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General Convergence Analysis for the Perturbation Iteration Technique

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Abstract. In this study, we propose a different approach of the newly developed perturbation iteration method to analyze its convergence properties when solving nonlinear equations. Our main goal is to give some theorems which prove that this technique is convergent under some special conditions. Error estimate is also provided as a result of related theorems. A few interesting problems are investigated to illustrate our arguments.

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1. Introduction

Many engineering problems are modelled by nonlinear differential equations and it is still hard to obtain closed form solutions for these problems. For this reason, the most common methods such as Adomian decomposition method, homotopy decomposition method, Taylor collocation method, differential transform method, optimal homotopy asymptotic method, optimal perturbation iteration method, variational iteration method are used to handle them [4–12, 14, 15, 17, 19–22, 25, 29, 30]. Convergence of these methods has also been investigated by many authors [23,24,27,28]. However, even these methods are not always enough to obtain satisfactory results. In other words, their convergence region may generally small according to the desired solution. Therefore, new methods are required to get more efficient results.

Lately, Pakdemirli and co-workers have modified well-known classical perturbation method to build perturbation iteration method (PIM). This method is based on algorithms which are classified with respect to the number of terms in the perturbation expansion (n) and the degrees of derivatives in the Taylor expansions (m). Briefly, this process has been named as PIA (n,m) [1]. These algorithms have become a famous technique for solving frontier problems in different fields of science and engineering. In many papers, authors have proved that PIAs yield very satisfactory results in solving many nonlinear problems [2, 16, 26].

In this study, we analyze the basic perturbation iteration algorithm (PIA(1,1)) to give the sufficient conditions for convergence of the approximation series generated by PIM. We also aim to address the error estimate of the obtained solution.

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2. Perturbation Iteration Method

In this section, we make some modifications to PIM to examine the convergence analysis and error estimate. Let us consider a second-order differential equation in closed form:

$$F(y'', y', y, \varepsilon) = 0 \tag{2.1}$$

where $y = y(x)$ and ε is the perturbation parameter. For the simplest algorithm one needs to take one correction term y_c from the perturbation expansion as:

$$y_{n+1} = y_n + \varepsilon\,(y_c)_n \tag{2.2}$$

where $n \in \mathrm{N} \cup \{0\}$. Substituting (2.2) into (2.1) then expanding in a Taylor series with first derivatives only yields the simplest perturbation iteration algorithm or briefly PIA(1,1) :

$$F(y_n^{''}, y_n^{'}, y_n, 0) + F_y(y_n^{''}, y_n^{'}, y_n, 0)\,(y_c)_n\,\varepsilon + F_{y'}(y_n^{''}, y_n^{'}, y_n, 0)\left(y_c^{'}\right)_n \varepsilon + F_{y''}(y_n^{''}, y_n^{'}, y_n, 0)\left(y_c^{''}\right)_n \varepsilon + F_\varepsilon \varepsilon = 0. \tag{2.3}$$

(2.3) may seem complicated at first, but it should not be forgotten that (2.3) is valid for the general second order differential equations. In many cases, this algorithm turns into some simple mathematical expressions because of the lack of the terms y, y', y''. Reorganizing (2.3) gives

$$\left(y_c^{''}\right)_n + \frac{F_{y'}}{F_{y''}}\left(y_c^{'}\right)_n + \frac{F_y}{F_{y''}}\,(y_c)_n = -\frac{\frac{F}{\varepsilon} + F_\varepsilon}{F_{y''}}. \tag{2.4}$$

Note also that all calculations in (2.4) are performed at $\varepsilon = 0$. In order to find the first correction term, a trial function y_0 satisfying the initial condition(s) is chosen. Substituting y_0 into (2.4), the first order problem arises as:

$$\left(y_c^{''}\right)_0 + \frac{F_{y'}}{F_{y''}}\left(y_c^{'}\right)_0 + \frac{F_y}{F_{y''}}\,(y_c)_0 = -\frac{\frac{F}{\varepsilon} + F_\varepsilon}{F_{y''}} \tag{2.5}$$

