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Local Antisymmetric Connectedness in Asymmetrically Normed Real Vector Spaces

Nezakat Javanshir[1] and Filiz Yıldız[2*]

[1]*Department of Mathematics, Faculty of Science, Ankara Yıldırım Beyazıt University, Ankara, Türkiye*
[2]*Department of Mathematics, Faculty of Science, Hacettepe University, Ankara, Türkiye*
[*]*Corresponding author*

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Abstract

In this paper, some properties of locally antisymmetrically connected spaces which are the localized version of the antisymmetrically connected T_0-quasi-metric spaces constructed as the natural counterparts of connected complementary graphs, are presented in terms of asymmetric norms.

According to that, we investigated some different aspects and examples of local antisymmetric connectedness in the framework of asymmetrically normed real vector spaces.

Specifically, it is proved that the structures of antisymmetric connectedness and local antisymmetric connectedness coincide for the T_0-quasi-metrics induced by the asymmetric norms which associate the theory of quasi-metrics with functional analysis.

1. Introduction and Preliminaries

The structure of antisymmetric connectedness of a T_0-quasi-metric space was first described in [1]. This theory was especially discussed in terms of graph theory [2, 3] as a suitable counterpart of the connectedness for complementary graph. It is also observed that there were natural relationships between the theory of antisymmetrically connected T_0-quasi-metric spaces and the theory of connected complementary graphs, with the help of symmetry graphs introduced in [1].

On the other hand, it is useful to localize some topological properties (see [4]) as is well-known from topology. In the light of these considerations; following the structure of antisymmetric connectedness constructed lately, the "locality" status of antisymmetric connectedness was investigated in [5] under the name local antisymmetric connectedness.

As for this paper, many interesting properties of locally antisymmetrically connected spaces will be presented in the context of asymmetrically normed real vector spaces.

Therefore, the paper is organized in the following format:

Some necessary background material for the remaining of paper is first given in Section 1 via some references. Particularly, Section 1 mostly consists of the information about the antisymmetrically connected T_0-quasi-metric spaces, besides the other types of spaces peculiar to asymmetric topology.

In Section 2, the required propositions and examples about the locally antisymmetrically connected T_0-quasi-metric spaces are reminded via [5], in detail.

After recalling all preliminary information, as the main purpose of paper; in Section 3, we studied some properties of the theory of local antisymmetric connectedness in the framework of asymmetric norms. Indeed, it is known from [1] that the problem to determine the antisymmetry components of points in X turns out to be easier when it is formulated for a T_0-quasi-metric induced by the asymmetric norm of an asymmetrically normed real vector space which is introduced by Cobzaş [6] in Functional Analysis. In the light of this idea, locally

Email addresses and ORCID numbers**: nezakat.javanshir@aybu.edu.tr, 0000-0002-1780-4684 (N. Javanshir), yfiliz@hacettepe.edu.tr, 0000-0002-2112-8949 (F. Yıldız),**
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antisymmetrically connected T_0-quasi-metric spaces are investigated first time in the context of asymmetrically normed real vector spaces in detail.

Specifically, in Section 3 it is proved that the antisymmetric connectedness and local antisymmetric connectedness coincide for the T_0-quasi-metrics induced by the asymmetric norms.

Finally, a conclusion part together with the two questions that could be the subject of a future work is presented as the last section of paper.

Now let us recall some required notions and also examples from [1].

Definition 1.1. *Let X be a set. Then the function $\rho : X \times X \to [0,\infty)$ is called a* quasi-pseudometric *on X if*

(a) $\rho(x,x) = 0$ *whenever* $x \in X$,
(b) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ *whenever* $x,y,z \in X$.

We will say that ρ is a T_0-quasi-metric *provided that ρ also satisfies the following condition:*

For each $x,y \in X$,

$$\rho(x,y) = 0 = \rho(y,x) \text{ implies that } x = y.$$

Remark 1.2. *If ρ is a T_0-quasi-metric on a set X then the function $\rho^{-1} : X \times X \to [0,\infty)$ defined by $\rho^{-1}(x,y) = \rho(y,x)$ whenever $x,y \in X$, is also a T_0-quasi-metric, called the* conjugate T_0-quasi-metric of ρ. *If $\rho = \rho^{-1}$ then ρ is a* metric. *In line with the usual notational conventions, we write*

$$\rho^s = \sup\{\rho, \rho^{-1}\} = \rho \vee \rho^{-1}$$

for the symmetrization metric of ρ.

