (F,A) $\overset{*}{\underset{\cap}{\sim}}$ (G,X)=(H,A) . Then, $\forall\omega\in A$;

$$H(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus X \\ F(\omega)\cap G(\omega), & \omega\in A\cap X \end{cases}$$

Let (G,X) $\overset{*}{\underset{\cap}{\sim}}$ (F,A)=(T,X) . Then $\forall\omega\in X$;

$$T(\omega)=\begin{cases} G'(\omega), & \omega\in X\setminus A \\ G(\omega)\cap F(\omega), & \omega\in X\cap A \end{cases}$$

Here, while the parameter set of the soft set of the left hand side is A; the parameter set of the soft set of the right hand side is X. Thus, (F,A) $\overset{*}{\underset{\cap}{\sim}}$ (G,X)≠(G,X) $\overset{*}{\underset{\cap}{\sim}}$ (F,A). Hence, the operation $\overset{*}{\underset{\cap}{\sim}}$ has not commutative property in the set $S_E(U)$. However it is easy to see that (F,A) $\overset{*}{\underset{\cap}{\sim}}$ (G,A)=(G,A) $\overset{*}{\underset{\cap}{\sim}}$ (F,A). That is to say, the operation $\overset{*}{\underset{\cap}{\sim}}$ has commutative property, where the parameter sets of the soft sets are the same.

5) (F,A) $\overset{*}{\underset{\cap}{\sim}}$ (F,A)=(F,A)

Proof: Let (F,A) $\overset{*}{\underset{\cap}{\sim}}$ (F,A)=(H,A), where $\forall\omega\in A$;

$$H(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus A=\emptyset \\ F(\omega)\cap F(\omega), & \omega\in A\cap A=A \end{cases}$$

Here $\forall\omega\in A$; H(ω)= F(ω) ∩F(ω)=F(ω), thus (H,A)=(F,A).

That is, the operation $\overset{*}{\underset{\cap}{\sim}}$ is idempotent in $S_E(U)$.

6) $(F,A)\overset{*}{\underset{\cap}{\sim}}\emptyset_A=\emptyset_A\overset{*}{\underset{\cap}{\sim}}(F,A)=\emptyset_A$

Proof: Let $\emptyset_A=(S,A)$. Then, $\forall\omega\in A$; $S(\omega)=\emptyset$. Let $(F,A)\overset{*}{\underset{\cap}{\sim}}(S,A)=(H,A)$, where $\forall\omega\in A$,

$$H(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash A=\emptyset \\ F(\omega)\cap S(\omega), & \omega\in A\cap A=A \end{cases}$$

Hence, $\forall\omega\in A$; $H(\omega)=F(\omega)\cap S(\omega)=F(\omega)\cap\emptyset=\emptyset$. Thus, $(H,A)=\emptyset_A$. Note that, for the soft sets whose parameter set is A, $\emptyset_A$ is the absorbing element for the operation $\overset{*}{\underset{\cap}{\sim}}$ in $S_E(U)$.

7) $(F,A)\overset{*}{\underset{\cap}{\sim}}\emptyset_E=\emptyset_A$

Proof: Let $\emptyset_E=(S,E)$. Hence $\forall\omega\in E$; $S(\omega)=\emptyset$. Let $(F,A)\overset{*}{\underset{\cap}{\sim}}(S,E)=(H,A)$. Thus, $\forall\omega\in A$,

$$H(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash E=\emptyset \\ F(\omega)\cap S(\omega), & \omega\in A\cap E=A \end{cases}$$

Hence, $H(\omega)=F(\omega)\cap S(\omega)=F(\omega)\cap\emptyset=\emptyset$, so $(H,A)=(F,A)$.

8) $(F,A)\overset{*}{\underset{\cap}{\sim}}U_A=U_A\overset{*}{\underset{\cap}{\sim}}(F,A)=(F,A)$.

Proof: Let $U_A=(T,A)$. Then, $\forall\omega\in A$; $T(\omega)=U$. Let $(F,A)\overset{*}{\underset{\cap}{\sim}}(T,A)=(H,A)$, where $\forall\omega\in A$;

$$H(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash A=\emptyset \\ F(\omega)\cap T(\omega), & \omega\in A\cap A=A \end{cases}$$

Thus, $\forall\omega\in A$; $H(\omega)=F(\omega)\cap T(\omega)=F(\omega)\cap U=F(\omega)$, hence $(H,A)=(F,A)$. Note that, U_A is the identity element for the operation $\overset{*}{\underset{\cap}{\sim}}$ in $S_A(U)$.

REMARK 1: By Theorem 3.3. (1), (2), (4) and (8), $(S_A(U),\overset{*}{\underset{\cap}{\sim}})$ is a commutative monoid with identity U_A.

9) $(F,A)\overset{*}{\underset{\cap}{\sim}}U_E=(F,A)$

Proof: Let $U_E=(T,E)$. Hence, $\forall\omega\in E$, $T(\omega)=U$. Let $(F,A)\overset{*}{\underset{\cap}{\sim}}(T,E)=(H,A)$, then $\forall\omega\in A$,

$$H(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash E=\emptyset \\ F(\omega)\cap T(\omega), & \omega\in A\cap E=A \end{cases}$$

Hence, $\forall\omega\in A$, $H(\omega)=F(\omega)\cap T(\omega)=F(\omega)\cap U=F(\omega)$, so $(H,A)=(F,A)$. Note that, for all the soft sets (no matter what the parameter set is), U_E is the right identity element for the operation $\overset{*}{\underset{\cap}{\sim}}$ in $S_E(U)$.

10) $U_E\overset{*}{\underset{\cap}{\sim}}(F,A)=\cup_A$

Proof: Let $U_E=(T,E)$. Then, $\forall\omega\in E$; $T(\omega)=U$. Let $(T,E)\overset{*}{\underset{\cap}{\sim}}(F,A)=(H,L)$, where $\forall\omega\in A$;

$$H(\omega)=\begin{cases} T'(\omega), & \omega\in E\backslash A \\ T(\omega)\cap F(\omega), & \omega\in E\cap A=A \end{cases}$$

Hence $\forall\omega\in A$; $H(\omega)=T(\omega)\cap F(\omega)=U\cap F(\omega)=F(\omega)$, thus $(H,A)=(F,A)$.

11) $(F,A)\overset{*}{\underset{\cap}{\sim}}(F,A)^r=(F,A)^r\overset{*}{\underset{\cap}{\sim}}(F,A)=\emptyset_A$

Proof: Let $(F,A)^r=(H,A)$. Hence, $\forall\omega\in A$; $H(\omega)=F'(\omega)$. Let $(F,A)\overset{*}{\underset{\cap}{\sim}}(H,A)=(T,A)$, where $\forall\omega\in A$,

$$T(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash A=\emptyset \\ F(\omega)\cap H(\omega), & \omega\in A\cap A=A \end{cases}$$

Hence, $\forall\omega\in A$; $T(\omega)=F(\omega)\cap H(\omega)=F(\omega)\cap F'(\omega)=\emptyset$, thus $(T,A)=\emptyset_A$.

12) $(F,A)\overset{*}{\underset{\cap}{\sim}}(G,X)]^r=(F,A)\,\tilde{*}\,(G,X)$.

Proof: Let $(F,A)\overset{*}{\underset{\cap}{\sim}}(G,X)=(H,A)$. Then, $\forall\omega\in A$,

$$H(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ F(\omega)\cap G(\omega), & \omega\in A\cap X \end{cases}$$

Let $(H,A)^r=(T,A)$, so $\forall\omega\in A$,

$$T(\omega)=\begin{cases} F(\omega), & \omega\in A\backslash X \\ F'(\omega)\cup G'(\omega), & \omega\in A\cap X \end{cases}$$

Thus, $(T,A)=(F,A)\,\tilde{*}\,(G,X)$.

16) $(F,A)\overset{*}{\underset{\cap}{\sim}}(G,A)=U_A\Leftrightarrow(F,A)=U_A$ and $(G,A)=U_A$.

Proof: Let $(F,A)\overset{*}{\underset{\cap}{\sim}}(G,A)=(T,A)$. Hence, $\forall\omega\in A$,

$$T(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash A=\emptyset \\ F(\omega)\cap G(\omega), & \omega\in A\cap A=A \end{cases}$$

Since, (T,A)= U_A , ∀ω∈A, T(ω)= U. Thus, ∀ω∈A, T(ω)= F(ω)∩G(ω)=U⇔ ∀ω ∈ A, F(ω)= U and G(ω)= U ⇔(F, A) = U_A and (G, A) = U_A.

17) $\emptyset_A \tilde{\subseteq}(F,A) \overset{*}{\underset{\cap}{\sim}} (G,X)$ and $\emptyset_B \tilde{\subseteq}(G,X) \overset{*}{\underset{\cap}{\sim}} ((F,A)$.

18) $(F,A) \overset{*}{\underset{\cap}{\sim}} (G,X) \tilde{\subseteq}\ U_A$ and $(G,X) \overset{*}{\underset{\cap}{\sim}} (F,A) \tilde{\subseteq}\ U_B$

19) $(F,A) \overset{*}{\underset{\cap}{\sim}} (G, A) \tilde{\subseteq} (F,A) \overset{*}{\underset{\cup}{\sim}} (G, A)$ (Yavuz, 2023)

20)$(F,A) \overset{*}{\underset{\cap}{\sim}} (G, A) = (F,A) \overset{*}{\underset{\cup}{\sim}} (G, A) \Leftrightarrow (F, A) = (G, A)$ (Yavuz, 2023)

21) $(F,A) \overset{*}{\underset{\cap}{\sim}} (G,A) \tilde{\subseteq}(F,A)$ and $(F,A) \overset{*}{\underset{\cap}{\sim}} (G,A) \tilde{\subseteq}(G,A)$.

Proof: Let $(F,A) \overset{*}{\underset{\cap}{\sim}} (G,A)=(H,A)$. First of all, A⊆ A. Moreover , ∀ω∈A,

$$H(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash A=\emptyset \\ F(\omega)\cap G(\omega), & \omega\in A\cap A=A \end{cases}$$

Since H(ω)=F(ω)∩G(ω) ⊆ F(ω) and H(ω)=F(ω)∩G(ω) ⊆ G(ω) , ∀ω∈A, the proof is completed.

22) (F, A) ⊆̃ (G, A) ⇔$(F,A) \overset{*}{\underset{\cap}{\sim}} (G,A)= (F,A)$.

Proof: Let (F, A) ⊆̃ (G, A), then, ∀ω∈A, F(ω)⊆ G(ω) and let $(F,A) \overset{*}{\underset{\cap}{\sim}} (G,A)=(H,A)$. Then, ∀ω∈A,

$$H(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash A=\emptyset \\ F(\omega)\cap G(\omega), & \omega\in A\cap A=A \end{cases}$$

Since ∀ω∈A, H(ω)=F(ω)∩G(ω) =F(ω), hence (H,A)= (F,A). For the converse, let $(F,A) \overset{*}{\underset{\cap}{\sim}} (G,A)= (F,A)$. Since, $(F,A) \overset{*}{\underset{\cap}{\sim}} (G,A) \tilde{\subseteq}(G,A)$ by (21), $(F,A) \overset{*}{\underset{\cap}{\sim}} (G,A)= (F,A) \tilde{\subseteq}(G,A)$.

23) If (F,A) ⊆̃ (G,A) and (H,A) ⊆̃ (T,A), then $(F,A) \overset{*}{\underset{\cap}{\sim}} (H,A) \tilde{\subseteq} (G,A) \overset{*}{\underset{\cap}{\sim}} (T,A)$.

