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Yeni Tip *q*-Stancu-Durrmeyer Operatörlerinin Lokal Yaklaşım Özellikleri

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ÖZ

Bu çalışmada, yeni tip *q*-Stancu-Durrmeyer operatörleri (kısaca *q*-SD operatörleri) tanımlanmıştır. Bu operatörlerin birinci dereceden ve ikinci dereceden momentleri $U_Z^{q,\alpha}(t^j,\tau)$, $j=0,1,2$ için hesaplanmıştır. İkinci dereceden merkezi momenti için de tahmini değer hesaplaması yapılmıştır. Ayrıca, ikinci dereceden süreklilik modülü kullanılarak *q*-SD operatörlerinin lokal yaklaşım özellikleri incelenmiş ve β derece Lipschitz sınıfı maksimal fonksiyon aracılığıyla tahmini yaklaşım hızı verilmiştir.

Local Approximation Properties for the New Type *q*-Stancu-Durrmeyer Operators

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ABSTRACT

In this article, new kind *q*-Stancu-Durrmeyer operators (or shortly *q*-SD operators) are defined. For these operators, first order and second order moments $U_Z^{q,\alpha}(t^j,\tau)$ are calculated for $j=0,1,2$. An estimation for the second order central moment $U_Z^{q,\alpha}\left(\left(t-\tau\right)^2,\tau\right)$, is given. Furthermore, through the use of the second modulus of continuity, approximation properties of the *q*-SD operators in the local sense are examined and by using β order Lipschitz kind maximal function, a direct estimation for the *q*-SD operators is given.

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1.Introduction

In 1967, an integral variation of the popular Bernstein polynomials is proposed by J.L. Durrmeyer for the aim of approximating the Lebesgue integrable functions on the closed interval $[0,1]$ (Durrmeyer, 1967). This integral variation was handled in many other forms by some other authors. One year later, in 1968, again inspired by the Bernstein polynomials, D.D. Stancu proposed the positive linear operators depending on the parameter α (Stancu, 1968). On the other hand, the applications of quantum calculus (or calculus without limitations) in the theory of approximation by positive linear operators have come to a very important place in the last few decades. There are many papers related with this topic (see for example: Gupta, 2008; Gupta and Finta, 2008; Gupta and Heping, 2008; Mahmudov and Sabancigil, 2010; Narayan and Patel, 2013; Neer, et. al., 2017; Nowak, 2009; Phillips, 1997; Sabancigil, 2023; Sabancigil, 2021; Sabancigil 2022 and etc.). Stancu's operators are modified by G. Nowak using the q-integers and thus in 2009, q-Stancu operators was born (Nowak, 2009). For the aim of approximating functions with the continuity property, the authors V. Gupta and W. Heping proposed a new variant of the Durrmeyer operators by using the concepts of the quantum calculus, (Gupta and Heping, 2008). These new operators are given explicitly as below:

$$\Delta_{z,q}(\zeta,\tau) := \zeta(0)\chi_{z,0}(q,\tau) + [z+1]\sum_{\kappa=1}^{z}\frac{1}{q^{\kappa-1}}\chi_{z,\kappa}(q,\tau)\int_0^1 \chi_{z,\kappa-1}(q,qt)\zeta(t)d_q t$$

where $\chi_{z,\kappa}(q,\tau) = \begin{bmatrix} z \\ \kappa \end{bmatrix}\tau^{\kappa}\prod_{\ell=0}^{z-\kappa-1}(1-q^{\ell}\tau)$.

