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TRADE RULES FOR UNCLEARED MARKETS WITH A VARIABLE POPULATION*

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Abstract

We analyze markets in open economies in which the price of a traded commodity is fixed and as a result of this stickiness, the demand and the supply are possibly unequal. In our model, the agents have single peaked preferences on their consumption and production choices. For such markets, we analyze the implications of population changes as formalized by the well-known "consistency" property. We first characterize the subclass of "Uniform trade rules" that satisfies Pareto optimality, no-envy, and consistency. Next, we add an informational simplicity property which is called "independence of trade volume" and we show that among the "Uniform trade rules" that satisfy Pareto optimality, no-envy, and consistency, only the one that clear either the short or long side of the market satisfies independence of trade volume.

Keywords: market disequilibrium, trade rule, variable population, efficiency, consistency

JEL Classification numbers: D5, D6, D7

I. INTRODUCTION

We analyze markets in open economies in which (i) the price of a traded commodity is fixed, (ii) the demand and the supply are possibly unequal, and (iii) the population is variable.

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There are many examples of such non-clearing markets. Mostly, these are regulated markets, that is, the price is determined by a central authority.

The agricultural sector such as the hazelnut market provides a typical example. For political reasons, the markets in this sector are usually regulated and because of these regulations, the demand and the supply may not be equal. In fact, there is usually an excess supply. For example, in hazelnut market, the prices are determined by the government and as a result, there is usually an excess supply. For example, in Turkey, the government sets a maximum amount of production for each farmer and up to that amount, it purchases all the supply. The public health sector provides another example. The prices of public hospitals are determined by a central authority and by law, the hospitals have to attend all the patients even though there is usually excess demand.

The main question is the following: in such markets, how should a central authority design a mechanism (hereafter, a trade rule) that determines the trade? In this paper, we characterize trade rules satisfying some good properties.

In our model, there is only one commodity to be traded. There are differentiated sets of buyers and sellers. We assume that buyers have single peaked preferences on their consumption of the commodity. This assumption is derived from a general assumption that buyers have strictly convex preferences on consumption bundles. Similarly, we assume that sellers have single peaked preferences on their production of the commodity. This is also derived from a general assumption that the sellers have strictly convex production sets.

A trade rule maps each economy to a feasible trade. In our model, it is made up of two components: a trade-volume rule and an allocation rule. The trade-volume rule determines the trade-volume that will be carried out in the economy and thus, the total consumption and the total production. Then, the allocation rule allocates the total consumption among the buyers and the total production among the sellers.

A trade-volume rule takes single peaked preferences of the buyers and sellers and it determines the trade-volume. When there are only one buyer and one seller in our model, this is like determining the level of public good production level when agents have single peaked preferences. In this sense, our model is related to (Moulin, 1980). However, when there are more than one buyer or seller, our model is more complicated because of the interaction or buyers and sellers.

An allocation rule takes the single peaked preferences of the buyers and sellers and also the trade volume and it allocates this volume among buyers and sellers. This problem is extensively analyzed by (Sprumont, 1991) who proposed and analyzed a “uniform rule” which later became a central rule of that literature (for example, see (Dagan, 1996), (Ching, 1992, 1994), (Thomson, 1994)). Since we analyze markets with multiple buyers and sellers, our domain is an extension of Sprumont's domain.

Our model is also related to (Thomson, 1995) and (Klaus, Peters & Storcken, 1997, 1998). They analyze the reallocation of an infinitely divisible commodity among agents with single peaked preferences and individual endowments. Suppliers are the agents whose endowments are greater than their peaks and demanders are the agents whose endowments are less than their peaks. Note that, in their model the suppliers and the demanders are not differentiated. A supplier by misrepresenting his preferences can turn into a demander or a demander can turn into a supplier. In our model, however, buyers and sellers are differentiated. This difference has important implications over the properties analyzed. For example, fairness properties are much weaker in our model since they only compare agents on the same side of the market. Also, in our model the agents do not have exogenously given endowments.

The following papers study the design of a mechanism that determines the trade in nonclearing markets. (Bénassy, 2002) analyzes nonclearing markets and the following properties: Pareto optimality, voluntary trade, and strategy proofness. However, he does not study designing a mechanism satisfying those properties. He rather uses a trade rule that clears the short side of the market and uniformly rations the long side of it.

(Barbera & Jackson, 1995) analyze allocation of goods in exchange economies with a finite number of agents and commodities. Each agent has a positive endowment of the commodities and a continuous, strictly convex, and monotonic preference relation on his consumption. The authors look for strategy proof rules that facilitate trade in this exchange economy.

Our model is closely related to (Kıbrıs & Küçükşenel, 2009). They analyze a class of trade rules each of which is a composition of the Uniform rule with a trade-volume rule that picks the median of total demand, total supply and an exogenous constant. They show that this class uniquely satisfies Pareto optimality, strategy proofness, no-envy, and an informational simplicity axiom called independence of trade-volume.

In all these papers, the authors analyze markets with a fixed population. In this paper, we allow the population to be variable and analyze the implications of these population changes. We introduce a class of Uniform trade rules each of which is a composition of the Uniform rule and a trade-volume rule. We axiomatically analyze Uniform trade rules on the basis of a property concerning variations of the population, namely, consistency and standard properties such as Pareto optimality and no-envy, and an informational simplicity property, strong independence of trade volume.