Then, first correction term is calculated from (2.5) by using prescribed initial condition(s). This procedure is repeated with the aid of Eqs. (2.2) and (2.4) until a satisfactory solution is reached.
To begin studying the convergence of the perturbation iteration method, we express the approximate solutions in a different way. To do that, let us define

$$C_0 = y_0, \quad C_{n+1} = (y_c)_n\,.$$

Thus, other solutions can be defined from the following iterations:

$$\begin{aligned}
y_0 &= C_0\\
y_1 &= y_0 + (y_c)_0 = C_0 + C_1\\
y_2 &= y_1 + (y_c)_1 = C_0 + C_1 + C_2\\
y_3 &= y_2 + (y_c)_2 = C_0 + C_1 + C_2 + C_3\\
&\vdots\\
y_{n+1} &= y_n + (y_c)_n = C_0 + C_1 + C_2 + \cdots + C_{n+1} = \textstyle\sum_{i=0}^{n+1} C_i
\end{aligned} \tag{2.6}$$

Consequently, we can represent the solution of (2.1) as

$$y(x) = \lim_{n\to\infty} y_{n+1}(x) = \sum_{i=0}^{\infty} C_i.$$

3. Convergence Analysis and Error Estimate

In this section, we study the convergence of the perturbation iteration method by giving some theorems with proofs.

Theorem 1:(Banach's fixed point theorem). *Suppose that B be a Banach space and*

$$A : B \to B$$

is a nonlinear mapping, and assume that

$$\left\|A\left[y\right]-A\left[\bar{y}\right]\right\| \leq \beta \left\|y-\bar{y}\right\|, y, \bar{y} \in B,$$

for some constant $\beta < 1$ *. Then A has a unique fixed point. Also, the sequence*

$$y_{n+1} = A\left[y_n\right],$$

with an arbitrary choice of $y_0 \in B$ *, converges to the fixed point of A and*

$$\|y_r - y_s\| \leq \|y_1 - y_0\| \sum_{j=s-1}^{r-2} \beta^j.$$

The following theorem, which is required for our analysis, can be deduced from the Banach fixed point theorem.

Theorem 2. *Let B be a Banach space denoted with a suitable norm* $\|.\|$ *over which the series* $\sum_{i=0}^{\infty} C_i$ *is defined and assume that the initial guess* $y_0 = C_0$ *remains inside the ball of the solution* $y(x)$*. The series solution* $\sum_{i=0}^{\infty} C_i$ *converges if there exists* β *such that*

$$\|C_{n+1}\| \leq \beta \|C_n\|.$$

Proof: We first define a sequence as:

$$\begin{aligned} A_0 &= C_0 \\ A_1 &= C_0 + C_1 \\ A_2 &= C_0 + C_1 + C_2 \\ &\vdots \\ A_n &= C_0 + C_1 + C_2 + \cdots + C_n \end{aligned}$$

Now, we need to show that $\{A_n\}_{n=0}^{\infty}$ is a Cauchy sequence in B. To do this, consider that

$$\|A_{n+1} - A_n\| = \|C_{n+1}\| \leq \beta \|C_n\| \leq \beta^2 \|C_{n-1}\| \leq \cdots \leq \beta^{n+1} \|C_0\|.$$

For every $n, k \in \mathrm{N}, n \geq k$, we have

$$\begin{aligned} &\|A_n - A_k\| = \|(A_n - A_{n-1}) + (A_{n-1} - A_{n-2}) + \cdots + (A_{k+1} - A_k)\| \\ &\leq \|A_n - A_{n-1}\| + \|A_{n-1} - A_{n-2}\| + \cdots + \|A_{k+1} - A_k\| \\ &\leq \beta^n \|C_0\| + \beta^{n-1} \|C_0\| + \cdots + \beta^{k+1} \|C_0\| = \tfrac{1-\beta^{n-k}}{1-\beta} \beta^{k+1} \|C_0\| \end{aligned} \tag{3.1}$$

Since we also have $0 < \beta < 1$, we can obtain from (3.1)

$$\lim_{n,k\to\infty} \|A_n - A_k\| = 0.$$

Thus, $\{A_n\}_{n=0}^{\infty}$ is a Cauchy sequence in B and this implies that the series solution (2.6) is convergent.

Theorem 3. *If the initial function* $y_0 = C_0$ *remains inside the ball of the solution* $y(x)$ *then* $A_n = \sum_{i=0}^{n} C_i$ *also remains inside the ball of the solution.*
Proof: Suppose that

$$C_0 \in B_r(y)$$

where

$$B_r(y) = \{C \in A \,|\, \|y - C\| < r\}.$$

is the ball of the solution $y(x)$*. By hypothesis* $y = \lim_{n\to\infty} A_n = \sum_{i=0}^{\infty} C_i$ *and from Theorem 2, we have*

$$\|y - A_n\| \leq \beta^{n+1} \|C_0\| < \|C_0\| < r$$

where $\beta \in (0, 1)$*and* $n \in \mathrm{N}$. This completes the proof.

