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CHARACTERIZATION OF ONE DIMENSIONAL PERIODIC BOUNDARY CELLULAR AUTOMATA

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Abstract

In this paper, we investigate one dimensional cellular automata under periodic boundary conditions. We use matrix algebra for calculating over on Zp. We obtain a formula for finding the reversibility of cellular automata. Finally, we give some important examples of cellular automata.

Keywords: Boundary condition, cellular automata, characteristic matrices, reversibility.

Öz

Bu çalışmada periyodik sınır şartı altında, bir boyutlu hücresel dönüşümleri inceliyoruz. Zp cismi üzerindeki hesaplamalar için matris cebirlerini kullanıyoruz. Hücresel dönüşümlerin tersini bulmak için bir formül elde ediyoruz. Son olarak, hücresel dönüşümlerin bazı önemli örneklerini veriyoruz.

Anahtar Kelimeler: Hücresel dönüşümler, karakteristik matrisler, sınır şartları, terslenebilirlik.

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1. INTRODUCTION

Cellular automata(shortly CA) were first studied by Neumann in the 1940s. Von Neumann framed CA as a cellular space capable of self-reproduction. Since then, many researchers have taken an interest in the study of CA for modeling the behavior of complex systems. Wolfram et al. (1983) investigated one-dimensional CA with the help of polynomial algebra. He used simple mathematical models to characterize cellular automata in statistical mechanics. One-dimensional CA was characterized using matrix algebras by Das et al. (1993) and a new method was developed for the examination of linear CA. The structure of CA was analyzed differently from other studies via polynomial algebras. They focused in the article, more on hybrid CAs. They gave an algorithm about the invertibility of the representation matrices of CA.l

While studying cellular automata, one of the most important problems is whether cellular automata is reversible or not. There are many studies made on this topic. Because if CA is reversible, we can obtain its initial form. Many disciplines can benefit from this situation. Martin del Rey et al. (2011) have studied the reversibility problem for null boundary cellular automata (shortlyNBCA). They used a pentadiagonal matrix over the binary field which was defined by a rule matrix. The existence of the inverse of one-dimensional cellular automata under the intermediate boundary condition was demonstrated by Chang et al., (2020) by obtaining various algorithms. The reversibility of linear cellular automata was investigated by observing the structure of the characteristic matrix on a finite field. Akın (2021) investigated the reversibility of 9-cycle one dimensional periodic boundary cellular automata (shortly 1D-PBCA). Radius was taken in 4 by the author. The reversibility criteria were determined with the help of coefficients and a rule matrix.

As mentioned above the problem of irreversibility in 1-D cellular automata is a difficult problem in general. In this article, we characterize 1D-PBCA over on $\mathbb{Z}_P$. We use elementary matrix operation for finding the reversibility of 1D-PBCA. We have implemented a new local rule for obtaining the characteristic matrix. We have developed a new algorithm that can calculate reversibility. Lastly, we have shared a few examples of algorithms.

2. CONSTRUCTION OF CELLULAR AUTOMATA OVER $\mathbb{Z}_P$

Definition 2.1 Let $\mathbb{Z}_m = \{0,1,2,\dots,m-1\}$. $x = (x_n)_{n=-\infty}^{n=\infty}$ is double-sided. infinite sequence. It denotes $\mathbb{Z}_m^{\mathbb{Z}}$. Let r be the radius and f is the local rule. So, we define the local rule as follows:

$$f\colon \mathbb{Z}_m^{2r+1} \to \mathbb{Z}_m$$

$$f(x_{-r},\dots,x_r) = \left(\sum_{i=-r}^{i=r} w_i x_i\right)(mod\, m)$$

where $w_i \in \mathbb{Z}_m$. $F\colon \mathbb{Z}_m^{\mathbb{Z}} \to \mathbb{Z}_m^{\mathbb{Z}}$ is generated by this local rule. So, we call it 1D additive CA. This transformation is defined as follows:

$$Fx = (y_n)_{n=-\infty}^{n=\infty}, y_n = f(x_{n-r},\dots,x_{n+r}) = \left(\textstyle\sum_{i=-r}^{i=r} w_i x_{n+i}\right)(mod\, m)$$

A 1D CA structure, whose definition is given on the $\mathbb{Z}_2$ field, can be thought of as a lattice of cells or blocks, where the value of each cell is taken as 0 or 1. The next transition state of the cell can be obtained depending on itself and its other two neighbors. Cells can evolve in discrete time steps according to local rules that depend only on the local neighborhood.