Our main objective in this paper is to understand the implications of an important property, consistency that is about the possible variations in the number of agents. Informally, a rule is consistent if any recommendation it makes for an economy always agrees with its recommendations for the associated reduced economies obtained by the departure of some of the agents with their promised shares. Consistency has been analyzed in many contexts such as bargaining, coalitional form games, and taxation (for a detailed discussion, see our Model). Consistency, however, is not well-defined for closed economies. Therefore, we analyze a specific type of an open economy by allowing possible transfers to/from outside the economy (for a detailed discussion, see our model).

We show in Theorem 1 that a particular subclass of Uniform trade rules uniquely satisfies consistency together with Pareto optimality and no-envy. Next, we add strong independence of trade volume to the list and characterize a smaller subclass that satisfies those properties. We note that each member of this subclass either clears the short side or the long side of any given market.

The paper is organized as follows. First, we introduce the model and then we analyze the implications of consistency.

II. MODEL

There are countable infinite universal sets, $\mathcal{B}$ of potential buyers and $\mathcal{S}$ of potential sellers. Let $\mathcal{B} \cap \mathcal{S} = \emptyset$. There is a perfectly divisible commodity that each seller produces and each buyer consumes. Let $\mathbb{R}_{++}$ be the consumption/ production space for each agent. Let R be a preference relation over $\mathbb{R}_{++}$ and P be the strict preference relation associated with R. The preference relation R is **single-peaked** if there is $p(R) \in \mathbb{R}_{++}$ called the peak of R, such that for all $x, y \in \mathbb{R}_{++}$, $x < y \leq p(R)$ or $x > y \geq p(R)$ implies $y\,P\,x$. Each $i \in \mathcal{B} \cap \mathcal{S}$ is endowed with a continuous single-peaked preference relation R_i over $\mathbb{R}_{++}$. Let $\mathcal{R}$ denote the set of all continuous and single-peaked preference relations on $\mathbb{R}_{++}$.

Given a finite set $B \subset \mathcal{B}$ of buyers and a finite set $S \subset \mathcal{S}$ of sellers such that either $B \neq \emptyset$ or $S \neq \emptyset$, let $N = B \cup S$ be a **society**. Let $\mathcal{N}$ be the set of all societies. A preference profile R_N for a society N is a list $(R_i)_{i \in N}$ such that for each $i \in N$, $R_i \in \mathcal{R}$. Let $\mathcal{R}^N$ denote the set of all profiles for the society N. Given $N' \subset N$ and $R_N \in \mathcal{R}^N$, let $R_{N'} = (R_i)_{i \in N'}$ denote the restriction of R_N to N'.

A **market for society N** $= B \cup S$ is a list (R_B, R_S, T) where $(R_B, R_S) \in \mathcal{R}^N$ is a profile of preferences for buyers and sellers and $T \in \mathbb{R}$ is a **transfer**. Note that T can both be positive and negative. A positive T represents a transfer made from outside. Thus, it is added to the production of the sellers and together they form the total supply. On the other hand, a negative T represents a transfer that must be made from the economy to the outside. Thus, it is considered as an addition to the total demand.

Given a market (R_B, R_S, T) for a society $N = B \cup S$, a **(feasible) trade** is a vector $z \in \mathbb{R}^N_{++}$ such that $\sum_B z_b = \sum_S z_s + T$. Let $Z(R_B, R_S, T)$ denote the set of all trades for (R_B, R_S, T).

There are two special subclasses of markets.

A market (R_B, R_S, T) is a **just-buyer market** if $B \neq \emptyset$ and $S = \emptyset$. For such markets, the feasible trades are as follows: If $T > 0$, $Z(R_B, R_S, T) = \{z \in \mathbb{R}^B_{++}: \sum_B z_b = T\}$. If $T \leq 0$, then $Z(R_B, R_S, T) = \emptyset$. This is trivial because if there is no seller, all the agents are demanders, and thus, the supply is zero. Thus, if the outside transfer is positive, it would be equal to the total supply and it is divided among the buyers. However, if there is a negative transfer (that is, a transfer must be made to outside), since there is no seller, the transfer cannot be realized. Thus, in that case there is no trade. Similarly, if there is no outside transfer, then the total supply is zero. Thus, there is again no trade.

A market (R_B, R_S, T) is a **just-seller market** if $B = \emptyset$ and $S \neq \emptyset$. For such markets, the feasible trades are as follows: If $T < 0$, $Z(R_B, R_S, T) = \{z \in \mathbb{R}^S_{++}: \sum_S z_s + T = 0\}$. If $T \geq 0$, then $Z(R_B, R_S, T) = \emptyset$. The explanation is similar to above. Note that just-buyer markets and just-seller markets mathematically coincide with the allocation problems analyzed by (Sprumont, 1991). Thus, his domain is a restriction of ours.