Theorem 4. *Assume that the obtained solution* $\sum_{i=0}^{\infty} C_i$ *is convergent to the solution* $y(x)$*. If the truncated series* $\sum_{i=0}^{k} C_i$ *is used as an approximation to the solution* $y(x)$ *of problem (2.1), then the maximum error is given as,*

$$E_k(x) \leq \frac{\beta^{k+1}}{1-\beta} \|C_0\|.$$

Proof: From (3.1), we have

$$\|A_n - A_k\| \leq \frac{1-\beta^{n-k}}{1-\beta}\beta^{k+1}\|C_0\|$$

for $n \geq k$. By knowing

$$y(x) = \lim_{n\to\infty} A_n(x) = \sum_{i=0}^{\infty} C_i$$

we can write

$$\left\| y(x) - \sum_{i=0}^{k} C_i \right\| \leq \frac{1-\beta^{n-k}}{1-\beta}\beta^{k+1}\|C_0\|$$

and also it can be written as

$$E_k(x) = \left\| y(x) - \sum_{i=0}^{k} C_i \right\| \leq \frac{\beta^{k+1}}{1-\beta}\|C_0\|$$

since $1-\beta^{n-k} < 1$. Here β is selected as $\beta = \max\{\beta_i, i = 0, 1, \ldots, n\}$ where

$$\beta_i = \frac{\|C_{n+1}\|}{\|C_n\|}.$$

4. Applications

In this section, some examples are given to verify the theorems.

Example 4.1. Consider the nonlinear problem

$$y'(x) - e^{-y(x)} = 0; \quad y(0) = 0; \quad 0 \leq x \leq 1. \tag{4.1}$$

Auxiliary parameter can be inserted as:

$$F(y', y, \varepsilon) = y'(x) - e^{-\varepsilon y(x)} = 0. \tag{4.2}$$

This problem does not involve the term y''. Thus, (2.3) simplifies to

$$F(y_n', y_n, 0) + F_y(y_n', y_n, 0)\,(y_c)_n\,\varepsilon + F_{y'}(y_n', y_n, 0)\left(y_c'\right)_n \varepsilon + F_\varepsilon \varepsilon = 0. \tag{4.3}$$

By using (4.2), we have

$$F = y_n' - 1, \; F_y = 0, F_{y'} = 1, \quad F_\varepsilon = y_n$$

for the formula (4.3) . Setting $\varepsilon = 1$ in (4.3) yields

$$({y_c}')_n = 1 - y_n - y_n'.$$

One can choose an initial function as:

$$y_0 = C_0 = 0$$

which satisfies the given initial conditions. Then first order problem becomes

$$({y_c}')_0 = 1, \;\; y(0) = 0 \tag{4.4}$$

Solving (4.4) and using the Eq. (2.6), we have

$$\begin{aligned} C_1 &= x \\ C_2 &= \tfrac{-x^2}{2} \\ C_3 &= \tfrac{x^3}{6} \\ C_4 &= \tfrac{-x^4}{24} \\ C_5 &= \tfrac{x^5}{120} \\ &\vdots \end{aligned}.$$

So, the nth approximate solution is written as

$$\sum_{i=0}^{n} C_n = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \frac{x^5}{120} + \cdots C_n \tag{4.5}$$

To make a decision on the convergence of the PIM, we compute β_i as:

$$\begin{aligned}
\beta_1 &= \frac{\|C_2\|}{\|C_1\|} = \frac{\|x^2/2\|}{\|x\|} = 0.5 < 1\\
\beta_2 &= \frac{\|C_3\|}{\|C_2\|} = \frac{\|x^3/6\|}{\|x^2/2\|} = 0.3333 < 1\\
\beta_3 &= \frac{\|C_3\|}{\|C_2\|} = \frac{\|x^4/24\|}{\|x^3/6\|} = 0.25 < 1\\
&\vdots\\
\beta_i &= \frac{\|C_{i+1}\|}{\|C_i\|} = \frac{\|x^{i+1}/(i+1)!\|}{\|x^i/i!\|} = \frac{1}{i+1} < 1
\end{aligned}.$$

where $0 \le x \le 1$.As it is seen, all β_i are less than one and this means that PIM is convergent. In other words, series solution $\sum_{i=0}^{\infty} C_i$ by perturbation iteration method converges to the exact solution for this problem.

