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Arf Numerical Semigroups with Multiplicity 8

Meral Süer [1]*, Halil İbrahim Karakaş[2], Sedat İlhan[3]
[1]*Batman University, Faculty of Science and Letters, Department of Mathematics, Batman, Türkiye*
[2]*Baskent University, Faculty of Commercial Science, Departmant of Banking and Finance, Ankara, Türkiye*
[3]*Dicle University, Faculty of Science, Department of Mathematics, Diyarbakır, Türkiye*
**meral.suer@batman.edu.tr, karakas@baskent.edu.tr, sedati@dicle.edu.tr*
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Abstract

In this study, we present a description of Arf numerical semigroups with multiplicity eight and given conductor.

Keywords: Arf numerical semigroups, conductor, embedding dimension, Frobenius number, multiplicity, numerical semigroups.

Katlılığı 8 Olan Arf Sayısal Yarıgrupları

Öz

Bu çalışmada, belirli iletecili ve katlılığı sekiz olan Arf sayısal yarı gruplarının tanımlamasını sunuyoruz.

Anahtar Kelimeler: Arf sayısal yarıgrupları, iletici, gömme boyutu, Frobenius sayısı, katlılık, sayısal yarıgruplar.

INTRODUCTION

A numerical semigroup S is a subset of $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$ such that S is closed under addition, $0 \in S$ and $\mathbb{N} \backslash S$ is finite (i.e. S has finite complement in $\mathbb{N}$). It is known that every numerical semigroup S is finitely generated; that is, there exist some elements $u_1, \cdots, u_p \in S$ $(p \in \mathbb{Z}^+)$ such that $S = \langle u_1, \cdots, u_p \rangle = u_1\mathbb{N} + \cdots + u_p\mathbb{N}$ (Barucci, Dobbs and Fontana, 1997; Fröberg, Gottlieb and H$\ddot{a}$ggkvist, 1987; Rosales and García-Sánchez, 2009). Moreover, $gcd\{u_1, \cdots, u_p\} = 1$ since this is equivalent to that the semigroup S has a finite complement in $\mathbb{N}$ where gcd the abbreviation for the greatest common divisor (Fröberg et al., 1987).

The set $A = \{u_1, \cdots, u_p\}$ is called the minimal system of generators for any semigroup S, if $S = \langle A \rangle$ and no proper subset A generates S. It is known that every numerical semigroup S has a unique minimal system of generators and the cardinality of the minimal system of generators of S is called the embedding dimension of S, denoted by $e(S)$. The least positive integer in S is called the multiplicity of S, denoted by $m(S)$. It is known that the minimal system of generators of S must contain $m(S)$, and that $e(S) \leq m(S)$ (García -Sánchez, Heredia, Karakaş, and Rosales, 2017; İlhan and Süer, 2017). Moreover, a numerical semigroup S is a numerical semigroup of maximal embedding dimension if $e(S) = m(S)$. Another notable element of a numerical semigroup is the ratio. The ratio of S, denoted by $R(S)$ (in short R), is defined as the least positive integer greater than the multiplicity of S in the minimal system of generators of S.

The greatest integer not in S is known as the Frobenius number of S, denoted by $F(S)$, through in the literature it is sometimes replaced by the conductor of S, denoted by $C(S)$ (in short C which is the least integer x such that $x + n \in S$ for all $n \in \mathbb{N}$. It is easy to see that $F(S) = C - 1$. If S is different from $\mathbb{N}$, it is traditional to denote the elements of S that are less than or equal to C by $s_0 = 0, s_1, \cdots, s_{n-1}, s_n = C$ with $s_{i-1} < s_i$for each $1 \leq i \leq n$, and write

$$S = \{s_0 = 0, s_1, \cdots, s_{n-1}, s_n = C, \rightarrow\}$$

where "$\rightarrow$" means that every integer greater than C belongs to the set. The elements $s_0 = 0, s_1, \cdots, s_{n-1}$ are called the small elements of S. Note that the first non-zero small element is $s_1 = m(S)$, the multiplicity

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of S, and $n = n(S) = |S \cap \{0, 1, \ldots, F(S)\}|$ is the number of small elements of S ($|A|$ denotes the cardinality of any set A).

If S is a numerical semigroup and $a \in S \setminus \{0\}$, the Apéry set of S with respect to a is the set $Ap(S, a) = \{s \in S : s - a \notin S\}$. It is easy to see that $Ap\ (S, a) = \{w_0 = 0, w_1, \cdots, w_{a-1}\}$ where w_i is the least element of S such that $w_i \equiv i \pmod a$ for each $1 \leq i \leq a - 1$. Moreover, $(Ap(S, a) \setminus \{0\}) \cup \{a\}$ generates S and $\max(Ap(S, a)) = F(S) + a = C(S) + a - 1$ for any $ka \in S \setminus \{0\}$ (Rosales, 2005; Rosales and García -Sánchez, 2009). Thus, if S is a numerical semigroup with multiplicity m, then S has maximal embedding dimension if and only if $(Ap(S, m) \setminus \{0\}) \cup \{m\}$ is the minimal system of generators for S.

Arf Numerical Semigroups

A numerical semigroup S is called Arf if $x + y - z \in S$ for all $x, y, z \in S$ where $x \geq y \geq z$. This definition was first given by C. Arf in 1949, and therefore, the condition in this definition is known as the Arf condition. For all $x, y, z \in S$ such that $x \geq y \geq z$ and $x \geq C$, clearly $x + y - z \geq C$ and so $x + y - z \in S$. Therefore, to check if a numerical semigroup is Arf, it is enough to check the Arf condition for only the small elements. There are many equivalent conditions to the Arf condition, one of them is specified as "a numerical semigroup is Arf if and only if $2x - y \in S$ where $x \geq y$" (García–Sánchez at al., 2017).