Proof: Let (F,A) ⊆̃ (G,A) and (H,A) ⊆̃ (T,A). Then, ∀ ω ∈A, F(ω)⊆ G(k) and H(ω) ⊆ T(k). Assume that $(F,A) \overset{*}{\underset{\cap}{\sim}} (H,A)=$ (K,A) where ∀ω∈A,

$$K(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash A=\emptyset \\ F(\omega)\cap H(\omega), & \omega\in A\cap A=A \end{cases}$$

Let $(G,A) \overset{*}{\underset{\cap}{\sim}} (T,A)= (S,A)$ where ∀w∈A,

$$S(\omega)=\begin{cases} G'(\omega), & \omega\in A\backslash A=\emptyset \\ G(\omega)\cap T(\omega), & \omega\in A\cap A=A \end{cases}$$

Since, ∀ω∈A, F(ω)⊆ G(ω) and H(ω) ⊆T(ω), then K(ω)= F(ω) ∩H(ω) ⊆G(ω) ∩T(ω). Hence, (K,A) ⊆̃ (S,A).

4. Distribution Rules

In this section, the distributions of complementary soft binary piecewise intersection (∩) operations over other soft set operations, including complementary extended soft set operations, complementary soft binary piecewise operations, soft binary piecewise operations, and restricted soft set operations, are thoroughly examined. As a result, several intriguing discoveries are made.

4.1. Distribution of Complementary Soft Binary Piecewise Intersection Operation over Extended Soft Set Operations

4.1.1. Left-distribution of complementary soft binary piecewise intersection operation over extended soft set operations

The followings are held where A∩X'∩L =∅.

1)$(F,A) \overset{*}{\underset{\cap}{\sim}} [(G,X)\cap_\varepsilon(H,L)]=[(F,A) \overset{*}{\underset{\cap}{\sim}} (G,X)]\tilde{\cap}[(H,L) \overset{*}{\underset{\cap}{\sim}} (F,A)]$.

Proof: Let's first take care of the left hand facet of the equality and let (G,X) $\cap_\varepsilon$ (H,L)=(M,X∪L) where ∀ω∈X∪L;

$$M(\omega)=\begin{cases} G(\omega), & \omega\in X\backslash L \\ H(\omega), & \omega\in L\backslash X \\ G(\omega)\cap H(\omega), & \omega\in X\cap L \end{cases}$$

Assume that $(F,A) \overset{*}{\underset{\cap}{\sim}} (M,X\cup L)=(N,A)$, where ∀ω∈A;

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash (X\cup L) \\ F(\omega)\cap M(\omega), & \omega\in A\cap(X\cup L) \end{cases}$$

Hence,

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash (X\cup L)=A\cap X'\cap L' \\ F(\omega)\cap G(\omega), & \omega\in A\cap(X\backslash L)= A\cap X\cap L' \\ F(\omega)\cap H(\omega), & \omega\in A\cap(L\backslash X)= A\cap X'\cap L \\ F(\omega)\cap\ [(G(\omega)\cap H(\omega)], & \omega\in A\cap X\cap L= A\cap X\cap L \end{cases}$$

Now let take care of the right hand facet of the equality: $[(F,A) \overset{*}{\underset{\cap}{\sim}} (G,X)]\tilde{\cap}[(H,L) \overset{*}{\underset{\cap}{\sim}} (F,A)]$. Assume that $(F,A) \overset{*}{\underset{\cap}{\sim}} (G,X)=(V,A)$, where ∀ω∈A;

$$V(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ F(\omega)\cap G(\omega), & \omega\in A\cap X \end{cases}$$

Let $(H,L)\overset{*}{\underset{\cap}{\sim}}(F,A)=(W,L)$, where $\forall\omega\in L$;

$$W(\omega)=\begin{cases} H'(\omega), & \omega\in L\backslash A \\ H(\omega)\cap F(\omega), & \omega\in L\cap A \end{cases}$$

Let $(V,A)\,\tilde{\cap}\,(W,L)=(T,A)$, where $\forall\omega\in A$;

$$T(\omega)=\begin{cases} V(\omega), & \omega\in A\backslash L \\ V(\omega)\cap W(\omega), & \omega\in A\cap L \end{cases}$$

Hence,

$$T(\omega)=\begin{cases} F'(\omega), & \omega\in(A\backslash X)\backslash L=A\cap X'\cap L' \\ F(\omega)\cap G(\omega), & \omega\in(A\cap X)\backslash L=A\cap X\cap L' \\ F'(\omega)\cap H'(\omega), & \omega\in(A\backslash X)\cap(L\backslash A)=\emptyset \\ F'(\omega)\cap[H(\omega)\cap F(\omega)], & \omega\in(A\backslash X)\cap(L\cap A)=A\cap X'\cap L \\ [F(\omega)\cap G(\omega)]\cap H'(\omega), & \omega\in(A\cap X)\cap(L\backslash A)=\emptyset \\ [F(\omega)\cap G(\omega)]\cap[H(\omega)\cap F(\omega)], & \omega\in(A\cap X)\cap(L\cap A)=A\cap X\cap L \end{cases}$$

It is seen that N=T.

2) $(F,A)\overset{*}{\underset{\cap}{\sim}}[(G,X)\cup_{\varepsilon}(H,L)]=[(F,A)\overset{*}{\underset{\cap}{\sim}}(G,X)]\tilde{\cup}[(H,L)\overset{*}{\underset{\cap}{\sim}}(F,A)]$.

3) $(F,A)\overset{*}{\underset{\cap}{\sim}}[(G,X)\lambda_{\varepsilon}(H,L)]=[(F,A)\overset{*}{\underset{\cap}{\sim}}(G,X)]\tilde{\cup}[(H,L)\overset{*}{\underset{\gamma}{\sim}}(F,A)]$.

Proof: Let's first take care of the left hand facet of the equality and let $(G,X)\,\lambda_{\varepsilon}(H,L)=(M,X\cup L)$ where $\forall\omega\in X\cup L$;

$$M(\omega)=\begin{cases} G(\omega), & \omega\in X\backslash L \\ H(\omega), & \omega\in L\backslash X \\ G(\omega)\cup H'(\omega), & \omega\in X\cap L \end{cases}$$

Assume that $(F,A)\overset{*}{\underset{\cap}{\sim}}(M,X\cup L)=(N,A)$, where $\forall\omega\in A$;

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash(X\cup L) \\ F(\omega)\cap M(\omega), & \omega\in A\cap(X\cup L) \end{cases}$$

Thus,

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash(X\cup L)=A\cap X'\cap L' \\ F(\omega)\cap G(\omega), & \omega\in A\cap(X\backslash L)=A\cap X\cap L' \\ F(\omega)\cap H(\omega), & \omega\in A\cap(L\backslash X)=A\cap X'\cap L \\ F(\omega)\cap[G(\omega)\cup H'(\omega)], & \omega\in A\cap X\cap L=A\cap X\cap L \end{cases}$$

Now let take care of the right hand facet of the equality:

$(F,A)\overset{*}{\underset{\cap}{\sim}}(G,X)]\tilde{\cup}[(H,L)\overset{*}{\underset{\gamma}{\sim}}(F,A)]$.

Assume that $(F,A)\overset{*}{\underset{\cap}{\sim}}(G,X)=(V,A)$, where $\forall\omega\in A$;

$$V(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ F(\omega)\cap G(\omega), & \omega\in A\cap X \end{cases}$$

Let $(H,L)\overset{*}{\underset{\gamma}{\sim}}(F,A)=(W,L)$, where $\forall\omega\in L$;

$$W(\omega)=\begin{cases} H'(\omega), & \omega\in L\backslash A \\ H'(\omega)\cap F(\omega), & \omega\in L\cap A \end{cases}$$

Let $(V,A)\,\tilde{\cup}\,(W,L)=(T,A)$, where $\forall\omega\in A$;

$$T(\omega)=\begin{cases} V(\omega), & \omega\in A\backslash L \\ V(\omega)\cup W(\omega), & \omega\in A\cap L \end{cases}$$

$$T(\omega)=\begin{cases} F'(\omega), & \omega\in(A\backslash X)\backslash L=A\cap X'\cap L' \\ F(\omega)\cap G(\omega), & \omega\in(A\cap X)\backslash L=A\cap X\cap L' \\ F'(\omega)\cup H'(\omega), & \omega\in(A\backslash X)\cap(L\backslash A)=\emptyset \\ F'(\omega)\cup[H'(\omega)\cap F(\omega)], & \omega\in(A\backslash X)\cap(L\cap A)=A\cap X'\cap L \\ [F(\omega)\cap G(\omega)]\cup H'(\omega), & \omega\in(A\cap X)\cap(L\backslash A)=\emptyset \\ [F(\omega)\cap G(\omega)]\cup[H'(\omega)\cap F(\omega)], & \omega\in(A\cap X)\cap(L\cap A)=A\cap X\cap L \end{cases}$$

It is seen that N=T.

4) $(F,A)\overset{*}{\underset{\cap}{\sim}}[(G,X)\backslash_{\varepsilon}(H,L)]=[(F,A)\overset{*}{\underset{\cap}{\sim}}(G,X)]\tilde{\cap}[(H,L)\overset{*}{\underset{\gamma}{\sim}}(F,A)]$

4.1.2. Right-distribution of complementary soft binary piecewise intersection operation over extended soft set operations

1) $[(F,A)\cup_{\varepsilon}(G,X)]\overset{*}{\underset{\cap}{\sim}}(H,L)=[(F,A)\overset{*}{\underset{\cap}{\sim}}(H,L)]\cup_{\varepsilon}[(G,X)\overset{*}{\underset{\cap}{\sim}}(H,L)$, where $A\cap X\cap L'=\emptyset$

Proof: Let's first take care of the left hand facet of the equality and let $(F,A)\cup_{\varepsilon}(G,X)=(M,A\cup X)$ where $\forall\omega\in A\cup X$;

$$M(\omega)=\begin{cases} F(\omega), & \omega\in A\backslash X \\ G(\omega), & \omega\in X\backslash A \\ F(\omega)\cup G(\omega), & \omega\in A\cap X \end{cases}$$

Let $(M,A\cup X)\overset{*}{\underset{\cap}{\sim}}(H,L)=(N,A\cup X)$, where $\forall\omega\in A\cup X$

$$N(\omega)=\begin{cases} M'(\omega), & \omega\in(A\cup X)\backslash L \\ M(\omega)\cap H(\omega), & \omega\in(A\cup X)\cap L \end{cases}$$

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in(A\backslash X)\backslash L=A\cap X'\cap L' \\ G'(\omega), & \omega\in(X\backslash A)\backslash L=A'\cap X\cap L' \\ F'(\omega)\cap G'(\omega), & \omega\in(A\cap X)\backslash L=A\cap X\cap L' \\ F(\omega)\cap H(\omega), & \omega\in(A\backslash X)\cap L=A\cap X'\cap L \\ G(\omega)\cap H(\omega), & \omega\in(X\backslash A)\cap L=A'\cap X\cap L \\ [F(\omega)\cup G(\omega)]\cap H(\omega), & \omega\in(A\cap X)\cap L=A\cap X\cap L \end{cases}$$

Now let's take care of the right hand facet of the equality:

$(F,A)\overset{*}{\underset{\cap}{\sim}}(H,L)]\cup_{\varepsilon}[(G,X)\overset{*}{\underset{\cap}{\sim}}(H,L)]$. Let $(F,A)\overset{*}{\underset{\cap}{\sim}}(H,L)=(V,A)$, where $\forall\omega\in A$;

$$V(\omega)=\begin{cases} F'(\omega) & \omega\in A\backslash L \\ F(\omega)\cap H(\omega) & \omega\in A\cap L \end{cases}$$

Suppose that $(G,X)\overset{*}{\underset{\cap}{\sim}}(H,L)=(W,X)$, where $\forall\omega\in X$;

$$W(\omega)=\begin{cases}G'(\omega), & \omega\in X\setminus L\\ G(\omega)\cap H(\omega), & \omega\in X\cap L\end{cases}$$

Let $(V,A)\cup_{\varepsilon}(W,X)=(T,A\cup X)$, where $\forall\omega\in A\cup X$;

$$T(\omega)=\begin{cases}V(\omega), & \omega\in A\setminus X\\ W(\omega), & \omega\in X\setminus A\\ V(\omega)\cup W(\omega), & \omega\in A\cap X\end{cases}$$

Hence,

$$T(\omega)=\begin{cases}F'(\omega), & \omega\in(A\setminus L)\setminus X=A\cap X'\cap L'\\ F(\omega)\cap H(\omega), & \omega\in(A\cap L)\setminus X=A\cap X'\cap L\\ G'(\omega), & \omega\in(X\setminus L)\setminus A=A'\cap X\cap L'\\ G(\omega)\cap H(\omega), & \omega\in(X\cap L)\setminus A=A'\cap X\cap L\\ F'(\omega)\cup G'(\omega), & \omega\in(A\setminus L)\cap(X\setminus L)=A\cap X\cap L'\\ F'(\omega)\cup[G(\omega)\cap H(\omega)], & \omega\in(A\setminus L)\cap(X\cap L)=\emptyset\\ [F(\omega)\cap H(\omega)]\cup G'(\omega), & \omega\in(A\cap L)\cap(X\setminus L)=\emptyset\\ [F(\omega)\cap H(\omega)]\cup[G(\omega)\cap H(\omega)], & \omega\in(A\cap L)\cap(X\cap L)=A\cap X\cap L\end{cases}$$

It is seen that N=T.