Since the markets with no feasible trade are trivial, we restrict ourselves to the set of markets for which the set of trades is nonempty. Let $\mathcal{M}^N = \{(R_B, R_S, T): (R_B, R_S) \in \mathcal{R}^N, T \in \mathbb{R}, \text{and } Z(R_B, R_S, T) \neq \emptyset\}$ be the set of all markets for society $N = B \cup S$ and let be the set of all markets.

$$\mathcal{M} = \bigcup_{N \in \mathcal{N}} \mathcal{M}^N$$

Let $\mathcal{M}_{\mathcal{B}} = \{(R_B, R_S, T) \in \mathcal{M}: B \neq \emptyset,\ S = \emptyset, and\ T > 0\}$ be the set of just-buyer markets and $\mathcal{M}_{\mathcal{S}} = \{(R_B, R_S, T) \in \mathcal{M}: S \neq \emptyset,\ B = \emptyset, and\ T < 0\}$ be the set of just-seller markets.

Let $h(R_B, R_S, T)$ denote the **short side of the market** $(\boldsymbol{R_B}, \boldsymbol{R_S}, \boldsymbol{T})$, that is,

$$h(R_B, R_S, T) = \begin{cases} B, & \sum_B p(R_b) \leq \sum_S p(R_s) + T, \\ S, & \sum_S p(R_s) + T \leq \sum_B p(R_b). \end{cases}$$

A trade $z \in Z(R_B, R_S, T)$ is **Pareto optimal with respect to** $(\boldsymbol{R_B}, \boldsymbol{R_S}, \boldsymbol{T})$ if there is no $z' \in Z(R_B, R_S, T)$ such that for all $i \in B \cup S$, $z_i'\ R_i\ z_i$ and for some $j \in B \cup S$, $z_j'\ P_j\ z_j$. The following lemma shows that in our framework, *Pareto optimality* is equivalent to the following three properties: (i) each agent in the short side of the market receives a share greater than or equal to his peak, (ii) each agent in the long side of the market receives a share less than or equal to his peak, and thus (iii) the total consumption is between the total supply and the total demand. Its proof is simple, see (Kıbrıs & Küçükşenel, 2009).

Lemma 1. For each $\boldsymbol{B} \cup \boldsymbol{S} \in \boldsymbol{\mathcal{N}}$ and $(\boldsymbol{R_B}, \boldsymbol{R_S}, \boldsymbol{T}) \in \boldsymbol{\mathcal{M}^{B \cup S}}$, a trade $\boldsymbol{z} \in \boldsymbol{Z}(\boldsymbol{R_B}, \boldsymbol{R_S}, \boldsymbol{T})$ is Pareto optimal with respect to $(\boldsymbol{R_B}, \boldsymbol{R_S}, \boldsymbol{T})$ if and only if for $\boldsymbol{K} \in \{\boldsymbol{B}, \boldsymbol{S}\}$, $\boldsymbol{h}(\boldsymbol{R_B}, \boldsymbol{R_S}, \boldsymbol{T}) = \boldsymbol{K}$ implies

(i) for each $k \in K$, $p(R_k) \leq z_k$,

(ii) for each $l \in N \backslash K$, $z_l \leq p(R_l)$,

(iii) $\sum_B z_b = \begin{cases} \sum_B p(R_b) & if \;\; h(R_B, R_S, T) = B, \\ \sum_S p(R_s) + T & if \;\; h(R_B, R_S, T) = S. \end{cases}$

A trade rule first determines the volume of trade that will be carried out in the economy and therefore, the total production and the total consumption. Then, it allocates the total production among the sellers and the total consumption among the buyers. Before defining a trade rule, we will first define a trade-volume rule.

A **trade-volume rule** $\Omega : \mathcal{M} \to \mathbb{R}^2_{++}$ associates each market (R_B, R_S, T) with a vector $\Omega(R_B, R_S, T) = (\Omega_B(R_B, R_S, T), \Omega_S(R_B, R_S, T))$ whose first coordinate, $\Omega_B(R_B, R_S, T)$ is the total consumption of the buyers and the second coordinate, $\Omega_S(R_B, R_S, T)$ is the total production of the sellers. Note that, for each market (R_B, R_S, T) and a trade-volume *rule* Ω, $\Omega_B(R_B, R_S, T) = \Omega_S(R_B, R_S, T) + T$. Thus, the volume of Ω_B determines the volume of Ω_S. Therefore, with an abuse of notation, we will sometimes call Ω_B a trade-volume rule.

In a market in which there are only buyers, the transfer is divided among the buyers. Thus, the total consumption is equal to the transfer. In a just-seller market, however, the sellers produce an amount that corresponds to the transfer. Thus, in that case, the total production is equal to the absolute value of the transfer.

Let $\mathcal{V}$ be the set of all trade-volume rules. Let $\mathcal{V}^{[\text{short,long}]}$ be the set of trade-volume rules, Ω each of which chooses a trade-volume between the total demand and supply of the market, that is, for each market $(\mathrm{R_B}, \mathrm{R_S}, \mathrm{T})$, $\Omega(\mathrm{R_B}, \mathrm{R_S}, \mathrm{T}) \in [\sum_{\mathrm{B}} \mathrm{p}(\mathrm{R_b}), \; \sum_{\mathrm{S}} \mathrm{p}(\mathrm{R_s}) + \mathrm{T}]$.