Any Arf numerical semigroup has maximal embedding dimension. Thus, if S is an Arf numerical semigroup with multiplicity m, then S is minimally generated by $(Ap(S, m) \setminus \{0\}) \cup \{m\}$.

A class of numerical semigroups Arf, the multiplicity of which is a p prime number, is given in (Çelik, 2022). However, García–Sánchez at al., in 2017 show that Arf numerical semigroups with multiplicity up to seven and given conductor are described parametrically. In this work, we obtain a description for Arf numerical semigroups with multiplicity eight, which is similar to the work in (García–Sánchez at al., 2017). We now recall some results that we will frequently use throughout the paper.

Lemma 1 [García–Sánchez at al., 2017, Lemma 11] Let S be an Arf numerical semigroup with multiplicity m and conductor C. Let $Ap\ (S, m) = \{w_0 = 0, w_1, \cdots, w_{m-1}\}$. For each $j = 2, 3, \cdots, m - 1$, we have

(a) $w_{j-1} < w_j \Rightarrow C \leq w_j - 1$

(b) $w_j < w_{j-1} \Rightarrow C \leq w_{j-1}$.

Lemma 1 shows that for each $j = 2, \cdots, m - 1$, at least one of w_{j-1} or w_j is not less than C.

Let S be a numerical semigroup with multiplicity m and conductor C. As every non-negative multiple of m is in S and $C - 1 \notin S$, it follows that $C \not\equiv 1 \pmod m$. The following lemma shows that w_1 and w_{m-1} are completely determined by the multiplicity and the conductor in any Arf numerical semigroup.

Lemma 2 [García–Sánchez at al., 2017, Lemma 13] Let S be an Arf numerical semigroup with multiplicity m and conductor C where $C \equiv k \pmod m$ and $k \in \{0, 2, \cdots, m - 1\}$. Then

(a)
$$w_1 = \begin{cases} C + 1 & if\ k = 0\ (C \equiv 0 \pmod m) \\ \mathrm{C - k + m + 1} & if\ k \neq 0\ (C \not\equiv 0 \pmod m) \end{cases}$$

(b) $w_{m-1} = \mathrm{C - k + m - 1}$

Lemma 3 [García–Sánchez at al., 2017, Lemma 15] Let S be an Arf numerical semigroup with multiplicity $m > 2$. For any $t \in \mathbb{N}$ with $t \leq \frac{m}{2}$, we have $w_{2t} \leq w_t + t$ and $w_{m-2t} \leq w_{m-t} + m - t$.

Arf Numerıcal Semıgroups With Multiplicity 8

Let S be an Arf numerical semigroup with multiplicity eight and conductor C. Then $C \equiv 0, 2, 3, 4, 5, 6$ or $7 \pmod 8$. Recall that the ratio R of S is the least element larger than the multiplicity in the minimal system of generators for S. It can be easily seen that

$R \leq C + 1$ if $C \equiv 0 \pmod 8$, $R \leq C$ if $C \not\equiv 0 \pmod 8$.

Remark 1 As a result of Lemma 3, if S is an Arf numerical semigroup with multiplicity eight and $Ap\ (S, 8) = \{w_0 = 0, w_1, \cdots, w_7\}$ then we have

(a) $w_6 \leq w_3 + 3$ (b) $w_4 \leq w_2 + 2$

(c) $w_2 \leq w_5 + 5$ (d) $w_4 \leq w_6 + 6$.

Theorem 4 Let S be a numerical semigroup with multiplicity eight and conductor C where $C > 8$ and $C \equiv 0 \pmod 8$. Then S is an Arf numerical semigroup if and only if S is one of the followings:

$\langle 8, C + 1, C + 2, C + 3, C + 4, C + 5, C + 6, C + 7 \rangle$;

$\langle 8, 8u + 2, 8u + 4, 8u + 6, C + 1, C + 3, C + 5, C + 7 \rangle$ for each $1 \leq u \leq \frac{C-8}{8}$;

$\langle 8, C - 5, C - 2, C + 1, C + 2, C + 4, C + 5, C + 7 \rangle$;

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$\langle 8, 8u+4, 8t+2, 8t+6, C+1, C+3, C+5, C+7\rangle$ for each $1 \leq u < t \leq \frac{C}{8}$;

$\langle 8, 8u+4, 8t-2, 8t+2, C+1, C+3, C+5, C+7\rangle$ for each $1 \leq u < t \leq \frac{C}{8}$;

$\langle 8, C-3, C+1, C+2, C+3, C+4, C+6, C+7\rangle$;

$\langle 8, 8u+6, 8u+10, 8u+12, C+1, C+3, C+5, C+7\rangle$ for each $1 \leq u \leq \frac{C-8}{8}$.

Proof (Necessity) Let S be an Arf numerical semigroup with multiplicity eight and conductor C where $C > 8$ and $C \equiv 0 \pmod 8$. It can be obtained from Lemma 2 that $w_1 = C+1$ and $w_7 = C+7$. In addition, the largest element of the set $Ap(S,8)$ is $w_7 = C+7$ which can be obtained from the equality of $\max(Ap(S,8)) = F(S)+8$. So the other elements of the Apéry set must be smaller than $w_7 = C+7$. In other words, $w_i < C+7$ *for* each $i \in \{0,1,2,3,4,5,6\}$. In this way, the ratio of S, R, is one of the elements w_1, w_2, w_3, w_4, w_5 or w_6.