Remark 1: One can also expand the region of convergence of the obtained series solution by enforcing the ratio in (3.18) to hold true in the infinite limit. For instance, the interval or region of convergence of the obtained solution (4.5) is taken as $0 \le x \le 1$ for this problem. However, to obtain larger interval we can choose $0 \le x \le \delta$ for the problem (4.1). So, we get

$$\beta_i = \frac{\|C_{i+1}\|}{\|C_i\|} = \frac{\|x^{i+1}/(i+1)!\|}{\|x^i/i!\|} = \frac{\delta}{i+1} < 1$$

under the maximum norm defined over R for $i > \delta$. As a result, we can say that series solution $\sum_{i=0}^{\infty} C_i$ is convergent for every positive δ.

Example 4.2. Consider the first order nonlinear differential equation [18]:

$$y' = x^2 + y^2, \;\; y(0) = 1, \; 0 \le x \le 1. \tag{4.6}$$

which does not have exact solution.

To construct the algorithm, we rewrite the Eq.(4.6) as

$$F(y', y, \varepsilon) = y' - x^2 - \varepsilon y^2 \tag{4.7}$$

where $\varepsilon = 1$ perturbation parameter.For this problem,we do not have the term y''. Therefore, Eq. (2.3) turns into:

$$F(y_n^{'}, y_n, 0) + F_y(y_n^{'}, y_n, 0)\,(y_c)_n\,\varepsilon + F_{y'}(y_n^{'}, y_n, 0)\left(y_c^{'}\right)_n \varepsilon + F_\varepsilon \varepsilon = 0. \tag{4.8}$$

Necessary terms in (4.8) are obtained

$$F = y_n^{'} - x^2, \;\; F_y = 0, F_{y'} = 1, \;\; F_\varepsilon = -(y_n)^2$$

from (4.7) and setting $\varepsilon = 1$ yields

$$({y_c}')_n = y_n^2 - y_n^{'} + x^2. \tag{4.9}$$

We start the iteration by taking a trivial solution which satisfies the given initial conditions:

$$y_0 = C_0 = 1. \tag{4.10}$$

Substituting (4.10) into (4.9) generates first order problem as:

$$({y_c}')_0 = 1 + x^2, \;\; y(0) = 0 \tag{4.11}$$

First correction term is obtained by solving the first order problem (4.11) and so on. Using the Eqs. (2.6) and (4.9),

$$\begin{aligned}
C_1 &= x + \tfrac{x^3}{3}\\
C_2 &= x^2 + \tfrac{x^3}{3} + \tfrac{x^4}{6} + \tfrac{2x^5}{15} + \tfrac{x^7}{63}\\
C_3 &= \tfrac{2x^3}{3} + \tfrac{2x^4}{3} + \tfrac{2x^5}{5} + \tfrac{29x^6}{90} + \tfrac{2x^7}{15} + \tfrac{41x^8}{630} + \tfrac{299x^9}{11340} + \tfrac{4x^{10}}{525}\\
&\vdots
\end{aligned}$$

Therefore, we have

TABLE 1. Numerical solution of the initial value problem in Example 4.2.

h	x=0.9	x=1
0.1	14.02182	735.0991
0.05	14.27117	175863
0.01	14.30478	2.09×10^{2893}

$$\begin{aligned}
\beta_0 &= \frac{\|C_1\|}{\|C_0\|} = 1.333 > 1\\
\beta_1 &= \frac{\|C_2\|}{\|C_1\|} = 1.2369 > 1\\
\beta_2 &= \frac{\|C_3\|}{\|C_2\|} = 1.2704 > 1\\
\beta_2 &= \frac{\|C_4\|}{\|C_3\|} = 1.3478 > 1\\
&\vdots
\end{aligned}$$

Thus, one can conclude that perturbation iteration method is not convergent for this problem. Actually, none of the most common analytical methods such as variational iteration method, Adomian decomposition method, differential transform method etc. is convergent for this problem. Because, solution of (4.6) has an asymptote on the interval $0 \le x \le 1$ in which the calculations are performed. Table 1 shows some numerical solutions obtained by Runge-Kutta method [13]. It is clear that there is huge different between the solution for *x=0.9* and *x=1*.
In view of Remark 1, we can select another interval by trial so that PIM is convergent for Example 4.2. For the interval $0 \le x \le 0.8$, β_i's are obtained as

$$\begin{aligned}
\beta_0 &= \frac{\|C_1\|}{\|C_0\|} = 0.97066 < 1\\
\beta_1 &= \frac{\|C_2\|}{\|C_1\|} = 0.943386 < 1\\
\beta_2 &= \frac{\|C_3\|}{\|C_2\|} = 0.92541 < 1\\
\beta_2 &= \frac{\|C_4\|}{\|C_3\|} = 0.89054 < 1\\
&\vdots
\end{aligned}$$

Now, we say that the series solution obtained by PIM is convergent for $0 \le x \le 0.8$.