The following subclass of $\mathcal{V}^{[\text{short,long}]}$ will be used extensively in rest of the paper. Let $\mathcal{V}^{\{\text{short,long}\}}$ be the set of trade-volume rules, Ω each of which alternates between picking the total demand/supply of the short and the long side of the market, that is, for each market $(\mathrm{R_B}, \mathrm{R_S}, \mathrm{T})$, $\Omega(\mathrm{R_B}, \mathrm{R_S}, \mathrm{T}) \in \{\sum_{\mathrm{B}} \mathrm{p}(\mathrm{R_b}), \; \sum_{\mathrm{S}} \mathrm{p}(\mathrm{R_s}) + \mathrm{T}\}$.

An **allocation rule** $f\colon \bigcup_{N \in (2^{\mathcal{B}} \cup 2^{\mathcal{S}}) \backslash \{\emptyset\}} \mathcal{R}^N \times \mathbb{R}_{++} \to \bigcup_{N \in (2^{\mathcal{B}} \cup 2^{\mathcal{S}}) \backslash \{\emptyset\}} \mathbb{R}^N_{++}$ allocates each trade volume among buyers and sellers in such a way that for each $N \in (2^{\mathcal{B}} \cup 2^{\mathcal{S}}) \backslash \{\emptyset\}$, $R_N \in \mathcal{R}^N$, and

$w \in \mathbb{R}_{++}$, $\sum_N f_i(R_N, w) = w$. For example, *uniform rule*, U, introduced by (Sprumont, 1991) is very central in the literature. In our paper, also, it will be used extensively. Formally, it is defined as follows: for each $N \in (2^{\mathcal{B}} \cup 2^{\mathcal{S}})\backslash\{\emptyset\}$, $R_N \in \mathcal{R}^N, w \in \mathbb{R}_{++}$, and $i \in N$,

$$U_i(R_N, w) = \begin{cases} \min\{p(R_i), \lambda\}, & if \ \ \sum_N p(R_i) \geq w \\ \max\{\mathrm{p}(R_i), \mu\}, & if \ \sum_N p(R_i) \leq w \end{cases}$$

where λ and μ are uniquely determined by the equations, $\sum_N \min\{p(R_i), \lambda\} = w$ and $\sum_N \max\{p(R_i), \mu\} = w$.

A **trade rule** $\mathrm{F}: \mathcal{M} \to \bigcup_{\mathrm{M} \in \mathcal{M}} \mathrm{Z}(\mathrm{M})$ is a composition of a trade-volume rule Ω and an allocation rule f: $\mathrm{F} = \mathrm{f} \circ \Omega$. More precisely, for each market $(\mathrm{R_B}, \mathrm{R_S}, \mathrm{T})$ and $\mathrm{K} \in \{\mathrm{B}, \mathrm{S}\}$, $\mathrm{F_K}(\mathrm{R_B}, \mathrm{R_S}, \mathrm{T}) = \mathrm{f}(\mathrm{R_K}, \Omega_{\mathrm{K}}(\mathrm{R_B}, \mathrm{R_S}, \mathrm{T}))$. A trade rule, $\mathrm{F} = \mathrm{U} \circ \Omega$, that is composed of the uniform rule and a trade-volume rule Ω is called the **uniform trade rule with respect to Ω**. (Kıbrıs & Küçükşenel, 2009) characterize a particular class of uniform trade rules for which Ω is the median of total demand, total supply, and an exogenous constant.

Now, we introduce properties of a trade rule. We start with efficiency. A trade rule F is **Pareto optimal** if for each $(\mathrm{R_B}, \mathrm{R_S}, \mathrm{T}) \in \mathcal{M}$, the trade $\mathrm{F}(\mathrm{R_B}, \mathrm{R_S}, \mathrm{T})$ is Pareto optimal with respect to $(\mathrm{R_B}, \mathrm{R_S}, \mathrm{T})$. Pareto optimality of an allocation rule is defined in a similar way.

Now, we present a fairness property. A trade is envy free if each buyer (respectively, seller) prefers his own consumption (respectively, production) to that of every other buyer (respectively, seller). A trade rule satisfies no-envy, if it always chooses an envy free trade. Formally, a trade rule satisfies **no-envy** if for each $\mathrm{N} = (\mathrm{B} \cup \mathrm{S}) \in \mathcal{N}$, $(\mathrm{R_B}, \mathrm{R_S}, \mathrm{T}) \in \mathcal{M}^{\mathrm{N}}$, $\mathrm{K} \in \{\mathrm{B}, \mathrm{S}\}$, and $\mathrm{i}, \mathrm{j} \in \mathrm{K}$, $\mathrm{F_i}(\mathrm{R_B}, \mathrm{R_S}, \mathrm{T}) \ \mathrm{R_i} \ \mathrm{F_j}(\mathrm{R_B}, \mathrm{R_S}, \mathrm{T})$. Since in our model the agents on different sides of the market are exogenously differentiated, this property only compares agents on the same side of the market.