The notation τ_{ρ^s} denotes the topology generated by the (symmetrization) metric ρ^s and it is called symmetrization topology of ρ.

An adequate background for the T_0-quasi-metric spaces can be obtained in the works [7–12]. Now, as for the main structures required for the paper:

Definition 1.3. *If (X,ρ) is a T_0-quasi-metric space then the pair $(x,y) \in X \times X$ is called*

(i) antisymmetric pair *whenever the condition $\rho(x,y) \neq \rho(y,x)$ is satisfied.*
(ii) symmetric pair *if it satisfies the condition $\rho(x,y) = \rho(y,x)$.*

Definition 1.4. *If (X,ρ) is a T_0-quasi-metric space then a finite sequence of the elements in X is called an* antisymmetric path *(*symmetric path*) $Q_{x,y} = (x = x_0, x_1, \ldots, x_{n-1}, x_n = y)$, $n \in \mathbb{N}$, from the starting point x to the ending point y provided that all the pairs (x_i, x_{i+1}) are antisymmetric pairs (symmetric pairs) for $i \in \{0,1,\ldots,n-1\}$.*

Now, let us recall the antisymmetric connectedness from [1]:

Definition 1.5. (i) *In a T_0-quasi-metric space (X,ρ), two points $x,y \in X$ will be called* antisymmetrically connected *if we have an antisymmetric path $Q_{x,y}$ starting at x and ending with y, or $x = y$.*

Obviously, the relation"antisymmetric connectedness" is an equivalence relation on the set X.

(ii) *The equivalence class of $x \in X$ according to the antisymmetric connectedness relation T_ρ is called the* antisymmetry component *of x, and the notation*

$$T_\rho(x) = \{y \in X : \text{there is an antisymmetric path } Q_{x,y} \text{ from } x \text{ to } y\}$$

will be used for it.

It is clear that $T_\rho(x)$ is the largest antisymmetrically connected subspace of X containing $x \in X$.

(iii) *If $T_\rho = X \times X$, that is $T_\rho(x) = X$ whenever $x \in X$, then (X,ρ) is called* antisymmetrically connected space.

Now, let us present a well-known antisymmetrically connected T_0-quasi-metric space as follows:

Example 1.6. *On the set $\mathbb{R}$ of the reals, take $\mu(x,y) = max\{x - y, 0\}$ whenever $x,y \in \mathbb{R}$. It is easy to verify that μ is a T_0-quasi-metric, called the* standard *T_0-quasi-metric on $\mathbb{R}$. Moreover, the space $(\mathbb{R},\mu)$ is antisymmetrically connected since $T_\mu(x) = \mathbb{R}$ for each $x \in \mathbb{R}$.*

From now on, we can turn our attention to some other notions and details, required for the paper. Then an opposite notion to that of "metric" can be recalled from [1]:

Definition 1.7. *A T_0-quasi-metric space (X,ρ) is called* antisymmetric *if*

$$\rho(x,y) \neq \rho(y,x) \text{ whenever } x \neq y$$

for all $x,y \in X$.

Therefore, by Definition 1.5 (iii) we have:

Proposition 1.8. *Each antisymmetric T_0-quasi-metric space will be antisymmetrically connected.*

Additionally, the dual notion of the antisymmetric connectedness can be recalled from [1], in the framework of T_0-quasi-metrics.

Definition 1.9. (i) *If (X,ρ) is a T_0-quasi-metric space then $x \in X$ is called* symmetrically connected to *$y \in X$ whenever there exists a symmetric path (see Definition 1.4) $Q_{x,y}$, from x to y.*

Trivially, the relation "symmetric connectedness" will be an equivalence relation on the points in X.