2) $[(F,A)\cap_{\varepsilon}(G,X)]\overset{*}{\underset{\cap}{\sim}}(H,L)=[(F,A)\overset{*}{\underset{\cap}{\sim}}(H,L)]\cap_{\varepsilon}[(G,X)\overset{*}{\underset{\cap}{\sim}}(H,L)]$, where $A\cap X\cap L'=\emptyset$.

3) $[(F,A)\setminus_{\varepsilon}(G,X)]\overset{*}{\underset{\cap}{\sim}}(H,L)=[(F,A)\overset{*}{\underset{\cap}{\sim}}(H,L)]\cap_{\varepsilon}[(G,X)\overset{*}{\underset{\gamma}{\sim}}(H,L)]$, where $A\cap X\cap L'=A'\cap X\cap L=\emptyset$.

Proof: Let first take care of the left hand facet of the equality and let $(F,A)\setminus_{\varepsilon}(G,X)=(M,A\cup X)$, where $\forall\omega\in A\cup X$,

$$M(\omega)=\begin{cases}F(\omega), & \omega\in A\setminus X\\ G(\omega), & \omega\in X\setminus A\\ F(\omega)\cap G'(\omega), & \omega\in A\cap X\end{cases}$$

Suppose that $(M,A\cup X)\overset{*}{\underset{\cap}{\sim}}(H,L)=(N,A\cup X)$, where $\forall\omega\in A\cup X$;

$$N(\omega)=\begin{cases}M'(\omega), & \omega\in(A\cup X)\setminus L\\ M(\omega)\cap H(\omega), & \omega\in(A\cup X)\cap L\end{cases}$$

Hence,

$$N(\omega)=\begin{cases}F'(\omega), & \omega\in(A\setminus X)\setminus L=A\cap X'\cap L'\\ G'(\omega), & \omega\in(X\setminus A)\setminus L=A'\cap X\cap L'\\ F'(\omega)\cup G(\omega), & \omega\in(A\cap X)\setminus L=A\cap X\cap L'\\ F(\omega)\cap H(\omega), & \omega\in(A\setminus X)\cap L=A\cap X'\cap L\\ G(\omega)\cap H(\omega), & \omega\in(X\setminus A)\cap L=A'\cap X\cap L\\ [F(\omega)\cap G'(\omega)]\cap H(\omega), & \omega\in(A\cap X)\cap L=A\cap X\cap L\end{cases}$$

Now let's take care of the right hand facet of the equality: $[(F,A)\overset{*}{\underset{\cap}{\sim}}(H,L)]\cap_{\varepsilon}[(G,X)\overset{*}{\underset{\gamma}{\sim}}(H,L)]$. Let $(F,A)\overset{*}{\underset{\cap}{\sim}}(H,L)=(V,A)$, where $\forall\omega\in A$;

$$V(\omega)=\begin{cases}F'(\omega), & \omega\in A\setminus L\\ F(\omega)\cap H(\omega), & \omega\in A\cap L\end{cases}$$

Let $(G,X)\overset{*}{\underset{\gamma}{\sim}}(H,L)=(W,X)$, where $\forall\omega\in X$;

$$W(\omega)=\begin{cases}G'(\omega), & \omega\in X\setminus L\\ G'(\omega)\cap H(\omega), & \omega\in X\cap L\end{cases}$$

Assume that $(V,A)\cap_{\varepsilon}(W,X)=(T,A\cup X)$, where $\forall\omega\in A\cup X$;

$$T(\omega)=\begin{cases}V(\omega), & \omega\in A\setminus X\\ W(\omega), & \omega\in X\setminus A\\ V(\omega)\cap W(\omega), & \omega\in A\cap X\end{cases}$$

Hence,

$$T(\omega)=\begin{cases}F'(\omega), & \omega\in(A\setminus L)\setminus X=A\cap X'\cap L'\\ F(\omega)\cap H(\omega), & \omega\in(A\cap L)\setminus X=A\cap X'\cap L\\ G'(\omega), & \omega\in(X\setminus L)\setminus A=A'\cap X\cap L'\\ G'(\omega)\cap H(\omega), & \omega\in(X\cap L)\setminus A=A'\cap X\cap L\\ F'(\omega)\cap G'(\omega), & \omega\in(A\setminus L)\cap(X\setminus L)=A\cap X\cap L'\\ F'(\omega)\cap[G'(\omega)\cap H(\omega)], & \omega\in(A\setminus L)\cap(X\cap L)=\emptyset\\ [F(\omega)\cap H(\omega)]\cap G'(\omega), & \omega\in(A\cap L)\cap(X\setminus L)=\emptyset\\ [F(\omega)\cap H(\omega)]\cap[G'(\omega)\cap H(\omega)], & \omega\in(A\cap L)\cap(X\cap L)=A\cap X\cap L\end{cases}$$

It is seen that N=T.

4) $[(F,A)\lambda_{\varepsilon}(G,X)]\overset{*}{\underset{\cap}{\sim}}(H,L)=[(F,A)\overset{*}{\underset{\cap}{\sim}}(H,L)]\cup_{\varepsilon}[(G,X)\overset{*}{\underset{\gamma}{\sim}}(H,L)]$, where $A\cap X\cap L'=A'\cap X\cap L=\emptyset$.

4.2. Distribution of Complementary Soft Binary Piecewise Intersection Operation over Complementary Extended Soft Set Operations

4.2.1. Left-distribution of complementary soft binary piecewise intersection operation over complementary extended soft set operations

1) $(F,A)\overset{*}{\underset{\cap}{\sim}}[(G,X)\overset{*}{*}_{\varepsilon}(H,L)]=[(F,A)\overset{*}{\underset{\setminus}{\sim}}(G,X)]\tilde{\cup}[(H,L)\overset{*}{\underset{\gamma}{\sim}}(F,A)]$

Proof: Let's first take care of the left hand facet of the equality. Let $(G,X)\overset{*}{*}_{\varepsilon}(H,L)=(M,X\cup L)$ where $\forall\omega\in X\cup L$;

$$M(\omega)=\begin{cases}G'(\omega), & \omega\in X\setminus L\\ H'(\omega), & \omega\in L\setminus X\\ G'(\omega)\cup H'(\omega), & \omega\in X\cap L\end{cases}$$

Let $(F,A)\overset{*}{\underset{\cap}{\sim}}(M,X\cup L)=(N,A)$, where $\forall\omega\in A$;

$$N(\omega)=\begin{cases}F'(\omega), & \omega\in A\setminus(X\cup L)\\ F(\omega)\cap M(\omega), & \omega\in A\cap(X\cup L)\end{cases}$$

Thus,

$$N(\omega)=\begin{cases}F'(\omega), & \omega\in A\setminus(X\cup L)=A\cap X\cap L'\\ F(\omega)\cap G'(\omega), & \omega\in A\cap(X\setminus L)=A\cap X\cap L'\\ F(\omega)\cap H'(\omega), & \omega\in A\cap(L\setminus X)=A\cap X'\cap L\\ F(\omega)\cap[(G'(\omega)\cup H'(\omega)], & \omega\in A\cap X\cap L=A\cap X\cap L\end{cases}$$

Now let's take care of the right hand facet of the equality $[(F,A)\overset{*}{\underset{\backslash}{\sim}}(G,X)]\tilde{\cup}\,[(H,L)\overset{*}{\underset{\gamma}{\sim}}(F,A)]$. Let $(F,A)\overset{*}{\underset{\backslash}{\sim}}(G,X)=(V,A)$, where $\forall\omega\in A$;

$$V(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ F(\omega)\cap G'(\omega), & \omega\in A\cap X \end{cases}$$

Let $(H,L)\overset{*}{\underset{\gamma}{\sim}}(F,A)=(W,L)$, where $\forall\omega\in L$;

$$W(\omega)=\begin{cases} H'(\omega), & \omega\in L\backslash A \\ H'(\omega)\cap F(\omega), & \omega\in L\cap A \end{cases}$$

Suppose that $(V,A)\tilde{\cup}\,(W,L)=(T,A)$, where $\forall\omega\in A$;

$$T(\omega)=\begin{cases} V(\omega), & \omega\in A\backslash L \\ V(\omega)\cup W(\omega), & \omega\in A\cap L \end{cases}$$

Thus,

$$T(w)=\begin{cases} F'(\omega), & \omega\in(A\backslash X)\backslash L=A\cap X'\cap L' \\ F(\omega)\cap G'(\omega), & \omega\in(A\cap X)\backslash L=A\cap X\cap L' \\ F'(\omega)\cup H'(\omega), & \omega\in(A\backslash X)\cap(L\backslash A)=\emptyset \\ F'(\omega)\cup[H'(\omega)\cap F(\omega)], & \omega\in(A\backslash X)\cap(L\cap A)=A\cap X'\cap L \\ [F(\omega)\cap G'(\omega)]\cup H'(\omega), & \omega\in(A\cap X)\cap(L\backslash A)=\emptyset \\ [F(\omega)\cap G'(\omega)]\cup[H'(\omega)\cap F(\omega)], & \omega\in(A\cap X)\cap(L\cap A)=A\cap X\cap L \end{cases}$$

It is seen that N=T.