Next, we present a property concerning variations in the number of agents. It is an adaptation of the standard consistency property to our domain. To explain consistency, consider a trade rule F and a market $(\mathrm{R_B}, \mathrm{R_S}, \mathrm{T})$. Suppose that F chooses the trade z. Imagine that some buyers and sellers leave with their shares they have been already assigned. This leads to a reduced market that the remaining agents are now facing. Consistency is about how the remaining agents' shares should be affected in this reduced market. If F is consistent, it should assign to them the same shares as in the initial market. However, without a transfer from outside, the recommendation for an economy may not be feasible for its reduced markets. This is one reason we consider open economies. This practice is similar to the analysis of consistency in economies with individual endowments (see (Thomson, 1992)). This leads to a reduced problem in which the remaining agents, $(B' \cup S')$ are now facing an updated transfer from T to $T - \sum_{B\backslash B'} z_b + \sum_{S\backslash S'} z_s$. Formally, given a trade rule F, for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, and $N' = (B' \cup S') \subseteq N$, a reduced market of (R_B, R_S, T) for N' at $z \equiv F(R_B, R_S, T)$ is $r^z_{N'}(R_B, R_S, T) = (R_{B'}, R_{S'}, T - \sum_{B\backslash B'} z_b + \sum_{S\backslash S'} z_s)$. A trade rule F is **consistent** if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, and $N' = (B' \cup S') \subseteq N$, if $z = F(R_B, R_S, T)$, then $z_{N'} = F(r^z_{N'}(R_B, R_S, T))$.

Consistency of a trade-volume rule can be defined in a similar way. It is about how the trade volume should be affected in the reduced market. If the trade-volume rule is consistent with respect to $F = f \circ \Omega$, then the trade volume in the reduced market should be the total consumption of the remaining buyers in the initial market (or equivalently, the total production of the remaining sellers in the initial market). Formally, a trade-volume rule Ω is **consistent with respect to F = f ∘ Ω** if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, $N' = (B' \cup S') \subseteq N$ and $z = F(R_B, R_S, T)$, $\Omega\left(r^z_{N'}(R_B, R_S, T)\right) = \sum_{B'} z_{b'}$ if $B' \neq \emptyset$ and $\Omega\left(r^z_{N'}(R_B, R_S, T)\right) = \sum_{S'} z_{s'}$ otherwise.

Lastly, we present the following informational simplicity property. Strong independence of trade volume requires the trade volume rule only to depend on the total demand and total supply but not on their individual components and the agents' identities. Formally, Ω satisfies **strong independence of trade volume** if for each $N = (B \cup S) \in \mathcal{N}$, $N' = (B' \cup S') \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, $(R_{B'}, R_{S'}, T) \in \mathcal{M}^{N'}$, $\sum_B p(R_b) = \sum_{B'} p(R_{b'})$, and $\sum_S p(R_s) = \sum_{S'} p(R_{s'})$ imply $\Omega(R_B, R_S, T) = \Omega(R_{B'}, R_{S'}, T)$.

III. RESULTS

The following theorem shows that the subclass of Uniform trade rules $F = U \circ \Omega$ where $\Omega \in \mathcal{V}^{[\text{short,long}]}$ is consistent with respect to F uniquely satisfies Pareto optimality, no-envy and consistency.

Theorem 1. A trade rule $F = f \circ \Omega$ satisfies Pareto optimality, no-envy, and consistency if and only if f = U and Ω satisfies the following:

(i) $\Omega \in \mathcal{V}^{[\text{short,long}]}$

(ii) Ω is consistent with respect to F.

Next, we add strong independence of trade volume to the list and we show in Theorem 2 that under strong independence of trade volume, the subclass of Uniform trade rules, $F = U \circ \Omega$ where $\Omega \in \mathcal{V}^{\{\text{short,long}\}}$ and Ω is consistent with respect to F uniquely satisfies Pareto optimality, no-envy and consistency.

Theorem 2. Let $\Omega \in \mathcal{V}$ satisfy strong independence of trade volume. A trade rule $F = f \circ \Omega$ satisfies Pareto optimality, no-envy, and consistency if and only if f = U and Ω satisfies the following:

(i) $\Omega \in \mathcal{V}^{\{\text{short,long}\}}$

(ii) Ω is consistent with respect to F.

IV. CONCLUSION

We analyze markets in open economies in which price is fixed and as a result the demand and the supply are possibly unequal and the population is variable. We characterize trade rules with respect to consistency property. We show that these rules either clear the short or the long side of the market.

In addition to consistency, there are other properties about population variation. Our next study will be the analysis of the other properties related to population variation, such as population monotonicity.

Another open question is the weakening of strong independence of trade volume. This property requires the trade volume rule to depend only on the total demand and total supply but not on their individual components and the agents' identities. One can study the implications of a weaker property which only relates two problems with the same set of agents.

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APPENDIX

To prove Theorem 1, we use the following two lemmas. The first one analyzes the relationship between the properties satisfied by a trade rule $F = f \circ \Omega$ and its component f. It shows that Pareto optimality, no-envy, and consistency satisfied by F passes on to f.

Lemma 2. If a trade rule $F = f \circ \Omega$ satisfies one of the following properties, then f also satisfies that property: Pareto optimality, no-envy, and consistency.