(i) If $R = w_1$, then it is obvious that

$$S = \langle 8, C+1, C+2, C+3, C+4, C+5, C+6, C+7\rangle.$$

(ii) If $R = w_2$, then $w_2 \leq C-6$. Consequently, $w_2 = 8u+2$, where $1 \leq u \leq \frac{C-8}{8}$. Since $w_2 < w_3$, $C+1 \leq w_3$. Then by Lemma 1, we have $w_3 = C+3$. Because we know that $w_2 < w_4$ and $w_4 \leq w_2+2$ given in Remark 1, we get $w_4 = w_2+2 = 8u+4$. The inequality $w_2 < w_5$ yields the inequality $w_4 = w_2+2 < w_5$. Then w_5 can be obtained from Lemma 1 to be $C+5$. Finally, we have $2w_4 - w_2 = 8u+6 \in S$ by the Arf condition, which $w_6 \leq 8u+6$. Since $w_2 = 8u+2 < w_6$, we get $w_6 = 8u+6$. It follows that

$S = \langle 8, 8u+2, 8u+4, 8u+6, C+1, C+3, C+5, C+7\rangle$ for each $1 \leq u \leq \frac{C-8}{8}$.

(iii) If $R = w_3$, then $w_3 \leq C-5$. Since $R = w_3$, w_2 and w_4 can be found from Lemma 1 as $w_2 = C+2$ and $w_4 = C+4$, respectively. In order see to that $C+4 = w_4 \leq w_6+6 \leq w_3+9 \leq C+4$, we use Remark 1 (d) and (a), respectively. Accordingly, we get $w_3 = C-5$ and $w_6 = C-2$. Since $R = w_3$ and $\max(Ap(S,8)) = C+7$, we can obtain $C-5 = w_3 < w_5 < C+7$. Thus, $w_5 = C+5$. Hence,

$$S = \langle 8, C-5, C-2, C+1, C+2, C+4, C+5, C+7\rangle.$$

(iv) If $R = w_4$, then $w_4 \leq C-4$. As a result of this, w_4 is equal to $8u+4$ for the interval of u indicated by $1 \leq u \leq \frac{C-8}{8}$. Then by Lemma 1, we have $w_3 = C+3$ and $w_5 = C+5$. The inequality of $w_4 \leq w_2+2$ is given in Remark 1 (b), and so $w_2+2 \in S$. Therefore, $2(w_2+2) - w_2 = w_2+4 \in S$ by the Arf condition. This implies that $w_6 \leq w_2+4$. Under these conditions, there are two cases: $w_4 < w_2 < w_6$ or $w_4 < w_6 < w_2$.

If $w_4 < w_2 < w_6$ then $w_2 = 8t+2$ and $w_6 = 8t+6$, for some $t \in \mathbb{N}$. Since $w_4 < w_6$ and $w_6 < C+7$, we have $1 \leq u < t \leq \frac{C}{8}$. It follows that

$S = \langle 8, 8u+4, 8t+2, 8t+6, C+1, C+3, C+5, C+7\rangle$ for each $1 \leq u < t \leq \frac{C}{8}$.

If $w_4 < w_6 < w_2$, then $w_2 = 8t+2$ and $w_6 = 8t-2$ for some $t \in \mathbb{N}$. Since $w_4 < w_2$ and $w_2 < C+7$, we have $1 \leq u < t \leq \frac{C}{8}$. It follows that

$S = \langle 8, 8u+4, 8t-2, 8t+2, C+1, C+3, C+5, C+7\rangle$ for each $1 \leq u < t \leq \frac{C}{8}$.

(v) If $R = w_5$, then $w_5 \leq C-3$. We can obtain $w_4 = C+4$ and $w_6 = C+6$ by using Lemma 1. From Remark 1 (a), (b) and (c), respectively, we have $C+6 = w_6 \leq w_3+3$, $C+4 = w_4 \leq w_2+2$ and $w_2 \leq w_5+5$. Accordingly, we can calculate $w_3 = C+3$, $w_2 = C+2$ and $w_5 = C-3$. As a result,

$$S = \langle 8, C-3, C+1, C+2, C+3, C+4, C+6, C+7\rangle.$$

(vi) If $R = w_6$, then $w_6 \leq C-2$. So $w_6 = 8u+6$ for $1 \leq u \leq \frac{C-8}{8}$. By using Lemma 1, we can calculate $w_5 = C+5$. By the Arf condition, $2w_6 - 8u = 8u+12 \in S$. Therefore, $w_4 \leq 8u+12$. This yields $w_4 = 8u+12$ due to $w_6 = 8u+6 < w_4$ $(R = w_6)$. In addition, by the Arf condition again, we obtain $2(8u+8) - (8u+6) = 8u+10 \in S$, and so $w_2 = 8u+10$. Furthermore, since $w_6 < w_3$, we can write $8u+11 \leq w_3$, which implies that $w_2 = 10u+10 < w_3$. Hence,

$S = \langle 8, 8u+6, 8u+10, 8u+12, C+1, C+3, C+5, C+7\rangle$ for each $1 \leq u \leq \frac{C-8}{8}$.

(Sufficiency) The definition of Arf numerical semigroup given in the Section 2 shows that each semigroup given in Theorem 4 satisfies the Arf property.

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Theorem 5 Let S be a numerical semigroup with multiplicity eight and conductor C where $C > 10$ and $C \equiv 2 \pmod 8$. Then S is an Arf numerical semigroup if and only if S is one of the followings:

$\langle 8, 8u+2, 8u+4, 8u+6, C+1, C+3, C+5, C+7 \rangle$ for each $1 \leq u \leq \frac{C-2}{8}$;

$\langle 8, 8u+4, 8t+2, 8t+6, C+1, C+3, C+5, C+7 \rangle$ for each $1 \leq u < t \leq \frac{C-2}{8}$;

$\langle 8, 8u+4, 8t-2, 8t+2, C+1, C+3, C+5, C+7 \rangle$ for each $1 \leq u < t \leq \frac{C-2}{8}$;

$\langle 8, C-5, C, C+1, C+2, C+4, C+5, C+7 \rangle$;

$\langle 8, 8u+6, 8u+10, 8u+12, C+1, C+3, C+5, C+7 \rangle$ for each $1 \leq u \leq \frac{C-10}{8}$.