In 2010, genuine type q-Bernstein-Durrmeyer operators are investigated by N. Mahmudov and P. Sabancigil (Mahmudov and Sabancigil, 2010). For $0 < q < 1$, a few direct local and global approximation properties of the q version of Durrmeyer operators are examined by V. Gupta and Z. Finta (Gupta and Finta, 2008). Stancu-Chlodowsky polynomials are studied by I. Buyukyazici (Buyukyazici, 2010). In 2012, approximation properties of two dimensional q-Bernstein-Chlodowsky-Durrmeyer operators are studied by I. Buyukyazici and H. Sharma (Buyukyazici and Sharma, 2012). The Stancu-Chlodowsky operators based on q-calculus are studied by E. Tas, C. Orhan and T. Yurdakadim in 2013 (Tas, Orhan and Yurdakadim, 2013). Stancu type modification of q-Durrmeyer operators are studied by V. Narayan and P. Patel (Narayan and Patel, 2013). They established the convergence rate and they constructed a Voronovskaja type asymptotic formula for their operators. In 2017, T. Neer, P.N. Agrawal and S. Araci proposed q-version of the Stancu-Durrmeyer operators by modifying q-Bernstein operators via q-Jackson integration (Neer, Agrawal and Araci, 2017). They established some estimations for the moments and invented properties of convergence in the local sense, by using β order Lipschitz kind maximal function, they gave a direct estimation. In 2018, as a continuation of the work in this article, T. Neer, A.M. Acu and P. Agrawal extended the previous work and they established q-SD operators based on two variables (Neer, Acu and Agrawal, 2018). Recently, in 2023, P. Sabancigil proposed the genuine type q-Stancu-Bernstein-Durrmeyer operators and consider them from different aspects (Sabancigil, 2023). Thus, we see that the theory of approximation using positive linear operators became an important area of research in the last few decades and these operators have many applications in other branches besides mathematics such as engineering, computer aided geometric design, statistics, etc. Here, in the present article, in order to provide a contribution to this area, we propose the latest q-SD type operators which are called $U_z^{q,\alpha}$. We calculate 1st and 2nd order moments explicitly and we present an estimation for the second order central moment. Furthermore, with the use of the second modulus of smoothness, we invent approximation properties of the q-SD operators in the local sense and by using Lipschitz kind maximal function of order β, we give a direct estimation for the q-SD operators again in the local sense.

We are going to start by presenting some symbols, definitions and notations related with the q-integers and more generally with quantum analysis.

Assume that q is a positive real number. For any $z \in \mathrm{N} \cup \{0\}$, the q-integer $[z] = [z]_q$ is defined by

$$[z] := q^{z-1} + ... + q + 1, \;\; [0] := 0,$$

and the q-factorial $[z]! = [z]_q!$ is defined by $[z]! = [z]...[2][1], \quad [0]! = 1.$

If k and z are integers satisfying $0 \le k \le z,$ the q-version of the binomial is expressed by

$$\begin{bmatrix} z \\ k \end{bmatrix} := \frac{[z]!}{[k]![z-k]!}.$$

Throughout the present article, We are going to make use of the below notations:

$$(1-\tau)_q^z := \prod_{\kappa=0}^{z-1} (1-q^\kappa \tau), \;\; p_{z,k}(q,\tau) := \begin{bmatrix} z \\ k \end{bmatrix} \tau^k (1-\tau)_q^{z-k},$$

$$p_{z,k}^{\alpha}(q,\tau) := \begin{bmatrix} z \\ k \end{bmatrix} \prod_{\ell=0}^{k-1} \left(\tau + \alpha[\ell]\right) \frac{\prod_{\kappa=0}^{z-1-k} \left(1 - q^\kappa \tau + \alpha[\kappa]\right)}{\prod_{\ell=0}^{z-1} \left(1 + \alpha[\ell]\right)}, \;\; 0 \le k \le z,$$

$$p_{z,k}^{\alpha}(q,1) = 0, \quad p_{z,z}^{\alpha}(q,1) = 1, \quad \prod_{s=0}^{-1} (*) = 1, \;\; \alpha \ge 0.$$

Assume that $\Omega > 0$, then the q-version of the integral on the closed interval $[0,\Omega]$ is given by

$$\int_0^{\Omega} \zeta(t) d_q t := \Omega(1-q) \sum_{\kappa=0}^{\infty} \zeta(\Omega q^\kappa) q^\kappa, \quad \text{where } 0 < q < 1.$$

2. *q*-SD Operators and Their Moments

Definition 1. Suppose that $\zeta \in C[0,1],\; \alpha \ge 0$ and $q \in (0,1)$. Explicit definition of the q-SD operators is stated as below:

$$U_z^{q,\alpha}(\zeta,\tau) = p_{z,0}^{\alpha}(q,\tau)\zeta(0) + [z+1] \sum_{k=1}^{z} q^{1-k} p_{z,k}^{\alpha}(q,\tau) \int_0^1 p_{z,k-1}(q,qt)\zeta(t) d_q t.$$

Explicit formulae of the moments $U_z^{q,\alpha}(t^j,\tau)$ for $j = 0,1,2$ and an estimation for the 2nd central moment $U_z^{q,\alpha}((t-\tau)^2,\tau)$ are obtained in the following lemma.