Proof. First, suppose for a contradiction $F = f \circ \Omega$ satisfies Pareto optimality whereas f does not. Then, there is $K \in (2^{\mathcal{B}} \cup 2^{\mathcal{S}}) \setminus \{\emptyset\}$, $R_K \in \mathcal{R}^K$, and $w \in \mathbb{R}_{++}$ such that $f(R_K, w)$ is not Pareto optimal with respect to (R_K, w). Then, there is $z \in \mathbb{R}_{++}^K$ such that for each $k \in K$, $z_k\ R_k\ f_k(R_K, w)$, for some $l \in K$, $z_l\ P_l\ f_l(R_K, w)$, and $\sum_K z_k = w$. First, suppose $K \subseteq \mathcal{B}$. Then, consider $(R_K, T) \in \mathcal{M}_{\mathcal{B}}$ such that $T = w$. Note that $F(R_K, T) = f\big(R_K, \Omega_B(R_K, T)\big) = f(R_K, T) = f(R_K, w)$. Then z also Pareto dominates $F(R_K, T)$, a contradiction to F being Pareto optimal. If $K \subseteq \mathcal{S}$, then consider $(R_K, T) \in \mathcal{M}_S$ such that $T = -w$. Note that, $F(R_K, T) = f\big(R_K, \Omega_S(R_K, T)\big) = f(R_K, -T) = f(R_K, w)$. Then z also Pareto dominates $F(R_K, T)$, a contradiction to F being Pareto optimal. The other properties can be proved similarly.

The second lemma is by (Dagan, 1996) on the allocation rule f. Bilateral consistency is a weaker consistency property restricted to subsocieties containing exactly two agents. For its proof, see (Dagan, 1996).

Lemma 3. (Dagan, 1996) If the potential number of agents is at least 4 and if an economy consists of at least 2 agents, then f satisfies Pareto optimality, no-envy, and bilateral-consistency if and only if f = U.

Proof. (Theorem 1) The if part is straightforward and thus, omitted. The only if part is as follows. Since F satisfies Pareto optimality, no-envy, and consistency, by Lemma 2, f also satisfies those properties. Then, by Lemma 3, f = U. (For markets with only one buyer or one seller, all the allocation rules choose the same allocation.)

Now, let $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$ and $(B' \cup S') \in \mathcal{N}$ be such that $N' = (B' \cup S') \subseteq (B \cup S)$. Let $z \equiv F(R_B, R_S, T)$ and $z' \equiv F\left(r^z_{N'}(R_B, R_S, T)\right)$. Since F is consistent, for each $i \in N'$, $z'_i = z_i$. Then, by the definition of Ω, $\Omega\left(r^z_{N'}(R_B, R_S, T)\right) = \sum_{B'} z'_{b'} = \sum_{B'} z_{b'}$. Thus, Ω is consistent with respect to F.

To prove Theorem 2, in addition to lemmas 2 and 3, we need the following lemma. It shows that for Pareto optimal rules, a reduced market has the same short side as the original.

Lemma 4. Let F be a Pareto optimal trade rule. Then, for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, and $N' = (B' \cup S') \subseteq N$, if $z \equiv F(R_B, R_S, T)$, then we have

$$h(r^z_{N'}(R_B, R_S, T)) = \begin{cases} B', & \text{if } \ h(R_B, R_S, T) = B \text{ and } B' \neq \emptyset, \\ S', & \text{if } \ h(R_B, R_S, T) = S \text{ and } S' \neq \emptyset. \end{cases}$$

Proof. Let $\mathbf{N} = (\mathbf{B} \cup \mathbf{S}) \in \boldsymbol{\mathcal{N}}$, $(\mathbf{R_B}, \mathbf{R_S}, \mathbf{T}) \in \boldsymbol{\mathcal{M}}^{\mathbf{N}}$, and $\mathbf{N'} = (\mathbf{B'} \cup \mathbf{S'}) \subseteq \mathbf{N}$. Let $\mathbf{z} \equiv \mathbf{F}(\mathbf{R_B}, \mathbf{R_S}, \mathbf{T})$. First, suppose $\mathbf{h}(\mathbf{R_B}, \mathbf{R_S}, \mathbf{T}) = \mathbf{B}$ and $\mathbf{B'} \neq \emptyset$. Since F is Pareto optimal, z is Pareto optimal with respect to $(\mathbf{R_B}, \mathbf{R_S}, \mathbf{T})$. Then, by Lemma 1, for each $\mathbf{b} \in \mathbf{B}$, $\mathbf{p}(\mathbf{R_b}) \leq \mathbf{z_b}$ and for each $\mathbf{s} \in \mathbf{S}$, $\mathbf{z_s} \leq \mathbf{p}(\mathbf{R_s})$. Then,

$$\begin{aligned} \sum_{B \setminus B'} z_b + \sum_{B'} p(R_{b'}) &\leq \sum_{B} z_b \\ &= \sum_{S} z_s + T \\ &\leq \sum_{S'} p(R_{s'}) + \sum_{S \setminus S'} z_s + T. \end{aligned}$$