Proof (Necessity) Let S be an Arf numerical semigroup with multiplicity eight and conductor C where $C > 10$ and $C \equiv 2 (\mathrm{mod}\ 8)$. We can calculate $w_1 = C + 7$ and $w_7 = C+5$ by using Lemma 2. We can see that the largest element of the set $Ap(S, 8)$ is $w_1 = C+7$ by using $\max(Ap(S,8)) = F(S)+8$. Thus, the other elements of the Apéry set must be smaller than $w_1 = C+7$, i.e. $w_i < C+7$ for each $i \in \{0,2,3,4,5,6,7\}$. Note that w_3 must be bigger than C. Otherwise, if $w_3 \leq C$, then $C-7 \in S$. It can also be found from Lemma 1 $w_4 = C+2$. In addition, by respectively using Remark 1 (d), (a), the following inequalities can be obtained $C+2 = w_4 \leq w_6 + 6 \leq w_3 + 9 \leq C+2$. Thus, $w_6 = C-4$. Accordingly, by the Arf condition $2(C-4)-(C-7) = C-1 \in S$. This is a contradiction. For this reason w_3 is equal to $C+1$. Under this condition the ratio of S, is one of the elements w_2, w_4, w_5 or w_6.

(i) If $R = w_2$, then $w_2 \leq C$. w_2 must be $8u+2$ for the interval of u indicated by $1 \leq u \leq \frac{C-2}{8}$. Thus, we can get $2w_2 - (8u) = 8u+4 \in S$ by using the Arf condition. Therefore, $w_4 \leq 8u+4$. This shows $w_4 = 8u+4$ as $w_2 = 8u+2 < w_4$. The Arf condition also gives $w_4 + w_2 - (8u) = 8u+6 \in S$, which yields $w_6 = 8u+6$. We can see that $C+1, 8u+4$ and $8u+2 \in S$ through information given above. There are two situations: In the first, if $C+1 > 8u+4 > 8u+2$, then $(C+1)+(8u+4)-(8u+2) = C+3 \in S$ can be obtained by the Arf condition. In the second, if $8u+4 > C+1 > 8u+2$, then $u = \frac{C-2}{8}$ and $2(8u+4)-(C+1) = 2(C+2)-(C+1) = C+3 \in S$ can be obtained by the Arf condition. Hence, $w_5 = C+3$ and $S = \langle 8, 8u+2, 8u+4, 8u+6, C+1, C+3, C+5, C+7 \rangle$ for each $1 \leq u \leq \frac{C-2}{8}$.

(ii) If $R = w_4$, then $w_4 \leq C-6$. In this case, $w_4 = 8u+4$ for the interval of u indicated by $1 \leq u \leq \frac{C-10}{8}$. It can easily be calculated by using Lemma 1 that $w_3 = C+1$ and $w_5 = C+3$. On the other hand, $w_4 \leq w_2 + 2$ is given in Remark 1 (b). Therefore, $2(w_2+2) - w_2 = w_2 + 4 \in S$ can be obtained from the Arf condition. We obtained that $w_6 \leq w_2 + 4$. Under these conditions, there are two cases: $w_4 < w_2 < w_6$ or $w_4 < w_6 < w_2$. If $w_4 < w_2 < w_6$, then $w_2 = 8t+2$ and $w_6 = 8t+6$ for some $t \in \mathbb{N}$. Since $w_4 < w_6$ and $w_6 < C+7$, we have $1 \leq u < t \leq \frac{C-2}{8}$. It follows that

$S = \langle 8, 8u+4, 8t+2, 8t+6, C+1, C+3, C+5, C+7 \rangle$

for each $1 \leq u < t \leq \frac{C-2}{8}$.

If $w_4 < w_6 < w_2$, then $w_2 = 8t+2$ and $w_6 = 8t-2$ for some $t \in \mathbb{N}$. Since $w_4 < w_2$ and $w_2 < C+7$, we have $1 \leq u < t \leq \frac{C-2}{8}$. It follows that

$S = \langle 8, 8u+4, 8t-2, 8t+2, C+1, C+3, C+5, C+7 \rangle$ for each $1 \leq u < t \leq \frac{C-2}{8}$.

(iii) If $R = w_5$, then $w_5 \leq C-5$. We can calculate by usig Lemma 1 that $w_4 = C+2$ and $w_6 = C+4$. By respectively applying Remark 1 (b), (a) and (c), we can write $C+2 = w_4 \leq w_2 + 2$, $C+4 = w_6 \leq w_3 + 3$ and $C = w_2 \leq w_5 + 5$. As a result, $w_4 = C, w_3 = C+1$ and $w_5 = C-5$. Hence,
$S = \langle 8, C-5, C, C+1, C+2, C+4, C+5, C+7 \rangle$.

(iv) If $R = w_6$, then $w_6 \leq C-4$. In this case, $w_6 = 8u+6$ for the interval of u indicated by $1 \leq u \leq \frac{C-10}{8}$. By using the Arf condition, we can write $2w_6 - 8u = 8u+12 \in S$. Thus, $w_4 \leq 8u+12$. Since $R = w_6$, $w_6 = 8u+6 < w_4$. These situations yield $w_4 = 8u+12$. The Arf condition also gives $2(8u+8)-(8u+6) = 8u+10 \in S$, which yields $w_2 = 8u+10$. Furthermore, it can easily be seen by Lemma 1 that $w_5 = C+3$. Moreover, $8u+6 < w_3$ since $w_6 < w_3$. This implies that $w_2 < w_3$. Under these conditions, $w_3 = C+1$ by using Lemma 1. It is follows that

$S = \langle 8, 8u+6, 8u+10, 8u+12, C+1, C+3, C+5, C+7 \rangle$ for each $1 \leq u \leq \frac{C-10}{8}$.

(Sufficiency) The definition of Arf numerical semigroup given in the Section 2 shows that each

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semigroup given in Theorem 5 satisfies the Arf property.