Lemma 1. $U_z^{q,\alpha}(1,\tau) = 1,\; U_z^{q,\alpha}(t,\tau) = \frac{[z]}{[z+2]}\tau,$

$$U_z^{q,\alpha}(t^2,\tau) = \frac{[z]}{[z+2][z+3]}\tau + \frac{q[z]^2}{[z+2][z+3](1+\alpha)} \left(\tau^2 + \alpha\tau + \frac{(\tau-\tau^2)}{[z]} \right),$$

$$U_z^{q,\alpha}((t-\tau)^2,\tau) \le \left(\alpha + \frac{6+\alpha}{[z+2]} \right) \left(\frac{(\tau-\tau^2)}{1+\alpha} + \frac{1}{[z+2]} \right) \text{ for } \tau \in [0,1].$$

Proof. Thanks to the explanation of the q version of the Beta function (Kac, 2002), we have for $\upsilon = 0,1,2...,$

$$\int_0^1 t^{\upsilon} p_{z-2,k-1}(q,qt)d_qt = \begin{bmatrix} z-2 \\ k-1 \end{bmatrix} q^{k-1}\int_0^1 t^{k+\upsilon-1}(1-qt)_q^{z-1-k}\, d_qt$$

$$= \frac{q^{k-1}[z-2]!}{[k-1]![z-1-k]!}\,\frac{[k+\upsilon-1]![z-1-k]!}{[z+\upsilon-1]!} = \frac{q^{k-1}[z-2]![k+\upsilon-1]!}{[k-1]![z+\upsilon-1]!}.$$

For obtaining the desired equalities and the inequality given in Lemma 1, we will get help from the following equalities which are available in the paper of Nowak (2009):

$$\sum_{k=0}^{z} p_{z,k}^{\alpha}(q,\tau)=1, \qquad \sum_{k=0}^{z} p_{z,k}^{\alpha}(q,\tau)\frac{[k]}{[z]}=\tau, \qquad \sum_{k=0}^{z} p_{z,k}^{\alpha}(q,\tau)\frac{[k]^2}{[z]^2}=\frac{1}{1+\alpha}\left(\tau^2+\alpha\tau+\frac{(\tau-\tau^2)}{[z]}\right).$$

$$U_z^{q,\alpha}(1,\tau)= p_{z,0}^{\alpha}(q,\tau)+[z+1]\sum_{k=1}^{z} q^{1-k} p_{z,k}^{\alpha}(q,\tau)\int_0^1 p_{z,k-1}(q,qt)d_qt.$$

Now, if we evaluate the integral on the right side of the last equality and by using the above identities,

we get, $U_z^{q,\alpha}(1,\tau)= p_{z,0}^{\alpha}(q,\tau)+[z+1]\sum_{k=1}^{z} q^{1-k} p_{z,k}^{\alpha}(q,\tau)\frac{q^{k-1}[z]![k-1]!}{[k-1]![z+1]!}$

$$= p_{z,0}^{\alpha}(q,\tau)+\sum_{k=1}^{z} p_{z,k}^{\alpha}(q,\tau) = \sum_{k=0}^{z} p_{z,k}^{\alpha}(q,\tau)=1.$$

$$U_z^{q,\alpha}(t,\tau)=[z+1]\sum_{k=1}^{z} q^{1-k} p_{z,k}^{\alpha}(q,\tau)\int_0^1 t p_{z,k-1}(q,qt)d_qt$$

$$=[z+1]\sum_{k=1}^{z} q^{1-k} p_{z,k}^{\alpha}(q,\tau)\frac{q^{k-1}[z]![k]!}{[k-1]![z+2]!}$$

$$=[z+1]\sum_{k=1}^{z} p_{z,k}^{\alpha}(q,\tau)\frac{[k]}{[z+1][z+2]}$$

$$=\sum_{k=0}^{z} p_{z,k}^{\alpha}(q,\tau)\frac{[k]}{[z+2]}=\frac{[z]}{[z+2]}\sum_{k=0}^{z} p_{z,k}^{\alpha}(q,\tau)\frac{[k]}{[z]}=\frac{[z]}{[z+2]}\tau.$$

$$U_z^{q,\alpha}(t^2,\tau)=[z+1]\sum_{k=1}^{z} q^{1-k} p_{z,k}^{\alpha}(q,\tau)\int_0^1 t^2 p_{z,k-1}(q,qt)d_qt$$

$$=[z+1]\sum_{k=1}^{z} q^{1-k} p_{z,k}^{\alpha}(q,\tau)\frac{q^{k-1}[z]![k+1]!}{[k-1]![z+3]!}$$