That is, $\sum_{\mathbf{B'}} \mathbf{p}(\mathbf{R_{b'}}) \leq \sum_{\mathbf{S'}} \mathbf{p}(\mathbf{R_{s'}}) + \mathbf{T} - \sum_{\mathbf{B} \setminus \mathbf{B'}} \mathbf{z_b} + \sum_{\mathbf{S} \setminus \mathbf{S'}} \mathbf{z_s}$. Note that $\mathbf{r^z_{N'}}(\mathbf{R_B}, \mathbf{R_S}, \mathbf{T}) = (\mathbf{R_{B'}}, \mathbf{R_{S'}}, \mathbf{T'})$ for $\mathbf{T'} = \mathbf{T} - \sum_{\mathbf{B} \setminus \mathbf{B'}} \mathbf{z_b} + \sum_{\mathbf{S} \setminus \mathbf{S'}} \mathbf{z_s}$. Thus, $\sum_{\mathbf{B'}} \mathbf{p}(\mathbf{R_{b'}}) \leq \sum_{\mathbf{S'}} \mathbf{p}(\mathbf{R_{s'}}) + \mathbf{T'}$. Therefore, $\mathbf{h}(\mathbf{r^z_{N'}}(\mathbf{R_B}, \mathbf{R_S}, \mathbf{T})) = \mathbf{B'}$. The proof of the other case is similar.

Proof. (Theorem 2) The if part is straightforward and thus, omitted. The only if part is as follows. Since F satisfies Pareto optimality, no-envy, and consistency, by Theorem 1, $F = U \circ \Omega$ where $\Omega \in$

$\mathcal{V}^{[\text{short,long}]}$ and Ω is consistent with respect to F. Now, by using strong independence of trade volume, we will show that $\Omega \in \mathcal{V}^{\{\text{short,long}\}}$.

For this, let $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$. First, assume that $h(R_B, R_S, T) = S$. Let $a = \sum_B p(R_b)$, $d = \sum_S p(R_s) + T$ and $c = \Omega(R_B, R_S, T)$. Since $\Omega \in \mathcal{V}^{[\text{short,long}]}$, $c \in [d, a]$. Suppose for a contradiction$c \notin \{a, d\}$, that is $c \in (d, a)$. Let $\varepsilon \in \mathbb{R}_+$ be such that $\mathcal{E} < \min\left\{\frac{c}{n}, \frac{2(a-c)}{(n-2)}, \frac{2(n-1)(c-d)}{(m-1)(n-2)}\right\}$. Also, let $m, n \in \mathbb{N}$ be such that $n \geq 3$ and $m > \max\left\{3, \frac{c-T}{d-T}\right\}$.

Let $(R_{B'}, R_{S'}, T) \in \mathcal{M}^{B' \cup S'}$ be such that $|B'| = n$, $|S'| = m$ and

$$p(R_{b'_1}) = \frac{c}{n} - \varepsilon,\ p(R_{b'_2}) = \cdots = p(R_{b'_n}) = \frac{a}{n-1} - \frac{c}{n(n-1)} + \frac{\varepsilon}{n-1},$$

$$p(R_{s'_1}) = \frac{c}{m} - \frac{T}{m} + \frac{\varepsilon(m-1)(n-2)}{2(m-2)(n-1)},\ p(R_{s'_2}) = \frac{d}{m-1} - \frac{T}{m} - \frac{c}{m(m-1)} + \frac{\varepsilon(m-3)(n-2)}{2(m-2)(n-1)},$$

$$p(R_{s'_3}) = \cdots = p(R_{s'_m}) = \frac{d}{m-1} - \frac{T}{m} - \frac{c}{m(m-1)} - \frac{\varepsilon(n-2)}{(m-2)(n-1)}.$$

Also, let $(R'_{B'}, R'_{S'}, T) \in \mathcal{M}^{B' \cup S'}$ be such that

$$p\left(R'_{b'_1}\right) = \frac{c}{n} - \frac{\varepsilon}{2},\ p\left(R'_{b'_2}\right) = \frac{a}{n-1} - \frac{c}{n(n-1)} - \frac{\varepsilon(n-3)}{2(n-1)},$$

$$p\left(R'_{b'_3}\right) = \cdots = p\left(R'_{b'_n}\right) = \frac{a}{n-1} - \frac{c}{n(n-1)} + \frac{\varepsilon}{(n-1)},$$

$$p\left(R'_{s'_1}\right) = \frac{c}{m} - \frac{T}{m} + \frac{\varepsilon(m-1)(n-2)}{(m-2)(n-1)}$$

$$p\left(R'_{s'_2}\right) = \cdots = p\left(R'_{s'_m}\right) = \frac{d}{m-1} - \frac{T}{m} - \frac{c}{m(m-1)} - \frac{\varepsilon(n-2)}{(m-2)(n-1)}$$

Note that by the choice of ε and m, for each $k' \in (B' \cup S')$, $p(R_{k'}) \geq 0$ and $p(R'_{k'}) \geq 0$. Also, $\sum_{B'} p(R_{b'}) = \sum_{B'} p\left(R'_{b'}\right) = a$ and $\sum_{S'} p(R_{s'}) = \sum_{S'} p\left(R'_{s'}\right) = d - T$. Then, by independence of trade volume, $\Omega(R_{B'}, R_{S'}, T) = \Omega(R'_{B'}, R'_{S'}, T) = c$.