Theorem 6 Let S be a numerical semigroup with multiplicity eight and conductor C where $C > 11$ and $C \equiv 3 \pmod 8$. Then S is an Arf numerical semigroup if and only if S is one of the followings:

$\langle 8, C, C+1, C+2, C+3, C+4, C+6, C+7\rangle$;

$\langle 8, 8u+4, C, C+2, C+3, C+4, C+6, C+7\rangle$ for each $1 \leq u \leq \frac{C-11}{8}$.

Proof (Necessity) Let S be an Arf numerical semigroup with multiplicity eight and conductor C where $C > 11$ and $C \equiv 3 \pmod 8$. Then, we can get $w_1 = C+6$ and $w_7 = C+4$ by using Lemma 2.We can see that the largest element of the set $Ap(S,8)$ is $w_2 = C+7$ by using $\max(Ap(S,8)) = F(S)+8$. So the other elements of the Apéry set must be smaller than $w_2 = C+7$, i.e. $w_i < C+7$ for each $i \in \{0,1,3,4,5,6,7\}$. If $w_6 < C$, then $C-5 \in S$. We also have $C-3 \in S$, and so $2(C-3)-(C-5) = C-1 \in S$ by the Arf condition. This contradicts with the fact that C is the conductor of S. Therefore, w_6 must be bigger than C. This means that $w_6 = C+3$. By respectively using Remark 1 (a) and (c) we see that $C+3 = w_6 \leq w_3+3$ and $C+7 = w_2 \leq w_5+5$. Thus, these results yield $w_3 = C$ and $w_5 = C+2$. Accordingly, the ratio of S, R, is one of the elements $w_3 = C$ or w_4.

(i) If $w_3 = C$, then it is clear that
$S = \langle 8, C, C+1, C+2, C+3, C+4, C+6, C+7\rangle$.

(ii) If $R = w_4$, then $w_4 \leq C-7$. Consequently $w_4 = 8u+4$, for the interval of u indicated by $1 \leq u \leq \frac{C-11}{8}$. It follows that

$S = \langle 8, 8u+4, C, C+2, C+3, C+4, C+6, C+7\rangle$ for each $1 \leq u \leq \frac{C-11}{8}$.

(Sufficiency) The definition of Arf numerical semigroup given in the Section 2 shows that each semigroup given in Theorem 6 satisfies the Arf property.

Theorem 7 Let S be a numerical semigroup with multiplicity eight and conductor C where $C > 12$ and $C \equiv 4 \pmod 8$. Then S is an Arf numerical semigroup if and only if S is one of the followings:

$\langle 8, 8u+2, 8u+4, 8u+6, C+1, C+3, C+5, C+7\rangle$ for each $1 \leq u \leq \frac{C-4}{8}$;

$\langle 8, 8u+4, 8t+2, 8t+6, C+1, C+3, C+5, C+7\rangle$ for each $1 \leq u < t \leq \frac{C-4}{8}$;

$\langle 8, 8u+4, 8t-2, 8t+2, C+1, C+3, C+5, C+7\rangle$ for each $1 \leq u < t \leq \frac{C+4}{8}$;

$\langle 8, 8u+6, 8u+10, 8u+12, C+1, C+3, C+5, C+7\rangle$ for each $1 \leq u \leq \frac{C-12}{8}$.

Proof (Necessity) Let S be an Arf numerical semigroup with multiplicity eight and conductor C where $C > 12$ and $C \equiv 4 \pmod 8$. We can calculate $w_1 = C+5$ and $w_7 = C+3$ by using Lemma 2. The largest element of the set $Ap(S,8)$ is $w_3 = C+7$ due to the fact that $\max(Ap(S,8))$ $F(S)+8$. So the other elements of the Apéry set must be smaller than $w_3 = C+7$, i.e. $w_i < C+7$ for each $i \in \{0,1,2,4,5,6,7\}$. Note that $w_5 > C$. Otherwise, $w_5 \leq C$ which implies that $w_5 \leq C-7 \in S$. Then $C-4 \in S$ as well, and so $2(C-4)-(C-7) = C-1 \in S$ by the Arf condition. This is a contradiction. Therefore, $w_5 > C$ and $w_5 = C+1$. Thus, the ratio of S, R, is one of the elements w_2, w_4 *or* w_6.

(i) If $R = w_2$, then $w_2 \leq C-2$. Consequently, $w_2 = 8u+2$ for the interval of u indicated by $1 \leq u \leq \frac{C-4}{8}$. The Arf condition gives $2w_2 - 8u = 8u+4 \in S$. Therefore, $w_4 \leq 8u+4$. Since $8u+2 = w_2 < w_4$. The Arf condition also gives $w_4 + w_2 - 8u = 8u+6 \in S$. This yields $w_6 = 8u+6$ due to $w_2 = 8u+2 < w_6$. Hence,

$S = \langle 8, 8u+2, 8u+4, 8u+6, C+1, C+3, C+5, C+7\rangle$ for each $1 \leq u \leq \frac{C-4}{8}$.

(ii) If $R = w_4$, then $w_4 \leq C$. As a result, $w_4 = 8u+4$ for the interval of u indicated by $1 \leq u \leq \frac{C-4}{8}$. Note that $w_2 + 2 \in S$ since $w_4 < w_2 + 2$. Then the Arf condition gives $2(w_2+2) - w_2 = w_2 + 4 \in S$. This implies $w_6 \leq w_2 + 4$. As a result, we get two cases: $w_4 < w_2 < w_6$ and $w_4 < w_6 < w_2$ are considered. If $w_4 < w_2 < w_6$, then $w_2 = 8t+2$ and $w_6 = 8t+6$ for some $t \in \mathbb{N}$. Since $w_4 < w_6$ and $w_6 < C+7$, we have $1 \leq u < t \leq \frac{C-4}{8}$. It follows that

$S = \langle 8, 8u+4, 8t+2, 8t+6, C+1, C+3, C+5, C+7\rangle$ for each $1 \leq u < t \leq \frac{C-4}{8}$.