$$=\sum_{k=0}^{z} p_{z,k}^{\alpha}(q,\tau)\frac{[k][k+1]}{[z+2][z+3]},$$

here in this last step we remember the trivial formula related with q-integers such that $[k+1]=1+[k]q$, so we obtain

$$U_z^{q,\alpha}(t^2,\tau)=\sum_{k=0}^{z} p_{z,k}^{\alpha}(q,\tau)\frac{[k](1+[k]q)}{[z+2][z+3]}$$

$$=\frac{1}{[z+2][z+3]}\left\{q[z]^2\sum_{k=0}^{z} p_{z,k}^{\alpha}(q,\tau)\frac{[k]^2}{[z]^2}+[z]\sum_{k=0}^{z} p_{z,k}^{\alpha}(q,\tau)\frac{[k]}{[z]}\right\}$$

$$=\frac{q[z]^2}{[z+2][z+3]}\frac{1}{1+\alpha}\left(\tau^2+\alpha\tau+\frac{(\tau-\tau^2)}{[z]}\right)+\frac{[z]\tau}{[z+2][z+3]}.$$

Now to estimate $U_z^{q,\alpha}((t-\tau)^2,\tau)$ for $\tau\in[0,1]$, we use the following estimation which is previously given in the paper of Gupta and Finta (2009):

$$(q[z]+[z])+(q[z]^2-q[z])-2[z+3][z]+[z+3][z+2]\le 6.$$

It follows that

$$U_z^{q,\alpha}((t-\tau)^2,\tau)=\frac{[z](\alpha+1)+[z]^2q\alpha+[z]q}{(\alpha+1)[z+2][z+3]}(\tau-\tau^2)$$

$$+\left(\frac{[z](q+1)}{[z+3][z+2]}+\frac{([z]-1)[z]q-2[z+3][z]+[z+3][z+2]}{[z+3][z+2]}\right)\tau^2$$

$$\le\left(\frac{[z]}{[z+2][z+3]}+\frac{[z]^2}{[z+2][z+3]}\frac{\alpha}{\alpha+1}+\frac{[z]}{[z+2][z+3](\alpha+1)}\right)(\tau-\tau^2)+\frac{6\tau^2}{[z+2][z+3]}$$

$$\le\left(\frac{(\alpha+2)[z]}{[z+2][z+3]}+1\right)(\tau-\tau^2)+\frac{6}{[z+2]^2}$$

$$\le\frac{\alpha}{\alpha+1}(\tau-\tau^2)+\frac{\alpha+2}{[z+2]}\frac{(\tau-\tau^2)}{\alpha+1}+\frac{6}{[z+2]^2}$$

$$\le\left(\alpha+\frac{\alpha+6}{[z+2]}\right)\left(\frac{(\tau-\tau^2)}{\alpha+1}+\frac{1}{[z+2]}\right).$$

■

3. Local Approximation for the *q*-SD Operators

Under the present section, we will proceed to prove a theorem on the local approximation for the newly defined *q*-SD operators. Let us remember the K-functional which is explained as:

$$K_2(\zeta,\delta):=\inf\left\{\|\zeta-\eta\|_{C[0,1]}+\delta\|\eta''\|:\eta\in C^2[0,1]\right\},\ \delta\ge 0,$$

where

$$C^2[0,1]:=\{\eta:\eta,\eta',\eta''\in C[0,1]\}.$$

Then, thanks to a previously proved result which can be found in the book of Ditzian and Totik (1987), a constant $M_1>0$ exists in such a way that satisfies the below inequality

$$K_2\left(\zeta,\delta^2\right)\le M_1\omega_2\left(\zeta,\delta\right) \quad (1)$$

where $\omega_2\left(\zeta,\delta\right):=\sup\limits_{0<h\le\delta}\ \sup\limits_{\tau\pm h\in[0,1]}\left|\zeta\left(\tau-h\right)-2\zeta\left(\tau\right)+\zeta\left(\tau+h\right)\right|$

is the second modulus of smoothness of $\zeta\in C[0,1]$. Also let us remember the definition of the ordinary modulus of continuity $\omega\left(\zeta,\delta\right)$ where

$$\omega\left(\zeta,\delta\right)=\sup_{0<h\le\delta}\ \sup_{\tau\in[0,1]}\left|\zeta\left(\tau+h\right)-\zeta\left(\tau\right)\right|.$$

Below, in Theorem 1, we obtain approximation properties of the q-SD operators, $U_z^{q,\alpha}\left(\zeta,\tau\right)$, in the local sense which is the main result of our article.