(ii) *The equivalence class of $x \in X$ according to the symmetric connectedness relation C_ρ is called* symmetry component of x, *and the notation*

$$C_\rho(x) = \{y \in X : \text{there is a symmetric path } Q_{x,y} \text{ from } x \text{ to } y\}$$

will be used for it. Obviously $C_\rho(x)$ is the largest symmetrically connected subspace of X containing $x \in X$.

(iii) *If $C_\rho = X \times X$, that is $C_\rho(x) = X$ whenever $x \in X$, then (X,ρ) is called* symmetrically connected space.

In the light of above considerations, the next proposition was established in [1] as Corollary 25, by using the following crucial result well-known from graph theory.

For any graph G, G is connected or $\overline{G}$ the complement of G is connected in terms of graph theory. (See [2,3])

Proposition 1.10. *If (X,ρ) is a T_0-quasi-metric space then (X,ρ) is antisymmetrically connected or symmetrically connected.*

Note here that even though we will not be interested in this theory for the remainder of Section 1, the detailed background on the theory of "symmetric connectedness" can be found in [1].

The next notions will be required for the remainder of paper.

Definition 1.11. *Let ρ be a T_0-quasi-metric on X and $x \in X$. In this case,*

(i) *The point x is called* antisymmetric *whenever (x,y) is antisymmetric pair for $y \in X \setminus \{x\}$,*

(ii) *The point x is called* symmetric *whenever (x,y) is symmetric pair for $y \in X$.*

Hence, the next proposition which completes Section 1 can be seen easily via Definition 1.11.

Proposition 1.12. *If (X,ρ) is a T_0-quasi-metric space then all points of X are antisymmetric if and only if the space (X,ρ) is antisymmetric.*

After giving the required information above, now we are in the position to recall from [5] the localized version of antisymmetrically connected spaces.

2. Local Antisymmetric Connectedness

The following notions and the all propositions are presented in [5].

Definition 2.1. *Let (X,ρ) be a T_0-quasi-metric space and $x_0 \in X$. Thus (X,ρ) is called* locally antisymmetrically connected at $x_0 \in X$ *if $T_\rho(x_0) \in \tau_{\rho^s}$.*

As mentioned in Section 1, τ_{ρ^s} denotes the symmetrization topology generated by the metric $\rho^s = \rho \vee \rho^{-1}$.

Definition 2.2. *A T_0-quasi-metric space (X,ρ) is called* locally antisymmetrically connected *if (X,ρ) is locally antisymmetrically connected at each point of X.*

Hence, we have obviously the next crucial characterization because of Definition 2.1 and Definition 2.2.

Proposition 2.3. *A T_0-quasi-metric space (X,ρ) is locally antisymmetrically connected if and only if $T_\rho(x)$ (the antisymmetry component of x) is τ_{ρ^s}-open for each $x \in X$.*

Example 2.4. *Consider the (bounded) Sorgenfrey T_0-quasi-metric space $(\mathbb{R},\upsilon)$ where $\upsilon(x,y) = \min\{x-y,1\}$ if $x \geq y$ and $\upsilon(x,y) = 1$ if $x < y$. It is easy to show that the space $(\mathbb{R},\upsilon)$ is antisymmetrically connected, but not antisymmetric. Also, all antisymmetry components $T_\upsilon(x)$ $(x \in \mathbb{R})$ in the space $(\mathbb{R},\upsilon)$ are $\mathbb{R}$, and so they are open w.r.t. the discrete topology τ_{υ^s} generated by the symmetrization metric (which is discrete metric) of υ. That is, the space $(\mathbb{R},\upsilon)$ is locally antisymmetrically connected by Proposition 2.3.*

The following propositions and the last result were proved in [5] by taking into account Proposition 2.3:

Proposition 2.5. *Let (X,ρ) be a T_0-quasi-metric space. If (X,ρ) is antisymmetrically connected then (X,ρ) is locally antisymmetrically connected.*

The converse of Proposition 2.5 is not true by virtue of the next example:

Example 2.6. *Consider a T_0-quasi-metric on the set $X = \{1,2,3,4\}$ by the matrix*

$$\mathbf{W} = \begin{pmatrix} 0 & 8 & 4 & 1 \\ 9 & 0 & 6 & 7 \\ 4 & 6 & 0 & 5 \\ 3 & 7 & 5 & 0 \end{pmatrix}$$

That is, $W = (w_{ij})$ where $w(i,j) = w_{ij}$ for $i,j \in X$. Clearly, w is a T_0-quasi-metric on X. Indeed, it satisfies the other conditions of Definition 1.1, so we just prove the triangle inequality as follows: $w(1,2) = 8 \leq 4+6 = w(1,3)+w(3,2)$, $w(1,2) = 8 \leq 1+7 = w(1,4)+w(4,2)$,
$w(1,3) = 4 \leq 8+6 = w(1,2)+w(2,3)$, $w(1,3) = 4 \leq 1+5 = w(1,4)+w(4,3)$,
$w(1,4) = 1 \leq 8+7 = w(1,2)+w(2,4)$, $w(1,4) = 1 \leq 4+5 = w(1,3)+w(3,4)$,
$w(2,3) = 6 \leq 9+4 = w(2,1)+w(1,3)$, $w(2,3) = 6 \leq 7+5 = w(2,4)+w(4,3)$,

$w(2,4)=7\leq 9+1=w(2,1)+w(1,4)$, $w(2,4)=7\leq 6+5=w(2,3)+w(3,4)$,

$\vdots$

Also, note that (X,w) is not antisymmetric since $w(4,3)=w(3,4)$, and moreover it is not antisymmetrically connected since there is no any antisymmetric path from 1 to 3. Despite these, it is locally antisymmetrically connected: Note that (X,w) is a finite T_0-quasi-metric space. Thus, the symmetrized topological space (X,τ_{w^s}) will be discrete since the unique topology which is T_1 on a finite set is discrete. Hence, (X,w) will be locally antisymmetrically connected by Proposition 2.3 as the antisymmetry components of all points in X are open w.r.t. the symmetrization topology τ_{w^s}.

Proposition 2.7. *If (X,ρ) is locally antisymmetrically connected T_0-quasi-metric space and the topological space (X,τ_{ρ^s}) is connected then (X,ρ) is antisymmetrically connected.*

Proposition 2.8. *If (X,ρ) is a T_0-quasi-metric space then (X,ρ) is symmetrically connected or locally antisymmetrically connected.*

Corollary 2.9. *If (X,ρ) is an antisymmetric space then (X,ρ) is locally antisymmetrically connected.*

3. Locally Antisymmetrically Connected Spaces in the Context of Asymmetric Norms

Asymmetrically normed real vector spaces in the sense of [6] are also investigated in [1] as a new approach to the theory of asymmetry measurement for T_0-quasi-metrics.

First of all, let us recall the notion of *asymmetric norm* from [6]:

Definition 3.1. *Let X be a real vector space equipped with a given map $\|\cdot|:X\to[0,\infty)$ satisfying the conditions:*

(a) $\|x|=\|-x|=0$ *if and only if* $x=\mathbf{0}$.
(b) $\|\lambda x|=\lambda\|x|$ *whenever* $\lambda\geq 0$ *and* $x\in X$.
(c) $\|x+y|\leq\|x|+\|y|$ *whenever* $x,y\in X$.

Then $\|\cdot|$ is called an asymmetric norm *and $(X,\|\cdot|)$ an* asymmetrically normed real vector space. *(In* (a), $\mathbf{0}$ denotes the zero vector of the vector space X.)

Obviously, an asymmetric norm induces a T_0-quasi-metric on X with the equality $\rho_{\|\cdot|}(x,y)=\|x-y|$ for each $x,y\in X$, where $(X,\|\cdot|)$ is an asymmetrically normed real vector space. But, naturally some T_0-quasi-metrics may not be induced by an asymmetric norm:

Example 3.2. *Consider the function υ on $\mathbb{R}$ as follows:*
$\upsilon(x,y)=\begin{cases}\min\{x-y,1\} & ;\ x\geq y\\ 1 & ;\ x<y\end{cases}$ *for each $x,y\in\mathbb{R}$. It is easy to show that υ is a T_0-quasi-metric, but it cannot be induced by an asymmetric norm.*

Incidentally, the notation $\rho_{\|\cdot|}$ will be used for the T_0-quasi-metric induced by the asymmetric norm $\|\cdot|$.