If $w_4 < w_6 < w_2$, then $w_2 = 8t+2$ and $w_6 = 8t-2$ for some $t \in \mathbb{N}$.

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Since $w_4 < w_2$ and $w_2 < C+7$, we have $1 \le u < t \le \frac{C+4}{8}$. It follows that

$S = \langle 8, 8u+4, 8t-2, 8t+2, C+1, C+3, C+5, C+7\rangle$ for each $1 \le u < t \le \frac{C+4}{8}$.

(iii) If $R = w_6$, then $w_6 \le C-6$. Accordingly, $w_6 = 8u+6$ for the interval of u indicated by $1 \le u \le \frac{C-12}{8}$. We can get $w_5 = C+1$ by Lemma 1. On the other hand, $w_4 \le w_6 + 6$ is given in Remark 1 (d). Since $w_6 < w_4$, we get $w_4 = w_6 + 6 = 8u + 12$. By the Arf condition, we also get $2(8u+8) - w_6 = 8u + 10 \in S$. This implies $w_2 \le 8u + 10$. Since $w_6 = 8u + 6 < w_2$, w_2 must be equal to $8u + 10$. It follows that

$S = \langle 8, 8u+6, 8u+10, 8u+12, C+1, C+3, C+5, C+7\rangle$ for each $1 \le u \le \frac{C-12}{8}$.

(Sufficiency) The definition of Arf numerical semigroup given in the Section 2 shows that each semigroup given in Theorem 7 satisfies the Arf property.

Theorem 8 Let S be a numerical semigroup with multiplicity eight and conductor C where $C > 13$ and $C \equiv 5 \pmod 8$. Then S is an Arf numerical semigroup if and only if S is one of the followings:

$\langle 8, C-2, C, C+1, C+2, C+4, C+5, C+7\rangle$;

$\langle 8, C, C+1, C+2, C+4, C+5, C+6, C+7\rangle$.

Proof (Necessity) Let S be an Arf numerical semigroup with multiplicity eight and conductor C where $C > 13$ and $C \equiv 5 (\mathrm{mod}\, 8)$. It can easily be calculated from Lemma 2 that $w_1 = C+4$ and $w_7 = C+2$. The largest element of the set $Ap(S,8)$ is $w_4 = C+7$ due to the fact that $\max(Ap(S,8)) = F(S) + 8$. So the other elements of the Apéry set must be smaller than $w_4 = C+7$, i.e. $w_i < C+7$ for each $i \in \{0,1,2,3,5,6,7\}$. Using respectively Remark 1 (b), (d), (c) and (a), we see that $C+7 = w_4 \le w_2 + 2 \Rightarrow w_2 = C+5$, $C+7 = w_4 \le w_6 + 6 \Rightarrow w_6 = C+1$, $C+5 = w_2 \le w_5 + 5 \Rightarrow w_5 \ge C$ and $C+1 = w_6 \le w_3 + 3 \Rightarrow w_3 \ge C-2$. According to the values obtained above, $R = w_3 = C-2$ or $R = w_5 = C$.

(i) If $R = w_3 = C-2$, then

$S = \langle 8, C-2, C, C+1, C+2, C+4, C+5, C+7\rangle$.

(ii) If $R = w_5 = C$, then

$S = \langle 8, C, C+1, C+2, C+4, C+5, C+6, C+7\rangle$.

(Sufficiency) The definition of Arf numerical semigroup given in the Section 2 shows that each semigroup given in Theorem 8 satisfies the Arf property.

Theorem 9 Let S be a numerical semigroup with multiplicity eight and conductor C where $C > 14$ and $C \equiv 6 \pmod 8$. Then S is an Arf numerical semigroup if and only if S is one of the followings:

$\langle 8, 8u+2, 8u+4, 8u+6, C+1, C+3, C+5, C+7\rangle$ for each $1 \le u \le \frac{C-6}{8}$;

$\langle 8, C-3, C, C+1, C+3, C+4, C+6, C+7\rangle$;

$\langle 8, 8u+4, 8t+2, 8t+6, C+1, C+3, C+5, C+7\rangle$ for each $1 \le u < t \le \frac{C-6}{8}$;

$\langle 8, 8u+4, 8t-2, 8t+2, C+1, C+3, C+5, C+7\rangle$ for each $1 \le u < t \le \frac{C+2}{8}$;

$\langle 8, 8u+6, 8u+10, 8u+12, C+1, C+3, C+5, C+7\rangle$ for each $1 \le u \le \frac{C-6}{8}$.

Proof (Necessity) Let S be an Arf numerical semigroup with multiplicity eight and conductor C where $C > 14$ and $C \equiv 6 \pmod 8$. We get $w_1 = C+3$ and $w_7 = C+1$ by using Lemma 2. The largest element of the set $Ap(S,8)$ is $w_5 = C+7$ due to the fact that $\max(Ap(S,8)) = F(S) + 8$. So the other elements of the Apéry set must be smaller than $w_5 = C+7$, i.e. $w_i \le C+7$ for each $i \in \{0,1,2,3,4,6,7\}$. Therefore, the ratio of S, R, is one of the elements w_2, w_3, w_4 or w_6.

(i) If $R = w_2$, then $w_2 \le C-4$. Accordingly, $w_2 = 8u+2$ for the interval of u indicated by $1 \le u \le \frac{C-6}{8}$. $w_4 \le w_2 + 2$ is given in by Remark 1 (b) and due to fact that $w_2 = 8u+2 < w_4$, we get $w_4 = 8u+4$. By using the Arf condition, $2w_4 - w_2 = 8u+6 \in S$ which implies $w_6 \le 8u+6$. Since $w_2 = 8u+2 \le w_6$, w_6 must be equal to $8u+6$. It follows from Lemma 1 that $w_3 = C+5$. Under these conditions, S can be written as follows:

$S = \langle 8, 8u+2, 8u+4, 8u+6, C+1, C+3, C+5, C+7\rangle$ for each $1 \le u \le \frac{C-6}{8}$.