Example 4.3. Consider the following differential equation [28]:

$$y' + y^2 = 1, \quad y(0) = 0, \ 0 \le x \le 1. \tag{4.12}$$

Eq.(4.12) is reconsidered as:

$$F(y', y, \varepsilon) = y' + \varepsilon y^2 - 1 = 0$$

Proceeding as described in Section 2, one can easily obtain the algorithm as:

$$({y_c}')_n = -y_n^2 - y_n^{'} + 1.$$

By choosing $y_0 = C_0 = 0$ as an initial function, we have

$$\begin{aligned}
C_1 &= x\\
C_2 &= -\frac{x^3}{3}\\
C_3 &= \frac{2x^5}{15} - \frac{x^7}{63}\\
&\vdots
\end{aligned}$$

which yields

$$\begin{aligned}
\beta_0 &= \frac{\|C_2\|}{\|C_1\|} = 0.3333 < 1\\
\beta_1 &= \frac{\|C_3\|}{\|C_2\|} = 0.4476 < 1\\
\beta_2 &= \frac{\|C_4\|}{\|C_3\|} = 0.6683 < 1\\
&\vdots
\end{aligned}$$

These results confirm that, obtained series solution by PIM is convergent. Additionally, one can easily show that interval of convergence can be extended to $|x| \leq \frac{\pi}{2}$ based on Remark 1.

Remark 2: One can also extend the interval of convergence by choosing different initial guesses. Let us choose the initial function as

$$y_0 = C_0 = 1 - e^{-2x}$$

which clearly satisfies the initial condition. By using absolute norm, one can demonstrate that

$$\lim_{i\to\infty} \frac{C_{i+1}}{C_i} = \lim_{i\to\infty} \frac{\left(1 - e^{-2x}\right)\left(1 - e^{-2ix}\right)}{(2 - e^{-2ix} + e^{-4ix} - \cdots)} = \frac{1 - e^{-2x}}{2}.$$

Thus, new obtained series converges for $\left|\frac{1-e^{-2x}}{2}\right| < 1$ or equivalently for all $x > -\ln\left(\sqrt{3}\right)$.

Example 4.4. Let us now consider the Van Der Pol oscillator problem [3]:

$$y''(x) + y^2(x) + (y'(x))^2 - y(x) = 1; \quad y(0) = 2, y'(0) = 0.$$

Perturbation parameter can be furnished as:

$$F(y'', y', y, \varepsilon) = y''(x) + \varepsilon\left\{y^2(x) + (y'(x))^2 - y(x)\right\} - 1 = 0. \tag{4.13}$$

After making computations for the formula (2.3) and setting $\varepsilon = 1$, we have

$$(y_c'')_n = -\left(y_n^{''} - y_n + y_n^2 + \left(y_n^{'}\right)^2 - 1\right)$$

Following the same procedure as in the previous examples, iterations are reached as

$$\begin{aligned} C_0 &= 2 \\ C_1 &= -\tfrac{x^2}{2} \\ C_2 &= \tfrac{x^4}{24} - \tfrac{x^6}{120} \\ &\vdots \end{aligned}$$

One can easily deduce that the solution$\sum_{i=0}^{\infty} C_i$ obtained by PIM converges to the exact solution $y(x) = 1 + \cos(x)$. Actually, we already have

$$\begin{aligned} \beta_0 &= \tfrac{\|C_1\|}{\|C_0\|} = 0.25 < 1 \\ \beta_1 &= \tfrac{\|C_2\|}{\|C_1\|} = 0.0666 < 1 \\ \beta_2 &= \tfrac{\|C_3\|}{\|C_2\|} = 0.00817 < 1 \\ &\vdots \end{aligned}$$

which proves also that the solution $\sum_{i=0}^{n} C_i$ converge at least for the interval $0 < x \leq 1$. One may extend this interval by starting with another guess function. However it is not very easy to choose such a function which also has to satisfy the initial conditions.