Moreover, the function $\|\cdot|^s=\|\cdot|\vee\|\cdot|^{-1}=\|\cdot\|$ describes the standard (symmetrization) norm on X, where $\|a|^{-1}=\|-a|$ for $a\in X$, and so $\rho^s_{\|\cdot|}=\rho_{\|\cdot|^s}=\rho_{\|\cdot\|}$.

Note also that each norm is an asymmetric norm. However we have:

Example 3.3. (i) *If we take the function $\|x|=x\vee 0$ on $\mathbb{R}$, then $\|\cdot|$ satisfies the above conditions and gives an asymmetric norm, not norm on $\mathbb{R}$.*
(ii) *The function $\|\cdot|$ described by the equality $\|(x,y)|=x\vee y\vee 0$ on $\mathbb{R}^2$, where $x,y\in\mathbb{R}$, satisfies the above conditions and thus, it is an asymmetric norm which is not norm on $\mathbb{R}^2$.*

Now we are in the position to present some new considerations peculiar to asymmetrically normed real vector spaces.

Lemma 3.4. *Let $(X,\|\cdot|)$ be an asymmetrically normed real vector space. If $(X,\rho_{\|\cdot|})$ has an antisymmetric point, then $\mathbf{0}$ is an antisymmetric point.*

Proof. Let $a\in X$ be an antisymmetric point, and $b\in X$. Thus $a-b\in X$ and so by assumption, we have $\rho_{\|\cdot|}(a,a-b)\neq\rho_{\|\cdot|}(a-b,a)$ by Definition 1.11 (i). This means that $\|b|\neq\|-b|$ by the definition of induced T_0-quasi-metric $\rho_{\|\cdot|}$. That is, $\rho_{\|\cdot|}(b,\mathbf{0})\neq\rho_{\|\cdot|}(\mathbf{0},b)$, and so $\mathbf{0}$ is an antisymmetric point. $\square$

Proposition 3.5. *Let $(X,\|\cdot|)$ be an asymmetrically normed real vector space. If $\mathbf{0}$ is antisymmetric point then each point in X is an antisymmetric point in $(X,\rho_{\|\cdot|})$.*

Proof. Suppose that $\mathbf{0}$ is antisymmetric point and $a\in X$. In order to show that a is antisymmetric, let us take $b\in X$. In this case, $a-b\in X$ and so, $\rho_{\|\cdot|}(\mathbf{0},a-b)\neq\rho_{\|\cdot|}(a-b,\mathbf{0})$ since $\mathbf{0}$ is antisymmetric point. That is, $\|b-a|\neq\|a-b|$. Thus, $\rho_{\|\cdot|}(a,b)\neq\rho_{\|\cdot|}(b,a)$, and the point a will be antisymmetric. $\square$

Therefore, with Lemma 3.4 and Proposition 3.5 the following characterization will be clear taking into account Proposition 1.12.

Corollary 3.6. *Let $(X,\|\cdot|)$ be an asymmetrically normed real vector space. The T_0-quasi-metric space $(X,\rho_{\|\cdot|})$ has an antisymmetric point if and only if $(X,\rho_{\|\cdot|})$ is antisymmetric space.*

Finally, by virtue of Corollary 2.9 and Corollary 3.6 we have the next result, trivially.

Corollary 3.7. *If the T_0-quasi-metric space $(X,\rho_{\|\cdot|})$ has an antisymmetric point then $(X,\rho_{\|\cdot|})$ is locally antisymmetrically connected.*

The following equality will be very useful for the remaining of paper.

Lemma 3.8. *Let $(X,\|\cdot|)$ be an asymmetrically normed real vector space. Then $T_{\rho_{\|\cdot|}}(x)=T_{\rho_{\|\cdot|}}(\mathbf{0})+x$, whenever $x\in X$.*

Proof. First of all, let us take $\rho=\rho_{\|\cdot|}$ for the simplicity in the proof.