Theorem 1 Let $\zeta\in C[0,1]$. Then for every τ, satisfying $0\le\tau\le1$, a constant $L>0$ exists such that

$$\left|U_z^{q,\alpha}\left(\zeta,\tau\right)-\zeta\left(\tau\right)\right|\le L\omega_2\left(\zeta,\sqrt{\delta_z\left(\tau\right)}\right)+\omega\left(\zeta,\beta_z\left(\tau\right)\right),$$

where $\delta_z(\tau)=U_z^{q,\alpha}\left(\left(t-\tau\right)^2,\tau\right)+\left(U_z^{q,\alpha}\left(\left(t-\tau\right),\tau\right)\right)^2$ and $\beta_z(\tau)=\left|U_z^{q,\alpha}\left(\left(t-\tau\right),\tau\right)\right|$.

Proof. To prove this theorem, we will define the star q-SD operators ${}^{*}U_z^{q,\alpha}$ as follows.

$${}^{*}U_z^{q,\alpha}\left(\zeta,\tau\right)=U_z^{q,\alpha}\left(\zeta,\tau\right)+\zeta\left(\tau\right)-\zeta\left(\rho_z(\tau)\right)$$

where ζ belongs to the space $C[0,1]$, $\rho_z(\tau)=U_z^{q,\alpha}\left(\left(t-\tau\right),\tau\right)+\tau=\dfrac{[z]}{[z+2]}\tau$.

One can easily realize that ${}^{*}U_z^{q,\alpha}\left((t-\tau),\tau\right)=0.$

If the method of Taylor is used, for $\eta\in C^2[0,1]$, we obtain

$$\eta\left(t\right)=\left(\int_\tau^t\left(t-r\right)\eta''\left(r\right)dr\right)+\eta'\left(\tau\right)\left(t-\tau\right)+\eta\left(\tau\right),\quad \eta\in C^2[0,1].$$

Applying star q-SD operators ${}^{*}U_z^{q,\alpha}$ to L.H.S and R.H.S of the last equation, we have

$${}^{*}U_z^{q,\alpha}\left(\eta,\tau\right)-\eta(\tau)={}^{*}U_z^{q,\alpha}\left(\left(t-\tau\right)\eta'\left(\tau\right),\tau\right)+{}^{*}U_z^{q,\alpha}\left(\int_\tau^t\left(t-r\right)\eta''\left(r\right)dr,\tau\right)$$

$$=\eta'\left(\tau\right){}^{*}U_z^{q,\alpha}\left(\left(t-\tau\right),\tau\right)+U_z^{q,\alpha}\left(\int_\tau^t\left(t-r\right)\eta''\left(r\right)dr,\tau\right)-\int_\tau^{\rho_z(\tau)}\left(\rho_z(\tau)-r\right)\eta''\left(r\right)dr$$

$$=U_z^{q,\alpha}\left(\int_\tau^t (t-r)\eta''(r)dr,\tau\right)-\int_\tau^{\rho_z(\tau)}(\rho_z(\tau)-r)\eta''(r)dr.$$

Besides, through the use of the properties of absolute value of an integral, we get

$$\left|\int_\tau^t (t-r)\eta''(r)dr\right|\le\int_\tau^t (t-r)|\eta''(r)|dr\le\|\eta''\|\int_\tau^t (t-r)dr\le\|\eta''\|(t-r)^2$$

and

$$\left|\int_\tau^{\rho_z(\tau)}(\rho_z(\tau)-r)\eta''(r)dr\right|\le\|\eta''\|(\rho_z(\tau)-\tau)^2=\|\eta''\|\left(U_z^{q,\alpha}(t-\tau,\tau)\right)^2$$

which implies

$$\left|{}^*U_z^{q,\alpha}(\eta,\tau)-\eta(\tau)\right|\le\left|U_z^{q,\alpha}\left(\int_\tau^t (t-r)\eta''(r)\,dr,\tau\right)\right|+\left|\int_\tau^{\rho_z(\tau)}(\rho_z(\tau)-r)\eta''(r)dr\right|$$
$$\le\|\eta''\|\left\{U_z^{q,\alpha}\left((t-\tau)^2,\tau\right)+\left(U_z^{q,\alpha}((t-\tau),\tau)\right)^2\right\}.$$

Thus we get

$$\left|{}^*U_z^{q,\alpha}(\eta,\tau)-\eta(\tau)\right|\le\|\eta''\|\delta_z(\tau). \tag{2}$$