2) $(F,A)\overset{*}{\underset{\cup}{\sim}}[(G,X)\overset{*}{\theta_{\varepsilon}}(H,L)]=[(F,A)\overset{*}{\underset{\backslash}{\sim}}(G,X)]\tilde{\cap}[(H,L)\overset{*}{\underset{\gamma}{\sim}}(F,A)]$

3) $(F,A)\overset{*}{\underset{\cap}{\sim}}[(G,X)\overset{*}{+_{\varepsilon}}(H,L)]=[(F,A)\overset{*}{\underset{\backslash}{\sim}}(G,X)]\tilde{\cup}[(H,L)\overset{*}{\underset{\cap}{\sim}}(F,A)]$

Proof: Let first take care of the lefthand facet of the equality.Assume $(G,X)\overset{*}{+_{\varepsilon}}(H,L)=(M,X\cup L)$ where $\forall\omega\in X\cup L$;

$$M(\omega)=\begin{cases} G'(\omega), & \omega\in X\backslash L \\ H'(\omega), & \omega\in L\backslash X \\ G'(\omega)\cup H(\omega), & \omega\in X\cap L \end{cases}$$

Let $(F,A)\overset{*}{\underset{\cap}{\sim}}(M,X\cup L)=(N,A)$, where $\forall\omega\in A$;

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash(X\cup L) \\ F(\omega)\cap M(\omega), & \omega\in A\cap(X\cup L) \end{cases}$$

Thus,

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash(X\cup L)=A\cap X'\cap L' \\ F(\omega)\cap G'(\omega), & \omega\in A\cap(X\backslash L)=A\cap X\cap L' \\ F(\omega)\cap H'(\omega), & \omega\in A\cap(L\backslash X)=A\cap X'\cap L \\ F(\omega)\cap[(G'(\omega)\cup H(\omega)], & \omega\in A\cap X\cap L=A\cap X\cap L \end{cases}$$

Now let's take care of the right hand facet of the equality $[(F,A)\overset{*}{\underset{\backslash}{\sim}}(G,X)]\tilde{\cup}[(H,L)\overset{*}{\underset{\cap}{\sim}}(F,A)]$.Assume that $(F,A)\overset{*}{\underset{\backslash}{\sim}}(G,X)=(V,A)$, where $\forall\omega\in A$;

$$V(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ F(\omega)\cap G'(\omega), & \omega\in A\cap X \end{cases}$$

Let $(H,L)\overset{*}{\underset{\cap}{\sim}}(F,A)=(W,L)$, where $\forall\omega\in A$;

$$W(\omega)=\begin{cases} H'(\omega), & \omega\in L\backslash A \\ H(\omega)\cap F(\omega), & \omega\in L\cap A \end{cases}$$

Suppose that $(V,A)\tilde{\cup}\,(W,L)=(T,A)$, where $\forall\omega\in A$;

$$T(\omega)=\begin{cases} V(\omega), & \omega\in A\backslash L \\ V(\omega)\cup W(\omega), & \omega\in A\cap L \end{cases}$$

Thus,

$$T(\omega)=\begin{cases} F'(\omega), & \omega\in(A\backslash X)\backslash L=A\cap X'\cap L' \\ F(\omega)\cap G'(\omega), & \omega\in(A\cap X)\backslash L=A\cap X\cap L' \\ F'(\omega)\cup H'(\omega), & w\in(A\backslash X)\cap(L\backslash A)=\emptyset \\ F'(\omega)\cup[H(\omega)\cap F(\omega)], & \omega\in(A\backslash X)\cap(L\cap A)=A\cap X'\cap L \\ [F(\omega)\cap G'(\omega)]\cup H'(\omega), & \omega\in(A\cap X)\cap(L/A)=\emptyset \\ [F(\omega)\cap G'(\omega)]\cup[H(\omega)\cap F(\omega)], & \omega\in(A\cap X)\cap(L\cap A)=A\cap X\cap L \end{cases}$$

It is seen that N=T.

4) $(F,A)\overset{*}{\underset{\cap}{\sim}}[(G,X)\overset{*}{\gamma_{\varepsilon}}(H,L)]=[(F,A)\overset{*}{\underset{\backslash}{\sim}}(G,X)]\tilde{\cap}[(H,L)\overset{*}{\underset{\cap}{\sim}}(F,A)]$.

4.2.2. Right-distribution of complementary soft binary piecewise intersection operation over complementary extended soft set operations

1) $[(F,A)\overset{*}{\theta_{\varepsilon}}(G,X)]\overset{*}{\underset{\cap}{\sim}}(H,L)=(F,A)\underset{\gamma}{\sim}(H,L)]\cap_{\varepsilon}[(G,X)\underset{\gamma}{\sim}(H,L)]$, where $A\cap X\cap L'=\emptyset$

Proof: Let's first take care of the left hand facet of the equality. Let $(F,A)\overset{*}{\theta_{\varepsilon}}(G,X)=(M,A\cup X)$, where $\forall\omega\in A\cup X$;

$$M(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ G'(\omega), & \omega\in X\backslash A \\ F'(\omega)\cap G'(\omega), & \omega\in A\cap X \end{cases}$$

Let $(M,A\cup X)\overset{*}{\underset{\cap}{\sim}}(H,L)=(N,A\cup X)$, where $\forall\omega\in A\cup X$;

$$N(\omega)=\begin{cases} M'(\omega), & \omega\in(A\cup X)\backslash L \\ M(\omega)\cap H(\omega), & \omega\in(A\cup X)\cap L \end{cases}$$

Thus,

$$N(\omega)=\begin{cases} F(\omega), & \omega\in(A\backslash X)\backslash L=A\cap X'\cap L' \\ G(\omega), & \omega\in(X\backslash A)\backslash L=A'\cap X\cap L' \\ F(\omega)\cup G(\omega), & \omega\in(A\cap X)\backslash L=A\cap X\cap L' \\ F'(\omega)\cap H(\omega), & \omega\in(A\backslash X)\cap L=A\cap X'\cap L \\ G'(\omega)\cap H(\omega), & \omega\in(X\backslash A)\cap L=A'\cap X\cap L \\ [F'(\omega)\cap G'(\omega)]\cap H(\omega), & \omega\in(A\cap X)\cap L=A\cap X\cap L \end{cases}$$

Now let take care of the right hand facet of the equality: $[(F,A)\underset{\gamma}{\sim}(H,L)]\cap_\varepsilon[(G,X)\underset{\gamma}{\sim}(H,L)$. Assume that $(F,A)\underset{\gamma}{\sim}(H,L)=(V,A)$, where $\forall\omega\in A$;

$$V(\omega)=\begin{cases} F(\omega), & \omega\in A\backslash L \\ F'(\omega)\cap H(\omega), & \omega\in A\cap L \end{cases}$$

Let $(G,X)\underset{\gamma}{\sim}(H,L)=(W,X)$, where $\forall\omega\in X$;

$$W(\omega)=\begin{cases} G(\omega), & \omega\in X\backslash L \\ G'(\omega)\cap H(\omega), & \omega\in X\cap L \end{cases}$$

Assume that $(V,A)\cap_\varepsilon(W,X)=(T,A\cup X)$, where $\forall\omega\in A\cup X$;

$$T(\omega)=\begin{cases} V(\omega), & \omega\in A\backslash X \\ W(\omega), & \omega\in X\backslash A \\ V(\omega)\cap W(\omega), & \omega\in A\cap X \end{cases}$$

Hence,

$$T(\omega)=\begin{cases} F(\omega), & \omega\in(A\backslash L)\backslash X=A\cap X'\cap L' \\ F'(\omega)\cap H(\omega), & \omega\in(A\cap L)\backslash X=A\cap X'\cap L \\ G(\omega), & \omega\in(X\backslash L)\backslash A=A'\cap X\cap L' \\ G'(\omega)\cap H(\omega), & \omega\in(X\cap L)\backslash A=A'\cap X\cap L \\ F(\omega)\cap G(\omega), & \omega\in(A\backslash L)\cap(X\backslash L)=A\cap X\cap L' \\ F(\omega)\cap[G'(\omega)\cap H(\omega)], & \omega\in(A\backslash L)\cap(X\cap L)=\emptyset \\ [F'(\omega)\cap H(\omega)]\cap G(\omega), & \omega\in(A\cap L)\cap(X\backslash L)=\emptyset \\ [F'(\omega)\cap H(\omega)]\cap[G'(\omega)\cap H(\omega)], & \omega\in(A\cap L)\cap(X\cap L)=A\cap X\cap L \end{cases}$$

It is seen that N=T.

2) $[(F,A)\overset{*}{*}_{\varepsilon}(G,X)]\overset{*}{\underset{\cap}{\sim}}(H,L)=[(F,A)\underset{\gamma}{\sim}(H,L)]\cup_\varepsilon[(G,X)\underset{\gamma}{\sim}(H,L)]$

where $A\cap X\cap L'=\emptyset$

3) $[(F,A)\overset{*}{\gamma}_{\varepsilon}(G,X)]\overset{*}{\underset{\cap}{\sim}}(H,L)=[(F,A)\underset{\gamma}{\sim}(H,L)]\cap_\varepsilon[(G,X)\underset{\cap}{\sim}(H,L)]$

where $A\cap X\cap L'=A'\cap X\cap L=\emptyset$.

Proof: Let's first take care of the left hand facet of the equality, let $(F,A)\overset{*}{\gamma}_{\varepsilon}(G,X)=(M,A\cup X)$, where $\forall\omega\in A\cup X$;

$$M(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ G'(\omega), & \omega\in X\backslash A \\ F(\omega)\cap G(\omega), & \omega\in A\cap X \end{cases}$$

Let $(M,A\cup X)\overset{*}{\underset{\cap}{\sim}}(H,L)=(N,A\cup X)$, where $\forall\omega\in A\cup X$;

$$N(\omega)=\begin{cases} M'(\omega), & \omega\in(A\cup X)\backslash L \\ M(\omega)\cap H(\omega), & \omega\in(A\cup X)\cap L \end{cases}$$

Therefore,

$$N(\omega)=\begin{cases} F(\omega), & \omega\in(A\backslash X)\backslash L=A\cap X'\cap L' \\ G(\omega), & \omega\in(X\backslash A)\backslash L=A'\cap X\cap L' \\ F(\omega)\cup G'(\omega), & \omega\in(A\cap X)\backslash L=A\cap X\cap L' \\ F'(\omega)\cap H(\omega), & \omega\in(A\backslash X)\cap L=A\cap X'\cap L \\ G'(\omega)\cap H(\omega), & \omega\in(X\backslash A)\cap L=A'\cap X\cap L \\ [F'(\omega)\cap G(\omega)]\cap H(\omega), & \omega\in(A\cap X)\cap L=A\cap X\cap L \end{cases}$$

Now let's take care of the right hand facet of the equality: $[(F,A)\underset{ɣ}{\sim}(H,L)]\cap_\varepsilon[(G,X)\underset{\cap}{\sim}(H,L)]$. Assume that $(F,A)\underset{ɣ}{\sim}(H,L)=(V,A)$, where $\forall\omega\in A$;

$$V(\omega)=\begin{cases} F(\omega), & \omega\in A\backslash L \\ F'(\omega)\cap H(\omega), & \omega\in A\cap L \end{cases}$$

Let $(G,X)\underset{\cap}{\sim}(H,L)=(W,X)$, where $\forall\omega\in X$;

$$W(\omega)=\begin{cases} G(\omega), & \omega\in X\backslash L \\ G(\omega)\cap H(\omega), & \omega\in X\cap L \end{cases}$$

Assume that $(V,A)\cap_\varepsilon(W,X)=(T,A\cup X)$, where $\forall\omega\in A\cup X$;

$$T(\omega)=\begin{cases} V(\omega), & \omega\in A\backslash X \\ W(\omega), & \omega\in X\backslash A \\ V(\omega)\cap W(\omega), & \omega\in A\cap X \end{cases}$$

Hence,

$$T(\omega)=\begin{cases} F(\omega), & \omega\in(A\backslash L)\backslash X=A\cap X'\cap L' \\ F'(\omega)\cap H(\omega), & \omega\in(A\cap L)\backslash X=A\cap X'\cap L \\ G(\omega), & \omega\in(X\backslash L)\backslash A=A'\cap X\cap L' \\ G(\omega)\cap H(\omega), & \omega\in(X\cap L)\backslash A=A'\cap X\cap L \\ F(\omega)\cap G(\omega), & \omega\in(A\backslash L)\cap(X\backslash L)=A\cap X\cap L' \\ F(\omega)\cap[G(\omega)\cap H(\omega)], & \omega\in(A\backslash L)\cap(X\cap L)=\emptyset \\ [F'(\omega)\cap H(\omega)]\cap G(\omega), & \omega\in(A\cap L)\cap(X\backslash L)=\emptyset \\ [F'(\omega)\cap H(\omega)]\cap[G(\omega)\cap H(\omega)], & \omega\in(A\cap L)\cap(X\cap L)=A\cap X\cap L \end{cases}$$

It is seen that N=T.

4) $[(F,A)\overset{*}{+}_{\varepsilon}(G,X)]\overset{*}{\underset{\cap}{\sim}}(H,L)=[(F,A)\underset{\gamma}{\sim}(H,L)]\cup_\varepsilon[(G,X)\underset{\cap}{\sim}(H,L)]$.

where $A\cap X\cap L'=A'\cap X\cap L=\emptyset$.