Mathematically, the next state transition of the i th cell can be represented as a function of the present states of the $(i-1)$th, i th and $(i+1)$th cells:

$$\wp_i(t+1) = f(\wp_i(t), \wp_{i+1}(t), \wp_{i-1}(t))$$

where $\wp$ is known as the rule of a CA.

Definition 2.2 If the same rule is applied to all the cells in a CA, then the CA is called a uniform or regular CA (Khan et al.,1999).

Definition 2.3 If different rules are applied to different cells in a CA, then the CA is called a hybrid CA (Khan et al.,1999).

Why should we limit the extreme cells of the lattice when working on cellular automata? In order to obtain better results, we take various boundary conditions on the extreme cells. There are many boundary conditions, we use periodic boundary conditions in this work.

Definition 2.3 A Periodic Boundary CA is the one in which the extreme cells are adjacent to each other (Khan et al.,1999).

Example 2.4 If we look carefully, as if a finite one-dimensional array of cells labeled b, e, r, r, i, n had joined a periodic loop.

n	b	e	r	r	i	n	b

In this article, we study only the linear rules and we will take radius 1. However, we will only 1D-CA defined by local rule (1) under modulo p addition and $p \geq 2$ is a prime number.

Definition 2.5 The vector $N^t = [x_1^t, x_2^t, \ldots, x_n^t]$ is called a configuration of the 1 D CA at time t, therefore $N^0$is the initial configuration.

We define the local rule as follows:

$$x_i^{t+1} = \begin{cases} w_1 x_n^t + w_2 x_1^t \ \ mod\, p\,,\ i = 1 \\ w_1 x_{i-1}^t + w_2 x_i^t \ \ mod\, p\,,\ 2 \leq i \leq n \end{cases} \tag{1}$$

where $w_1, w_2 \in \mathbb{Z}_p - \{0\}$. x_i^tstands for the state of the cell at a time$t$. Since the number of cells is finite.

Let us define the 1D-CA representation matrix T with PBC:

$$x_n^t [x_1^t, x_2^t, \ldots, x_n^t] x_1^t \underset{\rightarrow}{T} x_n^{t+1} [x_1^{t+1}, x_2^{t+1}, \ldots, x_n^{t+1}] x_1^{t+1} \tag{2}$$

$w_1 x_n^t + w_2 x_1^t \quad \vec{T} \ x_1^{t+1}$

$w_1 x_1^t + w_2 x_2^t \quad \vec{T} \ x_2^{t+1}$

$w_1 x_2^t + w_2 x_3^t \quad \vec{T} \ x_3^{t+1}$

$w_1 x_3^t + w_2 x_4^t \quad \vec{T} \ x_4^{t+1}$

$\dots\dots\dots\dots\dots\dots$

$w_1 x_{n-3}^t + w_2 x_{n-2}^t \, \vec{T} \ x_{n-2}^{t+1}$

$w_1 x_{n-2}^t + w_2 x_{n-1}^t \, \vec{T} \ x_{n-1}^{t+1}$

$w_1 x_{n-1}^t + w_2 x_n^t \quad \vec{T} \ x_n^{t+1}$

Therefore, we get the rule matrix as follows:

$$T_n = \begin{pmatrix} w_2 & 0 & 0 & 0 & \dots & 0 & 0 & w_1 \\ w_1 & w_2 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & w_1 & w_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & w_1 & w_2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & w_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & w_1 & w_2 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & w_1 & w_2 \end{pmatrix}_{n\times n} \quad (3)$$

Example 2.6 If we get $n = 5$, then we obtain the rule matrix T_5 of order 5. We take into consideration a configuration of size 1×5 with periodic boundary conditions.

$x_5^t[x_1^t \ x_2^t \ x_3^t \ x_4^t \ x_5^t]x_1^t$

This configuration is represented by an information matrix as follows**:**

$[X]_{1\times 5}^t = [x_1^t \quad x_2^t \quad x_3^t \quad x_4^t \quad x_5^t]$

If we carry out local rule to whole cells of the vector $[X]_{1\times 5}^t$, we obtain a new information matrix $[X]_{1\times 5}^{t+1}$ with entries as follows:

$w_1 x_5^t + w_2 x_1^t \, \vec{T} \ x_1^{t+1}$

$w_1 x_1^t + w_2 x_2^t \, \vec{T} \ x_2^{t+1}$

$w_1 x_2^t + w_2 x_3^t \, \vec{T} \ x_3^{t+1}$

$w_1 x_3^t + w_2 x_4^t \, \vec{T} \ x_4^{t+1}$

$w_1 x_4^t + w_2 x_5^t \, \vec{T} \ x_5^{t+1}$

In order to get the rule matrix T_5, we take advantage of the basis vectors.