For each $K \in \{B', S'\}$, let $z_K \equiv F_K(R_{B'}, R_{S'}, T) = U(R_K, c)$ and $z'_K \equiv F_K\left(R'_{B'}, R'_{S'}, T\right) = U(R'_K, c)$. Since for each $i = 2, \cdots, n$, $p\left(R_{b'_1}\right) < \frac{c}{n} < p\left(R_{b'_i}\right)$, $p\left(R'_{b'_1}\right) < \frac{c}{n} < p\left(R'_{b'_i}\right)$, and $\frac{1}{(n-1)}(c - p\left(R'_{b'_1}\right)) < p\left(R'_{b'_i}\right)$, we have

$$z_{b'_1} = p\left(R_{b'_1}\right) = \frac{c}{n} - \varepsilon,\ z_{b'_i} = \frac{1}{(n-1)}(c - p\left(R_{b'_1}\right)) = \frac{c}{n} + \frac{\varepsilon}{n-1},$$

$$z'_{b'_1} = p\left(R'_{b'_1}\right) = \frac{c}{n} - \frac{\varepsilon}{2}, \text{ and } z'_{b'_i} = \frac{1}{(n-1)}\left(c - p\left(R'_{b'_1}\right)\right) = \frac{c}{n} + \frac{\varepsilon}{2(n-1)}.$$

Since for each $= 2, \cdots, m$, $p\left(R_{s'_i}\right) < \frac{c-T}{m} < p\left(R_{b'_1}\right)$, $p\left(R'_{s'_i}\right) < \frac{c-T}{m} < p\left(R'_{s'_1}\right)$, and $\frac{1}{(m-1)}(c - T - p(R_{s'_1})) > p\left(R'_{s'_i}\right)$, we have

$$z_{s'_1} = p\left(R_{s'_1}\right) = \frac{c}{m} - \frac{T}{m} + \frac{\varepsilon(m-1)(n-2)}{2(m-2)(n-1)}, \quad z_{s'_i} = \frac{1}{(m-1)}\left(c - T - p\left(R_{s'_1}\right)\right) = \frac{c}{m} - \frac{T}{m} - \frac{\varepsilon(n-2)}{2(m-2)(n-1)},$$

$$z'_{s'_1} = p\left(R'_{s'_1}\right) = \frac{c}{m} - \frac{T}{m} + \frac{\varepsilon(m-1)(n-2)}{(m-2)(n-1)}, \quad z'_{s'_i} = \frac{1}{(m-1)}\left(c - T - p\left(R'_{s'_1}\right)\right) = \frac{c}{m} - \frac{T}{m} - \frac{\varepsilon(n-2)}{(m-2)(n-1)}.$$

Now, let $T' = \frac{2T}{m} + \frac{2(m-n)c}{mn} - \frac{3(n-2)\varepsilon}{2(n-1)}$ and consider the following two reduced problems:

(i) $r^{z}_{\{b'_1,b'_2,s'_1,s'_2\}}(R_{B'}, R_{S'}, T) = \left(R_{b'_1}, R_{b'_2}, R_{s'_1}, R_{s'_2}, T'\right)$

(ii) $r^{z'}_{\{b'_1,b'_2,s'_1,s'_2\}}(R'_{B'}, R'_{S'}, T) = \left(R'_{b'_1}, R'_{b'_2}, R'_{s'_1}, R'_{s'_2}, T'\right)$.

Note that, $p\left(R_{b'_1}\right) + p\left(R_{b'_2}\right) = p\left(R'_{b'_1}\right) + p\left(R'_{b'_2}\right)$ and $\left(R_{s'_1}\right) + p\left(R_{s'_2}\right) = p\left(R'_{s'_1}\right) + p\left(R'_{s'_2}\right)$.

Then, by strong independence of trade volume, $\Omega\left(r^{z}_{\{b'_1,b'_2,s'_1,s'_2\}}(R_{B'}, R_{S'}, T)\right) = \Omega(r^{z'}_{\{b'_1,b'_2,s'_1,s'_2\}}\left(R'_{B'}, R'_{S'}, T\right))$. By consistency, for $i = 1,2$, $F_{b'_i}\left(r^{z}_{\{b'_1,b'_2,s'_1,s'_2\}}(R_{B'}, R_{S'}, T)\right) = z_{b'_i}$ and $F_{b'_i}\left(r^{z'}_{\{b'_1,b'_2,s'_1,s'_2\}}(R'_{B'}, R'_{S'}, T)\right) = z'_{b'_i}$.

Then, $\Omega\left(r^{z}_{\{b'_1,b'_2,s'_1,s'_2\}}(R_{B'}, R_{S'}, T)\right) = z_{b'_1} + z_{b'_2} = \frac{2c}{n} + \frac{\varepsilon(2-n)}{n-1}$ and $\Omega\left(r^{z'}_{\{b'_1,b'_2,s'_1,s'_2\}}\left(R'_{B'}, R'_{S'}, T\right)\right) = z'_{b'_1} + z'_{b'_2} = \frac{2c}{n} + \frac{\varepsilon(2-n)}{2(n-1)}$. Thus, $\Omega\left(r^{z}_{\{b'_1,b'_2,s'_1,s'_2\}}(R_{B'}, R_{S'}, T)\right) \neq \Omega\left(r^{z'}_{\{b'_1,b'_2,s'_1,s'_2\}}\left(R'_{B'}, R'_{S'}, T\right)\right)$, a contradiction. Thus, $\Omega(R_B, R_S, T) \in \{\sum_B p(R_b), \sum_S p(R_s) + T\}$.