4.3. Distribution of Complementary Soft Binary Piecewise Intersection Operation over Soft Binary Piecewise Operations

4.3.1. Left-distribution of complementary soft binary piecewise intersection operation over soft binary piecewise operations

The followings are held where $A\cap X'\cap L=\emptyset$:

1) $(F,A)\overset{*}{\underset{\cap}{\sim}}[(G,X)\tilde{\cup}(H,L)]=[(F,A)\overset{*}{\underset{\cap}{\sim}}(G,X)]\tilde{\cup}[(H,L)\overset{*}{\underset{\cap}{\sim}}(F,A)]$

Proof: Let's first take care of the left hand facet of the equality, let $(G,X)\tilde{\cup}(H,L)=(M,X)$, where $\forall\omega\in X$;

$$M(\omega)=\begin{cases} G(\omega), & \omega\in X\backslash L \\ G(\omega)\cup H(\omega), & \omega\in X\cap L \end{cases}$$

Let $(F,A)\underset{\cap}{\overset{*}{\sim}}(M,X)=(N,A)$, where $\forall\omega\in A$;

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ F(\omega)\cap M(\omega), & \omega\in A\cap X \end{cases}$$

Thus,

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ F(\omega)\cap G(\omega), & \omega\in A\cap(X\backslash L)=A\cap X\cap L' \\ F(\omega)\cap[G(\omega)\cup H(\omega)], & \omega\in A\cap X\cap L=A\cap X\cap L \end{cases}$$

Now let take care of the right hand facet of the equality: $[(F,A)\underset{\cap}{\overset{*}{\sim}}(G,X)]\tilde{\cup}(H,L)\underset{\cap}{\overset{*}{\sim}}(F,A)]$. Let $(F,A)\underset{\cap}{\overset{*}{\sim}}(G,X)=(V,A)$, where $\forall\omega\in A$;

$$V(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ F(\omega)\cap G(\omega), & \omega\in A\cap X \end{cases}$$

Let $(H,L)\underset{\cap}{\overset{*}{\sim}}(F,A)=(W,L)$, where $\forall\omega\in L$;

$$W(\omega)=\begin{cases} H'(\omega), & \omega\in L\backslash A \\ H(\omega)\cap F(\omega), & \omega\in L\cap A \end{cases}$$

Suppose $(V,A)\tilde{\cup}(W,L)=(T,A)$, where $\forall\omega\in A$;

$$T(\omega)=\begin{cases} V(\omega), & \omega\in A\backslash L \\ V(\omega)\cup W(\omega), & \omega\in A\cap L \end{cases}$$

$$T(\omega)=\begin{cases} F'(\omega), & \omega\in(A\backslash X)\backslash L=A\cap X'\cap L' \\ F(\omega)\cap G(\omega), & \omega\in(A\cap X)\backslash L=A\cap X\cap L' \\ F'(\omega)\cup H'(\omega), & \omega\in(A\backslash X)\cap(L\backslash A)=\emptyset \\ F'(\omega)\cup[H(\omega)\cap F(\omega)], & \omega\in(A\backslash X)\cap(L\cap A)=A\cap X'\cap L \\ [F(\omega)\cap G(\omega)]\cup H'(\omega), & \omega\in(A\cap X)\cap(L\backslash A)=\emptyset \\ [F(\omega)\cap G(\omega)]\cup[H(\omega)\cap F(\omega)], & \omega\in(A\cap X)\cap(L\cap A)=A\cap X\cap L \end{cases}$$

Here let handle $\omega\in A\backslash X$ in the first equation of the first line. Since $A\backslash X=A\cap X'$, if $\omega\in X'$, then $\omega\in L\backslash X$ or $\omega\in(X\cup L)'$. Hence, if $\omega\in A\backslash X$, then $\omega\in A\cap X'\cap L'$ or $\omega\in A\cap X'\cap L$. Thus, it is seen that $N=T$.

2) $(F,A)\underset{\cap}{\overset{*}{\sim}}[(G,X)\tilde{\cap}(H,L)]=[(F,A)\underset{\cap}{\overset{*}{\sim}}(G,X)]\tilde{\cap}[(H,L)\underset{\cap}{\overset{*}{\sim}}(F,A)]$

3) $(F,A)\underset{\cap}{\overset{*}{\sim}}[(G,X)\tilde{\backslash}(H,L]=[(F,A)\underset{\cup}{\overset{*}{\sim}}(G,X)]\ \tilde{\cap}[(H,L)\underset{\gamma}{\overset{*}{\sim}}(F,A)]$.

Proof: Let's first take care of the left hand facet of the equality, let $(G,X)\tilde{\backslash}(H,L)=(M,X)$, where $\forall\omega\in X$;

$$M(\omega)=\begin{cases} G(\omega), & \omega\in X\backslash L \\ G(\omega)\cap H'(\omega), & \omega\in X\cap L \end{cases}$$

$(F,A)\underset{\cap}{\overset{*}{\sim}}(M,X)=(N,A)$, where $\forall\omega\in A$;

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ F(\omega)\cap M(\omega), & \omega\in A\cap X \end{cases}$$

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ F(\omega)\cap G(\omega), & \omega\in A\cap(X\backslash L)=A\cap X\cap L' \\ F(\omega)\cap[G(\omega)\cap H'(\omega)], & \omega\in A\cap X\cap L=A\cap X\cap L \end{cases}$$

Now let's take care of the right hand facet of the equality $[(F,A)\underset{\cap}{\overset{*}{\sim}}(G,X)]\tilde{\cap}[(H,L)\underset{\gamma}{\overset{*}{\sim}}(F,A)]$. Let $(F,A)\underset{\cap}{\overset{*}{\sim}}(G,X)=(V,A)$, where $\forall\omega\in A$;

$$V(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ F(\omega)\cap G(\omega), & \omega\in A\cap X \end{cases}$$

Let $(H,L)\underset{\gamma}{\overset{*}{\sim}}(F,A)=(W,L)$, where $\forall\omega\in L$;

$$W(\omega)=\begin{cases} H'(\omega), & \omega\in L\backslash A \\ H'(\omega)\cap F(\omega), & \omega\in L\cap A \end{cases}$$

Assume that $(V,A)\tilde{\cap}(W,L)=(T,A)$, where $\forall\omega\in A$;

$$T(\omega)=\begin{cases} V(\omega), & \omega\in A\backslash L \\ V(\omega)\cap W(\omega), & \omega\in A\cap L \end{cases}$$

Therefore,

$$T(\omega)=\begin{cases} F'(\omega), & \omega\in(A\backslash X)\backslash L=A\cap X'\cap L' \\ F(\omega)\cap G(\omega), & \omega\in(A\cap X)\backslash L=A\cap X\cap L' \\ F'(\omega)\cap H'(\omega), & \omega\in(A\backslash X)\cap(L\backslash A)=\emptyset \\ F'(\omega)\cap[H'(\omega)\cap F(\omega)], & \omega\in(A\backslash X)\cap(L\cap A)=A\cap X'\cap L \\ [F(\omega)\cap G(\omega)]\cap H'(\omega), & \omega\in(A\cap X)\cap(L\backslash A)=\emptyset \\ [F(\omega)\cap G(\omega)\cap H'(\omega), & \omega\in(A\cap X)\cap(L\cap A)=A\cap X\cap L \end{cases}$$

4) $(F,A)\underset{\cap}{\overset{*}{\sim}}[(G,X)\tilde{\lambda}(H,L]=[(F,A)\underset{\cap}{\overset{*}{\sim}}(G,X)]\ \tilde{\cup}\ [(H,L)\underset{\gamma}{\overset{*}{\sim}}(F,A)]$

4.3.2. Right-distribution of complementary soft binary piecewise intersection operation over soft binary piecewise operations

The followings are held where $A\cap X\cap L'=\emptyset$.

1) $[(F,A)\tilde{\cup}(G,X)]\underset{\cap}{\overset{*}{\sim}}(H,L)=[(F,A)\underset{\cap}{\overset{*}{\sim}}(H,L)]\tilde{\cup}[(G,X)\underset{\cap}{\overset{*}{\sim}}(H,L)]$.

Proof: Let first take care of the left hand facet of the equality. Suppose $(F,A)\tilde{\cup}(G,X)=(M,A)$, where $\forall\omega\in A$,

$$M(\omega)=\begin{cases} F(\omega), & \omega\in A\backslash X \\ F(\omega)\cup G(\omega), & \omega\in A\cap X \end{cases}$$

Let $(M,A)\underset{\cap}{\overset{*}{\sim}}(H,L)=(N,A)$, where $\forall\omega\in A$,

$$N(\omega)=\begin{cases} M'(\omega), & \omega\in A\backslash L \\ M(\omega)\cap H(\omega), & \omega\in A\cap L \end{cases}$$

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in(A\setminus X)\setminus L = A\cap X'\cap L' \\ F'(\omega)\cap G'(\omega), & \omega\in(A\cap X)\setminus L= A\cap X\cap L' \\ F(\omega)\cap H(\omega), & \omega\in(A\setminus X)\cap L= A\cap X'\cap L \\ [F(\omega)\cup G(\omega)]\cap H(\omega), & \omega\in(A\cap X)\cap L= A\cap X\cap L \end{cases}$$

Now let's take care of the right hand facet of the equality $[(F,A)\underset{\cap}{\overset{*}{\sim}}(H,L)]\,\tilde{\cup}\,[(G,X)\underset{\cap}{\overset{*}{\sim}}(H,L)]$. Let $(F,A)\underset{\cap}{\overset{*}{\sim}}(H,L)=(V,A)$, where $\forall\omega\in A$;

$$V(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus L \\ F(\omega)\cap H(\omega), & \omega\in A\cap L \end{cases}$$

Let $(G,X)\underset{\cap}{\overset{*}{\sim}}(H,L)=(W,X)$, where $\forall\omega\in X$;

$$W(\omega)=\begin{cases} G'(\omega), & \omega\in X\setminus L \\ G(\omega)\cap H(\omega), & \omega\in X\cap L \end{cases}$$

Suppose that $(V,A)\,\tilde{\cup}\,(W,X)=(T,A)$, where $\forall\omega\in A$;

$$T(\omega)=\begin{cases} V(\omega), & \omega\in A\setminus X \\ V(\omega)\cup W(\omega), & \omega\in A\cap X \end{cases}$$

Thus,

$$T(\omega)=\begin{cases} F'(\omega), & \omega\in(A\setminus L)\setminus X=A\cap X'\cap L' \\ F(\omega)\cap H(\omega), & \omega\in(A\cap L)\setminus X=A\cap X'\cap L \\ F'(\omega)\cup G'(\omega), & \omega\in(A\setminus L)\cap(X\setminus L)=A\cap X\cap L' \\ F'(\omega)\cup[G(\omega)\cap H(\omega)], & \omega\in(A\setminus L)\cap(X\cap L)=\emptyset \\ [F(\omega)\cap H(\omega)]\cup G'(\omega), & \omega\in(A\cap L)\cap(X\setminus L)=\emptyset \\ [F(\omega)\cup G(\omega)]\cap H(\omega)], & \omega\in(A\cap L)\cap(X\cap L)=A\cap X\cap L \end{cases}$$

It is seen that (1)=(2).

REMARK 2: In Yavuz (2023), it is proved that $(S_A(U),\tilde{\cup})$ is a commutative monoid with identity $\emptyset_A$. And in Remark1, we show that $(S_A(U),\underset{\cap}{\overset{*}{\sim}})$ is a commutative monoid. Moreover, $\emptyset_A\underset{\cap}{\overset{*}{\sim}}(F,A)=\emptyset_A$. That is to say, $\emptyset_A$ is the left-absorbing element for the operation $\underset{\cap}{\overset{*}{\sim}}$. Besides, by the subtitle 4.3.2. (1), $\underset{\cap}{\overset{*}{\sim}}$ satisfies the right distributive law over $\tilde{\cup}$. As a result we can conclude that $(S_A(U),\tilde{\cup},\underset{\cap}{\overset{*}{\sim}})$ is a (right) near-semiring. Moreover, since $(F,A)\underset{\cap}{\overset{*}{\sim}}\emptyset_A=\emptyset_A\underset{\cap}{\overset{*}{\sim}}(F,A)=\emptyset_A$, $(S_A(U),\tilde{\cup},\underset{\cap}{\overset{*}{\sim}})$ is a zero-symmetric near-semiring. One can similarly show that $(S_A(U),\tilde{\cup},\underset{\cap}{\overset{*}{\sim}})$ is also a hemiring.