$$T_5(E_1) = \; T_5(1 \quad 0 \quad 0 \quad 0 \quad 0) = (w_2 \quad w_1 \quad 0 \quad 0 \quad 0)$$

$$T_5(E_2) = \; T_5(0 \quad 1 \quad 0 \quad 0 \quad 0)^T = (0 \quad w_2 \quad w_1 \quad 0 \quad 0)^T$$

$$T_5(E_3) = \; T_5(0 \quad 0 \quad 1 \quad 0 \quad 0)^T = (0 \quad 0 \quad w_2 \quad w_1 \quad 0)^T$$

$$T_5(E_4) = \; T_5(0 \quad 0 \quad 0 \quad 1 \quad 0)^T = (0 \quad 0 \quad 0 \quad w_2 \quad w_1)^T$$

$$T_5(E_2) = \; T_5(0 \quad 0 \quad 0 \quad 0 \quad 1)^T = (w_1 \quad 0 \quad 0 \quad 0 \quad w_2)^T$$

Hence, we get the rule matrix T_5 of order 5 which is

$$T_5 = \begin{pmatrix} w_2 & 0 & 0 & 0 & w_1 \\ w_1 & w_2 & 0 & 0 & 0 \\ 0 & w_1 & w_2 & 0 & 0 \\ 0 & 0 & w_1 & w_2 & 0 \\ 0 & 0 & 0 & w_1 & w_2 \end{pmatrix}_{5\times5} .$$

3. REVERSIBILITY OF 1D PBCA

While studying CA, one of the basic problems is reversibility. In order to find the initial nature of CA, we must know whether CA is reversible or not. However, the problem of reversibility is a very hard problem for nonlinear situations. However, since finite linear cellular automata can be represented by a matrix, their inverse can be studied under various boundary conditions. Thus, the reversibility problem can be studied on finite fields using algebraic properties.

In order to characterize 1D PBCA on the finite field. We give the algorithm that determines under which conditions the cellular automata is reversible or not. Herein, we take advantage of matrix algebras.

We present a formula for determining the reversibility of cellular automata. We also find the criteria of reversibility of these 1D PBCA. The following Lemma gives a formula for computing the determinant of the matrix in (3).

Lemma 3.1 We can express the determinant of the matrix T_n as follows:

If $n = 2k$, $\det(T_n) = w_2^2 - w_1^2$ for $k \geq 1$

If $n = 2k + 1$, $\det(T_n) = w_2^2 + w_1^2$

Example 3.2 Let $n = 16, p = 7$. If we take $w_1 = 6$, $w_2 = 4$. Then we get the following matrix:

$$T_{16} = \begin{pmatrix}
4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&6\\
6&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&6&4&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&6&4&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&6&4&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&6&4&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&6&4&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&6&4&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&6&4&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&6&4&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&6&4&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&6&4&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&6&4&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&6&4&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&6&4&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&6&4
\end{pmatrix}_{16\times 16}$$

$\det(T_n) = w_2^n - w_1^n = 4^{16} - 6^{16} = -2816814940163 \equiv 3 \, mod \, 7.$

Since the determinant of the characteristic matrix is nonzero, the characteristic matrix is invertible. Thus, cellular automata that correspond to a characteristic matrix are reversible.

Example 3.2 Let $n = 25, p = 11$. If we take, $w_1 = 7$, $w_2 = 10$. Then we get the following matrix:

$$\boldsymbol{T_{25}} = \begin{pmatrix}
10&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&7\\
7&10&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&7&10&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&7&10&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&7&10&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&7&10&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&7&10&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&7&10&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&7&10&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&7&10&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&7&10&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&7&10&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&7&10&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&7&10&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&7&10&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&7&10&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&7&10&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&7&10&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&7&10&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&7&10&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&7&10&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&7&10&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&7&10&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&7&10&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&7&10
\end{pmatrix}_{25\times 25}$$

$\det(T_n) = w_2^n + w_1^n = 10^{25} + 7^{25} = 10001341068619663964900807 \equiv 9 \, mod \, 11.$

Similarly, we investigate example 3.2, we arrive that cellular automata are reversible.

4. CONCLUSION

In this work, we have studied 1D PBCA over finite fields with the help of matrix algebra. We have implemented a new local rule for obtaining the characteristic matrix. Consequently, we have acquired a new characteristic matrix. We have found a new formula for the reversibility criteria of cellular automata. Finally, we have given some examples of this formula. Further properties of 1-D CA other fields remain to be of great future research interest.

Statement of Research and Publication Ethics
Research and publication ethics were observed in the study.

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