Remark 3: The interval of convergence can also be changed by inserting the perturbation parameter to the problem in a different way. Let us reconsider the problem (4.13) as:

$$F(y'', y', y, \varepsilon) = y''(x) - y(x) + \varepsilon\left\{y^2(x) + (y'(x))^2\right\} - 1 = 0.$$

Due to this changing, algorithm becomes:

$$(y_c'')_n - (y_c)_n = -\left(y_n^{''} - y_n + y_n^2 + \left(y_n^{'}\right)^2 - 1\right).$$

Now we obtain

$$\begin{aligned} C_0 &= 2 \\ C_1 &= 1 - \tfrac{e^{-x}}{2} - \tfrac{e^{x}}{2} \\ C_2 &= -\tfrac{1}{6}e^{-2x}(-1 + e^x)(-1 - 15e^x + 15e^{2x} + e^{3x} - 9e^x x - 9e^{2x} x) \; \cdot \\ &\vdots \end{aligned}$$

It may be possible to get maximum length of the interval of convergence, but here we only try and get

$$\begin{aligned}
\beta_0 &= \frac{\|C_1\|}{\|C_0\|} = 4.535751152497 \times 10^{-7} < 1 \\
\beta_1 &= \frac{\|C_2\|}{\|C_1\|} = 7.208842143654 \times 10^{-7} < 1 \\
\beta_2 &= \frac{\|C_3\|}{\|C_2\|} = 2.066317865477 \times 10^{-8} < 1 \\
&\vdots
\end{aligned}.$$

for $0 \le x \le 100$ by using maximum norm.

Example 4.5. As a last example we consider the first order initial value problem

$$y'(x) - y^2(x) = -x^4 - 2x^3 - x^2 + 2x + 1; \quad y(0) = 0. \tag{4.14}$$

(4.14) can be rewritten as

$$F(y', y, \varepsilon) = y'(x) - \varepsilon y^2(x) + x^4 + 2x^3 + x^2 - 2x - 1.$$

Then algorithm (2.3) becomes

$$({y_c}')_n = -\left(y_n' - y_n^2 + x^4 + 2x^3 + x^2 - 2x - 1\right)$$

C_i ‘s are obtained as

$$\begin{aligned}
C_0 &= 0 \\
C_1 &= x + x^2 - \frac{x^3}{3} - \frac{x^4}{2} - \frac{x^5}{5} \\
C_2 &= \frac{x^3}{3} + \frac{x^4}{2} + \frac{x^5}{15} - \frac{5x^6}{18} - \frac{58x^7}{315} - \frac{x^8}{120} + \frac{23x^9}{540} + \frac{x^{10}}{50} + \frac{x^{11}}{275} \\
C_3 &= \frac{2x^5}{15} + \frac{5x^6}{18} + \frac{46x^7}{315} - \frac{17x^8}{180} - \frac{61x^9}{420} - \frac{737x^{10}}{12600} + \frac{5x^{11}}{1188} + \frac{269x^{12}}{16200} + \frac{108233x^{13}}{8108100} + \frac{34861x^{14}}{4365900} + \frac{21571x^{15}}{11907000} \\
&\vdots
\end{aligned}$$

For this problem, we obtain noise functions between each correction terms C_i. In other words, there are always some terms in C_{i-1} are canceled by some terms in C_i. We can underestimate them because their sum vanishes in the limit $\lim_{i\to\infty} \sum_{n=0}^{i} C_n$. That is, we do not need to analyze the interval convergence since we have directly the exact solution $y(x) = x^2 + x$. In such cases, we need to assume that the remaining term is the solution and try to substitute it in the problem.

5. Conclusion

In this work, we have aimed to study the convergence of the perturbation iteration algorithms for solving differential equations. We have used some important theorems which are derived with the help of Banach’s fixed point theorem to achieve our goals. Theorem 2 states that under certain assumptions PIM series solution is convergent to the exact solution of the problem. Theorem 3 shows the importance of selections of the initial functions to verify the expected convergence. Example 4.3 demonstrates that it is possible to make the PIM solution convergent by changing the initial guess function. It should be emphasized that there are no general theorems to choose initial functions for most common methods. Therefore, one need to start an appropriate function which satisfies the initial or boundary conditions. Theorem 4 gives the maximum error which may occur during the solution process. Other examples are also solved to reveal more opportunities to extend the region of convergence of the obtained solutions.

Finally, we can say that this paper presents a detailed study on the convergence of the newly developed perturbation iteration technique and it will be encouraging for further studies.

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