2) $[(F,A)\,\tilde{\cap}\,(G,X)]\underset{\cap}{\overset{*}{\sim}}(H,L)=[(F,A)\underset{\cap}{\overset{*}{\sim}}(H,L)]\,\tilde{\cap}\,[(G,X)\underset{\cap}{\overset{*}{\sim}}(H,L)]$

.**3)** $[(F,A)\,\tilde{\lambda}\,(G,X)]\underset{\cap}{\overset{*}{\sim}}(H,L)=[(F,A)\underset{\cap}{\overset{*}{\sim}}(H,L)]\,\tilde{\cup}\,[(G,X)\underset{\text{ɣ}}{\overset{*}{\sim}}(H,L)]$.

Proof: Let's first take care of the left hand facet of the equality. Suppose $(F,A)\,\tilde{\lambda}\,(G,X)=(M,A)$, where $\forall\omega\in A$,

$$M(\omega)=\begin{cases} F(\omega), & \omega\in A\setminus X \\ F(\omega)\cup G'(\omega), & \omega\in A\cap X \end{cases}$$

Let $(M,A)\underset{\cap}{\overset{*}{\sim}}(H,L)=(N,A)$, where $\forall\omega\in A$,

$$N(\omega)=\begin{cases} M'(\omega), & \omega\in A\setminus L \\ M(\omega)\cap H(\omega), & \omega\in A\cap L \end{cases}$$

Thus,

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in(A\setminus X)\setminus L = A\cap X'\cap L' \\ F'(\omega)\cap G(\omega), & \omega\in(A\cap X)\setminus L= A\cap X\cap L' \\ F(\omega)\cap H(\omega), & \omega\in(A\setminus X)\cap L= A\cap X'\cap L \\ [F(\omega)\cup G'(\omega)]\cap H(\omega), & \omega\in(A\cap X)\cap L= A\cap X\cap L \end{cases}$$

Now let's take care of the right hand facet of the equality: $[(F,A)\underset{\cap}{\overset{*}{\sim}}(H,L)]\,\tilde{\cup}\,[(G,X)\underset{\text{ɣ}}{\overset{*}{\sim}}(H,L)]$. Let $(F,A)\underset{\cap}{\overset{*}{\sim}}(H,L)=(V,A)$, where $\forall\omega\in A$;

$$V(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus L \\ F(\omega)\cap H(\omega), & \omega\in A\cap L \end{cases}$$

Let $(G,X)\underset{\text{ɣ}}{\overset{*}{\sim}}(H,L)=(W,X)$, where $\forall\omega\in X$;

$$W(\omega)=\begin{cases} G'(\omega), & \omega\in X\setminus L \\ G'(\omega)\cap H(\omega), & \omega\in X\cap L \end{cases}$$

Suppose that $(V,A)\,\tilde{\cup}\,(W,X)=(T,A)$, where $\forall\omega\in A$;

$$T(\omega)=\begin{cases} V(\omega), & \omega\in A\setminus X \\ V(\omega)\cup W(\omega), & \omega\in A\cap X \end{cases}$$

Hence,

$$T(\omega)=\begin{cases} F'(\omega), & \omega\in(A\setminus L)\setminus X=A\cap X'\cap L' \\ F(\omega)\cap H(\omega), & \omega\in(A\cap L)\setminus X=A\cap X'\cap L \\ F'(\omega)\cup G'(\omega), & \omega\in(A\setminus L)\cap(X\setminus L)=A\cap X\cap L' \\ F'(\omega)\cup[G'(\omega)\cap H(\omega)], & \omega\in(A\setminus L)\cap(X\cap L)=\emptyset \\ [F(\omega)\cap H(\omega)]\cup G'(\omega), & \omega\in(A\cap L)\cap(X\setminus L)=\emptyset \\ [F(\omega)\cup G'(\omega)]\cap H(\omega), & \omega\in(A\cap L)\cap(X\cap L)=A\cap X\cap L \end{cases}$$

It is seen that N=T.

4) $[(F,A)\,\tilde{\setminus}\,(G,X)]\underset{\cap}{\overset{*}{\sim}}(H,L)=[(F,A)\underset{\cap}{\overset{*}{\sim}}(H,L)]\,\tilde{\cap}\,[(G,X)\underset{\text{ɣ}}{\overset{*}{\sim}}(H,L)]$

4.4. Distribution of Complementary Soft Binary Piecewise Intersection Operation over Complementary Soft Binary Piecewise Operations

4.4.1. Left-distribution of complementary soft binary piecewise intersection operation over complementary soft binary piecewise operations

1) $(F,A)\underset{\cap}{\overset{*}{\sim}}[(G,X)\underset{*}{\overset{*}{\sim}}(H,L)]=[(F,A)\underset{\backslash}{\overset{*}{\sim}}(G,X)]\tilde{\cup}[(H,L)\underset{\backslash}{\overset{*}{\sim}}(F,A)]$

where $A\cap X\cap L=\emptyset$

Proof: Let's first take care of the left hand facet of the equality, let $(G,X)\underset{*}{\overset{*}{\sim}}(H,L)=(M,X)$, where $\forall\omega\in X$;

$$M(\omega)=\begin{cases} G'(\omega), & \omega\in X\backslash L \\ G'(\omega)\cup H'(\omega), & \omega\in X\cap L \end{cases}$$

Let $(F,A)\underset{\cap}{\overset{*}{\sim}}(M,X)=(N,A)$, where $\forall\omega\in A$;

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ F(\omega)\cap M(\omega), & \omega\in A\cap X \end{cases}$$

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ F(\omega)\cap G'(\omega), & \omega\in A\cap(X\backslash L)=A\cap X\cap L' \\ F(\omega)\cap[(G'(\omega)\cup H'(\omega)], & \omega\in A\cap X\cap L=A\cap X\cap L \end{cases}$$

Now let's take care of the right hand facet of the equality: $[(F,A)\underset{\backslash}{\overset{*}{\sim}}(G,X)]\tilde{\cup}[(H,L)\underset{\backslash}{\overset{*}{\sim}}(F,A)]$.Let $(F,A)\underset{\backslash}{\overset{*}{\sim}}(G,X)=(V,A)$, where $\forall\omega\in A$;

$$V(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ F(\omega)\cap G'(\omega), & \omega\in A\cap X \end{cases}$$

Suppose that $(H,L)\underset{\backslash}{\overset{*}{\sim}}(F,A)=(W,L)$, where $\forall\omega\in L$;

$$W(\omega)=\begin{cases} H'(\omega), & \omega\in L\backslash A \\ H(\omega)\cap F'(\omega), & \omega\in L\cap A \end{cases}$$

Let $(V,A)\tilde{\cup}(W,L)=(T,A)$, where $\forall\omega\in A$;

$$T(\omega)=\begin{cases} V(\omega), & \omega\in A\backslash L \\ V(\omega)\cup W(\omega), & \omega\in A\cap L \end{cases}$$

Hence,

$$T(\omega)=\begin{cases} F'(\omega), & \omega\in(A\backslash X)\backslash L=A\cap X'\cap L' \\ F(\omega)\cap G'(\omega), & \omega\in(A\cap X)\backslash L=A\cap X\cap L' \\ F'(\omega)\cup H'(\omega), & \omega\in(A\backslash X)\cap(L\backslash A)=\emptyset \\ F'(\omega)\cup[H(\omega)\cap F'(\omega)], & \omega\in(A\backslash X)\cap(L\cap A)=A\cap X'\cap L \\ [F(\omega)\cap G'(\omega)]\cup H'(\omega), & \omega\in(A\cap X)\cap(L\backslash A)=\emptyset \\ [F(\omega)\cap[G'(\omega)\cup H(\omega), & \omega\in(A\cap X)\cap(L\cap A)=A\cap X\cap L \end{cases}$$

Here let's handle $\omega\in A\backslash X$ in the first equation. Since $A\backslash X=A\cap X'$, if $\omega\in X'$, then $\omega\in L\backslash X$ or $\omega\in(X\cup L)'$. Hence, if $\omega\in A\backslash X$, $\omega\in A\cap X'\cap L'$ or $\omega\in A\cap X'\cap L$. Thus, it is seen that N=T.

2) $(F,A)\underset{\cap}{\overset{*}{\sim}}[(G,X)\underset{\theta}{\overset{*}{\sim}}(H,L)]=[(F,A)\underset{\backslash}{\overset{*}{\sim}}(G,X)]\tilde{\cup}[(H,L)\underset{\backslash}{\overset{*}{\sim}}(F,A)]$

where $A\cap X\cap L=\emptyset$

3) $(F,A)\underset{\cap}{\overset{*}{\sim}}[(G,X)\underset{\gamma}{\overset{*}{\sim}}(H,L)]=[(F,A)\underset{\backslash}{\overset{*}{\sim}}(G,X)]\tilde{\cap}[(H,L)\underset{\cap}{\overset{*}{\sim}}(F,A)]$

where $A\cap X'\cap L=\emptyset$

Proof: Let's first take care of the left hand facet of the equality, let $(G,X)\underset{\gamma}{\overset{*}{\sim}}(H,L)=(M,X)$, where $\forall\omega\in X$;

$$M(\omega)=\begin{cases} G'(\omega), & \omega\in X\backslash L \\ G'(\omega)\cap H(\omega), & \omega\in X\cap L \end{cases}$$

Let $(F,A)\underset{\cap}{\overset{*}{\sim}}(M,X)=(N,A)$, where $\forall\omega\in A$;

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ F(\omega)\cap M(\omega), & \omega\in A\cap X \end{cases}$$

Therefore,

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ F(\omega)\cap G'(\omega), & \omega\in A\cap(X\backslash L)=A\cap X\cap L' \\ F(\omega)\cap[(G'(\omega)\cap H(\omega)], & \omega\in A\cap X\cap L=A\cap X\cap L \end{cases}$$

Now let's take care of the right hand facet of the equality: $[(F,A)\underset{\backslash}{\overset{*}{\sim}}(G,X)]\tilde{\cap}[(H,L)\underset{\cap}{\overset{*}{\sim}}(F,A)]$. Let $(F,A)\underset{\backslash}{\overset{*}{\sim}}(G,X)=(V,A)$, where $\forall\omega\in A$;

$$V(\omega)=\begin{cases} F'(\omega), & \omega\in A\backslash X \\ F(\omega)\cap G'(\omega), & \omega\in A\cap X \end{cases}$$

Suppose that $(H,L)\underset{\cap}{\overset{*}{\sim}}(F,A)=(W,L)$, where $\forall\omega\in L$;

$$W(\omega)=\begin{cases} H'(\omega), & \omega\in L\backslash A \\ H(\omega)\cap F(\omega), & \omega\in L\cap A \end{cases}$$

Let $(V,A)\tilde{\cap}(W,L)=(T,A)$, where $\forall\omega\in A$;

$$T(\omega)=\begin{cases} V(\omega), & \omega\in A\backslash L \\ V(\omega)\cap W(\omega), & \omega\in A\cap L \end{cases}$$

$$T(\omega)=\begin{cases} F'(\omega), & \omega\in(A\backslash X)\backslash L=A\cap X'\cap L' \\ F(\omega)\cap G'(\omega), & \omega\in(A\cap X)\backslash L=A\cap X\cap L' \\ F'(\omega)\cap H'(\omega), & \omega\in(A\backslash X)\cap(L\backslash A)=\emptyset \\ F'(\omega)\cap[H(\omega)\cap F(\omega)], & \omega\in(A\backslash X)\cap(L\cap A)=A\cap X'\cap L \\ [F(\omega)\cap G'(\omega)]\cap H'(\omega), & \omega\in(A\cap X)\cap(L\backslash A)=\emptyset \\ [F(\omega)\cap G'(\omega)]\cap H(\omega), & \omega\in(A\cap X)\cap(L\cap A)=A\cap X\cap L \end{cases}$$