Assume that $y\in T_\rho(x)$. Then there exists an antisymmetric path $Q_{x,y}=(x_0,x_1,\ldots,x_n)$ from x to y, where $x_0=x$, $x_n=y$. Define the path $Q_{\mathbf{0},y-x}$ as $(x_0-x,x_1-x,\ldots,x_n-x)$. Then $Q_{\mathbf{0},y-x}$ is an antisymmetric path from $\mathbf{0}$ to $y-x$. Thus $y-x\in T_\rho(\mathbf{0})$. Therefore $y\in T_\rho(\mathbf{0})+x$ and $T_\rho(x)\subseteq T_\rho(\mathbf{0})+x$.

For the converse part, let $y\in T_\rho(\mathbf{0})+x$. Then there exists $t\in T_\rho(\mathbf{0})$ with $y=t+x$. Furthermore, for some $n\in\mathbb{N}$ there is an antisymmetric path $Q_{\mathbf{0},t}=(\mathbf{0},x_1\ldots,t)$ from $\mathbf{0}$ to t. Then define $Q_{x,x+t}$ as the path $(x,x+x_1,\ldots,x+t)$. Obviously $Q_{x,x+t}$ is an antisymmetric path from x to $x+t$. Therefore $y=t+x\in T_\rho(x)$ and we established that $T_\rho(\mathbf{0})+x\subseteq T_\rho(x)$. □

At this stage, we have the following characterizations with the help of Lemma 3.8:

Proposition 3.9. *Let $(X,\|\cdot|)$ be an asymmetrically normed real vector space. In this case, $T_{\rho_{\|\cdot|}}(\mathbf{0})$ is τ_{ρ^s}-open if and only if for each $x\in X$, the component $T_{\rho_{\|\cdot|}}(x)$ is τ_{ρ^s}-open.*

Proof. If $z\in T_{\rho_{\|\cdot|}}(x)$ then $z-x\in T_{\rho_{\|\cdot|}}(\mathbf{0})$ by Lemma 3.8. In addition, since $T_{\rho_{\|\cdot|}}(\mathbf{0})$ is τ_{ρ^s}-open, there exists $\varepsilon>0$ such that $B_{\rho^s_{\|\cdot|}}(z-x,\varepsilon)\subseteq T_{\rho_{\|\cdot|}}(\mathbf{0})$. Therefore, in a similar manner it is easy to verify that $B_{\rho^s_{\|\cdot|}}(z,\varepsilon)\subseteq T_{\rho_{\|\cdot|}}(x)$ with the help of the fact that $\rho^s_{\|\cdot|}(x,y)=\|x-y\|=\rho_{\|\cdot|^s}(x,y)$. Finally, $T_{\rho_{\|\cdot|}}(x)$ will be τ_{ρ^s}-open.

The converse part is clear. □

Proposition 3.10. *For an asymmetrically normed real vector space $(X,\|\cdot|)$, $T_{\rho_{\|\cdot|}}(\mathbf{0})=X$ if and only if $T_{\rho_{\|\cdot|}}(x)=X$ for each $x\in X$.*

Proof. Straightforward. □

Incidentally, the following characterization will also be obvious via Proposition 2.3 and Proposition 3.9.

Corollary 3.11. *Let $(X,\|\cdot|)$ be an asymmetrically normed real vector space. In this case, $T_{\rho_{\|\cdot|}}(\mathbf{0})$ is τ_{ρ^s}-open that is $(X,\rho_{\|\cdot|})$ is locally antisymmetrically connected at the point $\mathbf{0}$ if and only if $(X,\rho_{\|\cdot|})$ is locally antisymmetrically connected.*

Now, let us present the main theorem related to local antisymmetric connectedness, in the context of asymmetrically normed real vector spaces.