(ii) If $R = w_3$, then $w_3 \le C-3$. We can evaluate from Lemma 1 that $w_4 = C+6$ and $w_2 = C+4$. As given in Remark 1 (d), $C+6 = w_4 \le w_6 + 6$. Therefore, $w_6 = C$. On the other hand, $w_6 \le w_3 + 3$ is given in Remark 1 (a). Thus, $w_3 = C-3$. So, S can be written as

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$S = \langle 8, C-3, C, C+1, C+3, C+4, C+6, C+7\rangle$.

(iii) If $R = w_4$, then $w_4 \leq C-2$. Accordingly, $w_4 = 8u+4$ for the interval of u indicated by $1 \leq u \leq \frac{C-6}{8}$. By using Lemma 1, we can get $w_3 = C+5$. $w_4 \leq w_2 + 2$ is given in Remark 1 (b). By using the Arf condition $2(w_2+2) - w_2 = w_2 + 4 \in S$. This implies that $w_6 \leq w_2 + 4$. As a result, we get two cases: $w_4 < w_2 < w_6$ and $w_4 < w_6 < w_2$.

If $w_4 < w_2 < w_6$, then $w_2 = 8t+2$ and $w_6 = 8t+6$ for some $t \in \mathbb{N}$. Since $w_4 < w_6$ and $w_6 < C+7$, we have $1 \leq u < t \leq \frac{C-6}{8}$. It follows that

$S = \langle 8, 8u+4, 8t+2, 8t+6, C+1, C+3, C+5, C+7\rangle$ for each $1 \leq u < t \leq \frac{C-6}{8}$.

If $w_4 < w_6 < w_2$, then $w_2 = 8t+2$ and $w_6 = 8t-2$ for some $t \in \mathbb{N}$.

Since $w_4 < w_2$ and $w_2 < C+7$, we have $1 \leq u < t \leq \frac{C+2}{8}$. It follows that

$S = \langle 8, 8u+4, 8t-2, 8t+2, C+1, C+3, C+5, C+7\rangle$ for each $1 \leq u < t \leq \frac{C+2}{8}$.

(iv) If $R = w_6$, then $w_6 \leq C$. As a result, $w_6 = 8u+6$ for the interval of u indicated by $1 \leq u \leq \frac{C-6}{8}$. In this case, $2(8u+8) - w_6 = 8u+10 \in S$ by the Arf condition. Thus, $w_4 \leq 8u+12$. On the other hand, we know that $w_6 = 8u+6 < w_4$. Thus, w_4 must be equal to $8u+12$. Furthermore, $w_6 < w_3 \Rightarrow 8u + 6 < w_3 \Rightarrow w_2 < w_3$. So $w_3 = C+5$ is obtained by using Lemma 1. Hence,

$S = \langle 8, 8u+6, 8u+10, 8u+12, C+1, C+3, C+5, C+7\rangle$ for each $1 \leq u \leq \frac{C-6}{8}$.

(Sufficiency) The definition of Arf numerical semigroup given in the Section 2 shows that each semigroup given in Theorem 9 satisfies the Arf property.

Theorem 10 Let S be a numerical semigroup with multiplicity eight and conductor C where $C > 15$ and $C \equiv 7 (\mathrm{mod}\, 8)$. Then S is an Arf numerical semigroup if and only if S is one of the followings:

$\langle 8, 8u+4, C, C+2, C+3, C+4, C+6, C+7\rangle$ for each $1 \leq u \leq \frac{C-7}{8}$;

$\langle 8, C-2, C, C+2, C+3, C+4, C+5, C+7\rangle$;

$\langle 8, C, C+2, C+3, C+4, C+5, C+6, C+7\rangle$.

Proof (Necessity) Let S be an Arf numerical semigroup with multiplicity eight and conductor C where $C > 15$ and $C \equiv 7\ (\mathrm{mod}\ 8)$. Under these conditions, we can calculate $w_1 = C+2$ and $w_7 = C$ by using Lemma 2. The largest element of the set $Ap(S,8)$ is $w_6 = C+7$ due to the fact tha $\max(Ap(S,8)) = F(S) + 8$. So the other elements of the Apéry set must be smaller than $w_6 = C+7$, i.e. $w_i < C+7$ for each $i \in \{0,1,2,3,4,5,7\}$. Using Remark 1 (a), we get $w_3 = C+4$. Note that $w_2 > C$. Otherwise, if $w_2 < C$, then $w_2 \leq C-5 \in S$. Thus, by the Arf condition we get $2(C-5) - (C-7) = C-3 \in S$, and $2(C-3) - (C-5) = C-1 \in S$. This is a contradiction. Therefore, $w_2 = C+3$. Besides, $C+3 = w_2 \leq w_5 + 5$ is given in Remark 1 (c). This implies that $w_5 \geq C-2$. Therefore, the ratio of S, R, is one of the elements w_4, $w_5 = C-2$ or $w_7 = C$.

(i) If $R = w_4$, then $w_4 \leq C-3$. Accordingly, $w_4 = 8u+4$ for the interval of u indicated by $1 \leq u \leq \frac{C-7}{8}$. We can write $w_5 = C+6$ by using Lemma 1. Hence,

$S = \langle 8, 8u+4, C, C+2, C+3, C+4, C+6, C+7\rangle$ for each $1 \leq u \leq \frac{C-7}{8}$.

(ii) If $R = w_5 = C-2$, then

$S = \langle 8, C-2, C, C+2, C+3, C+4, C+5, C+7\rangle$.