4) $(F,A)\underset{\cap}{\overset{*}{\sim}}[(G,X)\underset{+}{\overset{*}{\sim}}(H,L)]=[(F,A)\underset{\backslash}{\overset{*}{\sim}}(G,X)]\tilde{\cup}[(H,L)\underset{\cap}{\overset{*}{\sim}}(F,A)]$

where $A\cap X'\cap L=\emptyset$

4.4.2. Right-distribution of complementary soft binary piecewise intersection operation over complementary soft binary piecewise operations

The followings are held where $A\cap X\cap L'=\emptyset$:

1) $[(F,A)\underset{\theta}{\overset{*}{\sim}}(G,X)]\underset{\cap}{\overset{*}{\sim}}(H,L)=[(F,A)\underset{\gamma}{\sim}(H,L)]\tilde{\cap}[(G,X)\underset{\gamma}{\sim}(H,L)]$

Proof: Let's first take care of the left hand facet of the equality, let $(F,A)\underset{\theta}{\overset{*}{\sim}}(G,X)=(M,A)$, where $\forall\omega\in A$,

$$M(\omega)=\begin{cases}F'(\omega), & \omega\in A\backslash X\\ F'(\omega)\cap G'(\omega), & \omega\in A\cap X\end{cases}$$

Let $(M,A)\underset{\cap}{\overset{*}{\sim}}(H,L)=(N,A)$, where$\forall\omega\in A$,

$$N(\omega)=\begin{cases}M'(\omega), & \omega\in A\backslash L\\ M(\omega)\cap H(\omega), & \omega\in A\cap L\end{cases}$$

Thus,

$$N(\omega)=\begin{cases}F(\omega), & \omega\in(A\backslash X)\backslash L=A\cap X'\cap L'\\ F(\omega)\cup G(\omega) & \omega\in(A\cap X)\backslash L=A\cap X\cap L'\\ F'(\omega)\cap H(\omega) & \omega\in(A\backslash X)\cap X=A\cap X'\cap L\\ [F'(\omega)\cap G'(\omega)]\cap H(\omega) & \omega\in(A\cap X)\cap L=A\cap X\cap L\end{cases}$$

Now let's take care of the right hand facet of the equality: $[(F,A)\underset{\gamma}{\sim}(H,L)]\tilde{\cap}[(G,X)\underset{\gamma}{\sim}(H,L)]$. Let $(F,A)\underset{\gamma}{\sim}(H,L)=(V,A)$, where $\forall\omega\in A$;

$$V(\omega)=\begin{cases}F(\omega), & \omega\in A\backslash L\\ F'(\omega)\cap H(\omega), & \omega\in A\cap L\end{cases}$$

Assume that $(G,X)\underset{\gamma}{\sim}(H,L)=(W,X)$, where $\forall\omega\in X$;

$$W(\omega)=\begin{cases}G(\omega), & \omega\in X\backslash L\\ G'(\omega)\cap H(\omega), & \omega\in X\cap L\end{cases}$$

Let $(V,A)\tilde{\cap}(W,X)=(T,A)$, where $\forall\omega\in A$;

$$T(\omega)=\begin{cases}V(\omega) & \omega\in A\backslash X\\ V(\omega)\cap W(\omega) & \omega\in A\cap X\end{cases}$$

Therefore,

$$T(\omega)=\begin{cases}F(\omega), & \omega\in(A\backslash L)\backslash X=A\cap X'\cap L'\\ F'(\omega)\cap H(\omega), & \omega\in(A\cap L)\backslash X=A\cap X'\cap L\\ F(\omega)\cap G(\omega), & \omega\in(A\backslash L)\cap(X\backslash L)=A\cap X\cap L'\\ F(\omega)\cap[G'(\omega)\cap H(\omega)], & \omega\in(A\backslash L)\cap(X\cap L)=\emptyset\\ [F'(\omega)\cap H(\omega)]\cap G(\omega), & \omega\in(A\cap L)\cap(X\backslash L)=\emptyset\\ [F'(\omega)\cap G'(\omega)\cap H(\omega)], & \omega\in(A\cap L)\cap(X\cap L)=A\cap X\cap L\end{cases}$$

It is seen that N=T.

2) $[(F,A)\underset{*}{\overset{*}{\sim}}(G,X)]\underset{\cap}{\overset{*}{\sim}}(H,L)=[(F,A)\underset{\gamma}{\sim}(H,L)]\tilde{\cup}[(G,X)\underset{\gamma}{\sim}(H,L)]$

3) $[(F,A)\underset{+}{\overset{*}{\sim}}(G,X)]\underset{\cap}{\overset{*}{\sim}}(H,L)=[(F,A)\underset{\gamma}{\sim}(H,L)]\tilde{\cup}[(G,X)\underset{\cap}{\sim}(H,L)]$

Proof: Let's first take care of the left hand facet of the equality, let $(F,A)\underset{+}{\overset{*}{\sim}}(G,X)=(M,A)$, where $\forall\omega\in A$,

$$M(\omega)=\begin{cases}F'(\omega), & \omega\in A\backslash X\\ F'(\omega)\cup G(\omega), & \omega\in A\cap X\end{cases}$$

Let $(M,A)\underset{\cap}{\overset{*}{\sim}}(H,L)=(N,A)$, where$\forall\omega\in A$,

$$N(\omega)=\begin{cases}M'(\omega), & \omega\in A\backslash L\\ M(\omega)\cap H(\omega), & \omega\in A\cap L\end{cases}$$

Hence,

$$N(\omega)=\begin{cases}F(\omega), & \omega\in(A\backslash X)\backslash L=A\cap X'\cap L'\\ F(\omega)\cap G'(\omega), & \omega\in(A\cap X)\backslash L=A\cap X\cap L'\\ F'(\omega)\cap H(\omega), & \omega\in(A\backslash X)\cap X=A\cap X'\cap L\\ [F'(\omega)\cup G(\omega)]\cap H(\omega), & \omega\in(A\cap X)\cap L=A\cap X\cap L\end{cases}$$

Now let's take care of the right hand facet of the equality: $[(F,A)\underset{\gamma}{\sim}(H,L)]\tilde{\cup}[(G,X)\underset{\cap}{\sim}(H,L)]$. Let $(F,A)\underset{\gamma}{\sim}(H,L)=(V,A)$, where $\forall\omega\in A$;

$$V(\omega)=\begin{cases}F(\omega), & \omega\in A\backslash L\\ F'(\omega)\cap H(\omega), & \omega\in A\cap L\end{cases}$$

Assume that $(G,X)\underset{\cap}{\sim}(H,L)=(W,X)$, where $\forall\omega\in X$;

$$W(\omega)=\begin{cases}G(\omega), & \omega\in X\backslash L\\ G(\omega)\cap H(\omega), & \omega\in X\cap L\end{cases}$$

Let $(V,A)\tilde{\cup}(W,X)=(T,A)$, where $\forall\omega\in A$;

$$T(\omega)=\begin{cases}V(\omega) & \omega\in A\backslash X\\ V(\omega)\cup W(\omega) & \omega\in A\cap X\end{cases}$$

Therefore,

$$T(\omega)=\begin{cases}F(\omega), & \omega\in(A\backslash L)\backslash X=A\cap X'\cap L'\\ F'(\omega)\cap H(\omega), & \omega\in(A\cap L)\backslash X=A\cap X'\cap L\\ F(\omega)\cup G(\omega), & \omega\in(A\backslash L)\cap(X\backslash L)=A\cap X\cap L'\\ F(\omega)\cup[G(\omega)\cap H(\omega)], & \omega\in(A\backslash L)\cap(X\cap L)=\emptyset\\ [F'(\omega)\cap H(\omega)\cup G(\omega), & \omega\in(A\cap L)\cap(X\backslash L)=\emptyset\\ [F'(\omega)\cup G(\omega)]\cap H(\omega), & \omega\in(A\cap L)\cap(X\cap L)=A\cap X\cap L\end{cases}$$

It is seen that N=T.

4) $[(F,A)\underset{\gamma}{\overset{*}{\sim}}(G,X)]\underset{\cap}{\overset{*}{\sim}}(H,L)=[(F,A)\underset{\gamma}{\sim}(H,L)]\tilde{\cap}[(G,X)\underset{\cap}{\sim}(H,L)]$

4.5. Distribution of Complementary Soft Binary Piecewise Intersection Operation over Restricted Soft Set Operations

The followings are held where $A\cap X\cap L=\emptyset$.

1) $(F,A)\overset{*}{\underset{\cap}{\sim}}[(G,X)\cap_R(H,L)]=[(F,A)\overset{*}{\underset{\gamma}{\sim}}(G,X)]\cup_R[(F,A)\overset{*}{\underset{\gamma}{\sim}}(H,L)$

Proof: Let's first take care of the left hand facet of the equality, suppose $(G,X)\cap_R(H,L)=(M,X\cap L)$ and so $\forall\omega\in X\cap L$, $M(\omega)=G(\omega)\cap H(\omega)$. Let $(F,A)\overset{*}{\underset{\cap}{\sim}}(M,X\cap L)=(N,A)$, so $\forall\omega\in A$,

$$N(\omega)=\begin{cases}F'(\omega), & \omega\in A\backslash(X\cap L)\\ F(\omega)\cap M(\omega), & \omega\in A\cap(X\cap L)\end{cases}$$

$$N(\omega)=\begin{cases}F'(\omega), & \omega\in A\backslash(X\cap L)\\ F(\omega)\cap[G(\omega)\cap H(\omega)], & \omega\in A\cap(X\cap L)\end{cases}$$

Now let's take care of the right hand facet of the equality: $[(F,A)\overset{*}{\underset{\gamma}{\sim}}(G,X)]\cup_R[(F,A)\overset{*}{\underset{\gamma}{\sim}}(H,L)]$. Let $(F,A)\overset{*}{\underset{\gamma}{\sim}}(G,X)=(V,A)$, so $\forall\omega\in A$,

$$V(\omega)=\begin{cases}F'(\omega), & \omega\in A\backslash X\\ F'(\omega)\cap G(\omega), & \omega\in A\cap X\end{cases}$$

Let $(F,A)\overset{*}{\underset{\gamma}{\sim}}(H,L)=(W,A)$, so $\forall\omega\in A$,

$$W(\omega)=\begin{cases}F'(\omega), & \omega\in A\backslash L\\ F'(\omega)\cap H(\omega), & \omega\in A\cap L\end{cases}$$

Assume that $(V,A)\cup_R(W,A)=(T,A)$, and so $\forall\omega\in A$, $T(\omega)=V(\omega)\cup W(\omega)$. Thus,

$$T(\omega)=\begin{cases}F'(\omega)\cup F'(\omega), & \omega\in(A\backslash X)\cap(A\backslash L)\\ F'(\omega)\cup[F'(\omega)\cap H(\omega)], & \omega\in(A\backslash X)\cap(A\cap L)\\ [F'(\omega)\cap G(\omega)]\cup F'(\omega), & \omega\in(A\cap X)\cap(A\backslash L)\\ [F'(\omega)\cap G(\omega)]\cup[F'(\omega)\cap H(\omega)], & \omega\in(A\cap X)\cap(A\cap L)\end{cases}$$

Hence,

$$T(\omega)=\begin{cases}F'(\omega), & \omega\in A\cap X'\cap L'\\ F'(\omega), & \omega\in A\cap X'\cap L\\ F'(\omega), & \omega\in A\cap X\cap L'\\ [F'(\omega)\cap G(\omega)]\cup[F'(\omega)\cap H(\omega)], & \omega\in A\cap X\cap L\end{cases}$$

Considering the parameter set of the first equation of the first row, that is, $A\backslash(X\cap L)$; since $A\backslash(X\cap L)=A\cap(X\cap L)'$, an element in $(X\cap L)'$ may be in $X\backslash L$, in $L\backslash X$ or $(X\cup L)$. Then, $A\backslash(X\cap L)$ is equivalent to the following 3 states: $A\cap(X\cap L')$, $A\cap(X'\cap L)$ and $A\cap(X'\cap L')$. Hence, (1)=(2).