Theorem 3.12. *For an asymmetrically normed real vector space $(X,\|\cdot|)$, the T_0-quasi-metric space $(X,\rho_{\|\cdot|})$ is locally antisymmetrically connected if and only if $(X,\rho_{\|\cdot|})$ is antisymmetrically connected.*

Proof. Suppose that $(X,\rho_{\|\cdot|})$ is locally antisymmetrically connected. Now, let us note that for any asymmetric norm $\|\cdot|$, the topology $\tau_{\|\cdot|^s}=\tau_{\rho^s_{\|\cdot|}}=\tau_{\|\cdot\|}$ generated by the symmetrization norm $\|\cdot|^s=\|\cdot|\vee\|\cdot|^{-1}=\|\cdot\|$, will be path-connected.

That is, the (normed) topological space $(X,\tau_{\|\cdot|^s})$ is path-connected, and so the same topological space $(X,\tau_{\rho^s_{\|\cdot|}})$ is connected. In this case, by Proposition 2.7, the T_0-quasi-metric space $(X,\rho_{\|\cdot|})$ will be antisymmetrically connected.

Conversely, the assertion is clear due to Proposition 2.5. □

Let us also recall a crucial proposition proved in [1, Proposition 58], as follows:

Proposition 3.13. *Each asymmetrically normed real vector space that is not normed is antisymmetrically connected.*

Consequently, we may state the next result by Theorem 3.12 and Proposition 3.13:

Corollary 3.14. *Each asymmetrically normed real vector space that is not normed is locally antisymmetrically connected.*

Even if a space with the T_0-quasi-metric induced by an asymmetric norm is locally antisymmetrically connected, its subspace need not be locally antisymmetrically connected in the context of asymmetrically normed real vector spaces. Indeed, we have the following example for this fact, moreover even when the subspace is dense w.r.t. the symmetrization topology on the space.

Example 3.15. *Consider the plane $\mathbb{R}^2$ with the T_0-quasi metric ρ induced by the maximum asymmetric norm $\|(x,y)|=x\vee y\vee 0$ (see Example 3.3* (ii)). It is easy to see that the space $(\mathbb{R}^2,\rho)$, where $\rho=\rho_{\|\cdot|}$, is locally antisymmetrically connected from Corollary 3.14 since $\|\cdot|$ is not a norm.

Now take the subset $C=\{(x,-x)\mid x\in\mathbb{R}\}\subseteq\mathbb{R}^2$. It is easy to show that this set is not dense w.r.t. the topology τ_{ρ^s} generated by symmetrization metric $\rho^s=\rho^s_{\|\cdot|}$, which is the Euclidean topology on $\mathbb{R}^2$. Moreover, the subspace (C,ρ_C) is a metric space, and so $T_\rho((a,b))=\{(a,b)\}$ for all $(a,b)\in C$ since all points of the subset C are symmetric. In addition, the topology $\tau_{\rho^s_C}$ generated by the restricted symmetrization metric $\rho^s_C=(\rho^s)_C$ on C, is homeomorphic to the usual real line topology. Thus, the sets $T_\rho((a,b))=\{(a,b)\}$ are not open w.r.t. the restricted topology $(\tau_{\rho^s})_C=\tau_{\rho^s_C}$.

Finally, the subspace (C,ρ_C) cannot be locally antisymmetrically connected.

4. Conclusion

After the theory of antisymmetrically connected T_0-quasi-metric spaces has been constructed as the suitable counterpart of connected complementary graphs in graph theory, the authors defined and studied in [5] the localized form of antisymmetric connectedness, in the context of quasi-metrics. According to that various topological characterizations of local antisymmetric connectedness for T_0-quasi-metric spaces are mentioned with the help of metrics, particularly.

Following these ideas, in this paper the theory of local antisymmetric connectedness is investigated first-time in the context of asymmetric norms, as a different approach to the asymmetric structure of a T_0-quasi-metric not metric. Thus, some relationships between the theories of antisymmetric connectedness and local antisymmetric connectedness are discussed through various propositions, results and examples in the environment of asymmetrically normed real vector spaces.

As the future work; it is very natural to observe the following questions.

How does the local antisymmetric connectedness behaves for subspaces, superspaces, products and unions in the context of T_0-quasi-metrics?

Is the image of locally antisymmetrically connected spaces preserved under an isometric isomorphism ?

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