(iii) If $R = w_7 = C$, then

$S = \langle 8, C, C+2, C+3, C+4, C+5, C+6, C+7\rangle$.

(Sufficiency) The definition of Arf numerical semigroup given in the Section 2 shows that each semigroup given in Theorem 10 satisfies the Arf property.

For any rational number x, the greatest integer less than or equal to x is denoted by $\lfloor x \rfloor$. We denote the set of Arf numerical semigroups and the number of Arf numerical semigroups with multiplicity eight and conductor C, where $\lfloor \frac{C}{8} \rfloor > 1$, by $S_{ARF}\,(8, C)$ and $N_{ARF}\,(8, C)$, respectively. The above theorems can be used to calculate the number of Arf numerical semigroups with multiplicity eight and given conductor.

Corollary 11 Let C be a positive integer such that $\lfloor \frac{C}{8} \rfloor > 1$. The number of Arf numerical semigroups with multiplicity eight and conductor C is

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$$N_{ARF}(8,C)=\begin{cases}\left(\frac{C}{8}\right)^2+\left(\frac{C}{8}\right)+1 & if\ C\equiv 0 \pmod 8,\\ \left(\frac{C-2}{8}\right)^2+\left(\frac{C-2}{8}\right) & if\ C\equiv 2 \pmod 8,\\ \frac{C-3}{8} & if\ C\equiv 3 \pmod 8,\\ \left(\frac{C-4}{8}\right)\left(\frac{C+4}{8}\right)+\left(\frac{C-4}{8}\right)-1 & if\ C\equiv 4 \pmod 8,\\ 2 & if\ C\equiv 5 \pmod 8,\\ \left(\frac{C-6}{8}+1\right)^2 & if\ C\equiv 6 \pmod 8,\\ \frac{C-7}{8}+2 & if\ C\equiv 7 \pmod 8.\end{cases}$$

Proof We give the proof for the case $C\equiv 0 \pmod 8$ where $\lfloor\frac{C}{8}\rfloor>1$. The proofs of the remaining cases are similar. By Theorem 4, there are seven types of semigroups in $S_{ARF}(8,C)$ Following the order in Theorem 4, there is exactly one single semigroup of the first type, while there are precisely $\frac{C-8}{8}$ semigroups of the second type. Similarly, there is one single semigroup of the third type, one single semigroup of the sixth type; there are precisely $\frac{1}{2}\left(\frac{C}{8}-1\right)\left(\frac{C}{8}\right)$ semigroups of the fourth type, precisely $\frac{1}{2}\left(\frac{C}{8}-1\right)\left(\frac{C}{8}\right)$ semigroups of the fifth type, and precisely $\frac{C-8}{8}$ semigroups of the seventh type. Thus the number of elements in $S_{ARF}(8,C)$ is

$$N_{ARF}(8,C)=1+\left(\frac{C-8}{8}\right)+1+\frac{1}{2}\left(\frac{C}{8}-1\right)\left(\frac{C}{8}\right)+\frac{1}{2}\left(\frac{C}{8}-1\right)\left(\frac{C}{8}\right)+1+\left(\frac{C-8}{8}\right)=\left(\frac{C}{8}\right)^2+\left(\frac{C}{8}\right)+1.$$

Example 1 There are two Arf numerical semigroups with multiplicity eight and conductor 53:

$\langle 8,51,53,54,55,57,58,60\rangle=\{0,8,16,24,32,40,48,51,53,\rightarrow\}$,

$\langle 8,53,54,55,57,58,59,60\rangle=\{0,8,16,24,32,40,48,53,\rightarrow\}$.

There are two Arf numerical semigroups with multiplicity eight and conductor 853, too:

$\langle 8,851,853,854,855,857,858,860\rangle=\{0,8,16,24,32,40,48,\dots,840,848,851,853,\rightarrow\}$,

$\langle 8,853,854,855,857,858,859,860\rangle=\{0,8,16,24,32,40,48,\dots,840,848,853,\rightarrow\}$.

Example 2 Let's find all Arf numerical semigroups with eight and conductor 47.

For $u=1,2,3,3,5$ and $C=47\equiv 7 \pmod 8$ we write five different semigroups as follows:

$\langle 8,12,47,49,50,51,53,54\rangle=\{0,8,12,16,20,24,28,32,36,40,44,47,\rightarrow\}$,

$\langle 8,20,47,49,50,51,53,54\rangle=\{0,8,16,20,24,28,32,36,40,44,47,\rightarrow\}$,

$\langle 8,28,47,49,50,51,53,54\rangle=\{0,8,16,24,28,32,36,40,44,47,\rightarrow\}$,

$\langle 8,36,47,49,50,51,53,54\rangle=\{0,8,16,24,32,36,40,44,47,\rightarrow\}$,

$\langle 8,44,47,49,50,51,53,54\rangle=\{0,8,16,24,32,40,44,47,\rightarrow\}$.

In addition, two different Arf numerical semigroups with eight and conductor 47 can be written, these are as follows:

$\langle 8,45,47,49,50,51,52,54\rangle=\{0,8,16,24,32,40,45,47,\rightarrow\}$,

$\langle 8,47,49,50,51,52,53,54\rangle=\{0,8,16,24,32,40,47,\rightarrow\}$.

CONCLUSION

In this study, all Arf numerical semigroups with multiplicity 8 were characterized when given a specific conductor. In addition, the number of all Arf numerical semigroups with conductor C and multiplicity 8 were formulated. In the continuation of this study, Arf numerical semigroups with certain multiplicities will be obtained, and it will be an important resource for those working in the field of numerical semigroup applications.

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CONFLICT OF INTEREST

The Author reports no conflict of interest relevant to this article.

RESEARCH AND PUBLICATION ETHICS STATEMENT

The Author declares that this study complies with research and publication ethics.

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