2) $(F,A)\overset{*}{\underset{\cap}{\sim}}[(G,X)\cup_R(H,L)]=[(F,A)\overset{*}{\underset{+}{\sim}}(G,X)]\cap_R[(F,A)\overset{*}{\underset{+}{\sim}}(H,L)$

3) $(F,A)\overset{*}{\underset{\cap}{\sim}}[(G,X)\theta_R(H,L)]=[(F,A)\overset{*}{\underset{\theta}{\sim}}(G,X)]\cup_R[(F,A)\overset{*}{\underset{\theta}{\sim}}(H,L)$

Proof: Let's first take care of the left hand facet of the equality, suppose $(G,X)\theta_R(H,L)=(M,X\cap L)$ and so $\forall\omega\in X\cap L$, $M(\omega)=G'(\omega)\cap H'(\omega)$. Let $(F,A)\overset{*}{\underset{\cap}{\sim}}(M,X\cap L)=(N,A)$, so $\forall\omega\in A$,

$$N(\omega)=\begin{cases}F'(\omega), & \omega\in A\backslash(X\cap L)\\ F(\omega)\cap M(\omega), & \omega\in A\cap(X\cap L)\end{cases}$$

Thus,

$$N(\omega)=\begin{cases}F'(\omega), & \omega\in A\backslash(X\cap L)\\ F(\omega)\cap[G'(\omega)\cap H'(\omega)], & \omega\in A\cap(X\cap L)\end{cases}$$

Now let's take care of the right hand facet of the equality, $[(F,A)\overset{*}{\underset{\theta}{\sim}}(G,X)]\cup_R[(F,A)\overset{*}{\underset{\theta}{\sim}}(H,L)]$. Let $(F,A)\overset{*}{\underset{\theta}{\sim}}(G,X)=(V,A)$, and $\forall\omega\in A$,

$$V(\omega)=\begin{cases}F'(\omega), & \omega\in A\backslash X\\ F'(\omega)\cap G'(\omega), & \omega\in A\cap X\end{cases}$$

Let $(F,A)\overset{*}{\underset{\theta}{\sim}}(H,L)=(W,A)$ and $\in A$,

$$W(\omega)=\begin{cases}F'(\omega), & \omega\in A\backslash L\\ F'(\omega)\cap H'(\omega), & \omega\in A\cap L\end{cases}$$

Assume that $(V,A)\cup_R(W,A)=(T,A)$, so $\forall\omega\in T(\omega)=V(\omega)\cup W(\omega)$,

$$T(\omega)=\begin{cases}F'(\omega)\cup F'(\omega), & \omega\in(A\backslash X)\cap(A\backslash L)\\ F'(\omega)\cup[F'(\omega)\cap H'(\omega)], & \omega\in(A\backslash X)\cap(A\cap L)\\ [F'(\omega)\cap G'(\omega)]\cup F'(\omega), & \omega\in(A\cap X)\cap(A\backslash L)\\ [F'(\omega)\cap G'(\omega)]\cup[F'(\omega)\cap H'(\omega)], & \omega\in(A\cap X)\cap(A\cap L)\end{cases}$$

Thus,

$$T(\omega)=\begin{cases}F'(\omega), & \omega\in A\cap X'\cap L'\\ F'(\omega), & \omega\in A\cap X'\cap L\\ F'(\omega), & \omega\in A\cap X\cap L'\\ [F'(\omega)\cap G'(\omega)]\cup[F'(\omega)\cap H'(\omega)], & \omega\in A\cap X\cap L\end{cases}$$

4) $(F,A)\overset{*}{\underset{\cap}{\sim}}[(G,X)*_R(H,L)]=[F,A)\overset{*}{\underset{*}{\sim}}(G,X)]\cap_R[(F,A)\overset{*}{\underset{*}{\sim}}(H,L)$

5) $(F,A)\overset{*}{\underset{\cap}{\sim}}(G,X)\gamma_R(H,L)]=[(F,A)\overset{*}{\underset{\gamma}{\sim}}(G,X)]\cup_R[(F,A)\overset{*}{\underset{\theta}{\sim}}(H,L)]$

Proof: Let's first take care of the left hand facet of the equality, suppose $(G,X)\gamma_R(H,L)=(M,X\cap L)$ and so $\forall\omega\in X\cap L$, $M(\omega)=G'(\omega)\cap H(\omega)$. Let $(F,A)\overset{*}{\underset{\cap}{\sim}}(M,X\cap L)=(N,A)$, so $\forall\omega\in A$,

$$N(\omega)=\begin{cases}F'(\omega), & \omega\in A\backslash(X\cap L)\\ F(\omega)\cap M(\omega), & \omega\in A\cap(X\cap L)\end{cases}$$

Thus,

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus(X\cap L) \\ F(\omega)\cap[G'(\omega)\cap H(\omega)], & \omega\in A\cap(X\cap L)\end{cases}$$

Now let's take care of the right hand facet of the equality: $[(F,A)\underset{\gamma}{\overset{*}{\sim}}(G,X)]\cap_R[(F,A)\underset{\theta}{\overset{*}{\sim}}(H,L)]$. Let $(F,A)\underset{\gamma}{\overset{*}{\sim}}(G,X)=(V,A)$, so $\forall\omega\in A$,

$$V(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus X \\ F'(\omega)\cap G(\omega), & \omega\in A\cap X\end{cases}$$

Let $(F,A)\underset{\theta}{\overset{*}{\sim}}(H,L)=(W,A)$, so $\forall\omega\in A$,

$$W(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus L \\ F'(\omega)\cap H'(\omega), & \omega\in A\cap L\end{cases}$$

Assume that $(V,A)\cup_R(W,A)=(T,A)$, and so $\forall\omega\in A$, $T(\omega)=V(\omega)\cup W(\omega)$,. Thus,

$$T(\omega)=\begin{cases} F'(\omega)\cup F'(\omega), & \omega\in(A\setminus X)\cap(A\setminus L) \\ F'(\omega)\cup[F'(\omega)\cap H'(\omega)], & \omega\in(A\setminus X)\cap(A\cap L) \\ [F'(\omega)\cap G(\omega)]\cup F'(\omega), & \omega\in(A\cap X)\cap(A\setminus L) \\ [F'(\omega)\cap G(\omega)]\cup[F'(\omega)\cap H'(\omega)], & \omega\in(A\cap X)\cap(A\cap L)\end{cases}$$

Hence,

$$T(\omega)=\begin{cases} F'(\omega), & \omega\in A\cap X'\cap L' \\ F'(\omega), & \omega\in A\cap X'\cap L \\ F'(\omega), & \omega\in A\cap X\cap L' \\ [F'(\omega)\cap G(\omega)]\cup[F'(\omega)\cap H'(\omega)], & \omega\in A\cap X\cap L\end{cases}$$

6)$(F,A)\underset{\cap}{\overset{*}{\sim}}[(G,X)\lambda_R(H,L)]=[(F,A)\underset{+}{\overset{*}{\sim}}(G,X)]\cap_R[(F,A)\underset{*}{\overset{*}{\sim}}(H,L)]$

7$(F,A)\underset{\cap}{\overset{*}{\sim}}[(G,X)\setminus_R(H,L)]=[(F,A)\underset{*}{\overset{*}{\sim}}(G,X)]\cap_R[(F,A)\underset{+}{\overset{*}{\sim}}(H,L)]$

Proof: Let first take care of the left hand facet of the equality, suppose $(G,X)\setminus_R(H,L)=(M,X\cap L)$ and so $\forall\omega\in X\cap L$, $M(\omega)=G(\omega)\cap H'(\omega)$. Let $(F,A)\underset{\cap}{\overset{*}{\sim}}(M,X\cap L)=(N,A)$, so $\forall\omega\in A$,

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus(X\cap L) \\ F(\omega)\cap M(\omega), & \omega\in A\cap(X\cap L)\end{cases}$$

Thus,

$$N(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus(X\cap L) \\ F(\omega)\cap[G(\omega)\cap H'(\omega)], & \omega\in A\cap(X\cap L)\end{cases}$$

Now let's take care of the right hand facet of the equality, $(F,A)\underset{*}{\overset{*}{\sim}}(G,X)]\cap_R[(F,A)\underset{+}{\overset{*}{\sim}}(H,L)]$, Let $(F,A)\underset{*}{\overset{*}{\sim}}(G,X)=(V,A)$, and $\forall\omega\in A$,

$$V(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus X \\ F'(\omega)\cup G'(\omega), & \omega\in A\cap X\end{cases}$$

Let $(F,A)\underset{+}{\overset{*}{\sim}}(H,L)=(W,A)$ and $\forall\omega\in A$,

$$W(\omega)=\begin{cases} F'(\omega), & \omega\in A\setminus L \\ F'(\omega)\cup H(\omega), & \omega\in A\cap L\end{cases}$$

Assume that $(V,A)\cup_R(W,A)=(T,A)$, so $\forall\omega\in T(\omega)=V(\omega)\cup W(\omega)$. Thus,

$$T(\omega)=\begin{cases} F'(\omega)\cap F'(\omega), & \omega\in(A\setminus X)\cap(A\setminus L) \\ F'(\omega)\cap[F'(\omega)\cup H(\omega)], & \omega\in(A\setminus X)\cap(A\cap L) \\ [F'(\omega)\cup G'(\omega)]\cap F'(\omega), & \omega\in(A\cap X)\cap(A\setminus L) \\ [F'(\omega)\cup G'(\omega)]\cap[F'(\omega)\cup H(\omega)], & \omega\in(A\cap X)\cap(A\cap L)\end{cases}$$

Thus,

$$T(\omega)=\begin{cases} F'(\omega), & \omega\in A\cap X'\cap L' \\ F'(\omega), & \omega\in A\cap X'\cap L \\ F'(\omega), & \omega\in A\cap X\cap L' \\ [F'(\omega)\cup G'(\omega)]\cap[F'(\omega)\cup H(\omega)], & \omega\in A\cap X\cap L\end{cases}$$

8)$(F,A)\underset{\cap}{\overset{*}{\sim}}[(G,X)+_R(H,L)]=[(F,A)\underset{\theta}{\overset{*}{\sim}}(G,X)]\cup_R[(F,A)\underset{\gamma}{\overset{*}{\sim}}(H,L)]$

5. Conclusion

In this paper, we have contributed to the literature on soft sets by defining a novel form of soft set operation, which we call complementary soft binary piecewise intersection operation. The basic algebraic properties of the operations are examined. By examining the distribution rules, we determine the connections between this new soft set operation and other soft set operations, including extended soft set operations, complementary extended soft set operations, soft binary piecewise operations, complementary soft binary piecewise operations, and restricted soft set operations. Additionally, we demonstrate that the set of all the soft sets with a fixed parameter set together with the complementary soft binary piecewise intersection operation and the soft binary piecewise union operation is a zero-symmetric near-semiring and also a hemiring. In the future studies, new types of soft set operations may be established. Moreover, since soft set is powerful mathematical tool for uncertain object detection, with this study, researchers may suggest some new encryption or decision making methods based on soft sets. Also, studies on the soft algebraic structures may be handled again as regards the algebraic properties by the operation defined in this paper.

Author Contributions

The percentage of the author(s) contributions is presented below. All authors reviewed and approved the final version of the manuscript.

	A.S.	F.N.A.	A.O.A.
C	34	33	33
D	34	33	33
S	34	33	33
DCP	34	33	33
DAI	34	33	33
L	34	33	33
W	34	33	33
CR	34	33	33
SR	34	33	33
PM	34	33	33
FA	34	33	33

C=Concept, D= design, S= supervision, DCP= data collection and/or processing, DAI= data analysis and/or interpretation, L= literature search, W= writing, CR= critical review, SR= submission and revision, PM= project management, FA= funding acquisition.

Conflict of Interest

The authors declared that there is no conflict of interest.

Ethical Consideration

Ethics committee approval was not required for this study because of there was no study on animals or humans.

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