We also have

$$\left|{}^*U_z^{q,\alpha}(\zeta,\tau)\right|\le\left|U_z^{q,\alpha}(\zeta,\tau)\right|+|\zeta(\tau)|+|\zeta(\rho_z(\tau))|\le U_z^{q,\alpha}\left(|\zeta|,\tau\right)+2\|\zeta\|\le 3\|\zeta\|.$$

Now, by using the uniform boundedness property of the star q-SD operators ${}^*U_z^{q,\alpha}$ and using (2), we can state the following inequality,

$$\left|U_z^{q,\alpha}(\zeta,\tau)-\zeta(\tau)\right|\le\left|{}^*U_z^{q,\alpha}((\zeta-\eta),\tau)\right|+\left|{}^*U_z^{q,\alpha}(\eta,\tau)-\eta(\tau)\right|$$
$$+\left|\zeta(\tau)-\eta(\tau)\right|+\left|\zeta(\rho_z(\tau))-\zeta(\tau)\right|$$

$$\le 4\|\zeta-\eta\|+\|\eta''\|\delta_z(\tau)+\omega(\zeta,\beta_z(\tau)).$$

At this step, if we take the infimum of the R.H.S for all $\eta\in C^2[0,1]$, we may write

$$\left|U_z^{q,\alpha}(\zeta,\tau)-\zeta(\tau)\right|\le 4K_2\left(\zeta;\delta_z(\tau)\right)+\omega\left(\zeta,\beta_z(\tau)\right),$$

which together with the inequality (1) completes the proof of the theorem. ■

Now, before the statement of the second theorem, let us remember that a function $\zeta\in C[0,1]$ is said to belong to the Lipschitz type space $Lip_D(\beta)$, $(D>0$ and $0<\beta\le 1)$, if it satisfies

$$\left|\zeta(t)-\zeta(\tau)\right| \le D\left|t-\tau\right|^{\beta}, \text{ for all } t,\tau \in [0,1]. \qquad (3)$$

Taking into account the above explanation, we obtain Theorem 2 below.

Theorem 2 Suppose that $0 \le \tau \le 1$. Then, for all $\zeta \in Lip_D(\beta)$, $z \in \mathrm{N}$ where N is the set of natural numbers, $\alpha \ge 0$ and $0 < q < 1$, we have

$$\left|U_z^{q,\alpha}(\zeta,\tau)-\zeta(\tau)\right| \le D\left\{\Phi_{\alpha,z}(\tau,q)\right\}^{\frac{\beta}{2}},$$

where

$$\Phi_{\alpha,z}(\tau,q) = \left(\alpha + \frac{\alpha+6}{[z+2]}\right)\left(\frac{(1-\tau)\tau}{\alpha+1} + \frac{1}{[z+2]}\right)$$

and D is a constant which depends on β and ζ.

Proof. Think of a function ζ such that $\zeta \in Lip_D(\beta)$, $(0<\beta \le 1)$. By (3), we can write

$$\begin{aligned}\left|U_z^{q,\alpha}(\zeta,\tau)-\zeta(\tau)\right| &\le U_z^{q,\alpha}\left(\left|\zeta(t)-\zeta(\tau)\right|,\tau\right)\\ &\le DU_z^{q,\alpha}\left(\left|t-\tau\right|^{\beta},\tau\right).\end{aligned}$$

Now, let us consider the inequality of Hölder and set $p_1 = 2\beta^{-1}$ and $p_2 = 2(2-\beta)^{-1}$ in this inequality. After applying this to the last step, we will get

$$\begin{aligned}\left|U_z^{q,\alpha}(\zeta,\tau)-\zeta(\tau)\right| &\le D\left\{\left[U_z^{q,\alpha}\left(\left|t-\tau\right|^{\beta p_1},\tau\right)\right]^{\frac{1}{p_1}}\left[U_z^{q,\alpha}\left(1^{p_2},\tau\right)\right]^{\frac{1}{p_2}}\right\}\\ &= D\left\{\left[U_z^{q,\alpha}\left(\left|t-\tau\right|^{2},\tau\right)\right]^{\frac{\beta}{2}}\right\}.\end{aligned}$$

In the last step, through the use of the estimation for $U_z^{q,\alpha}((t-\tau)^2,\tau)$ which is given in Lemma 1, we continue as follows

$$\begin{aligned}\left|U_z^{q,\alpha}(\zeta,\tau)-\zeta(\tau)\right| &\le D\left\{\left[\left(\alpha+\frac{\alpha+6}{[z+2]}\right)\left(\frac{(\tau-\tau^2)}{\alpha+1}+\frac{1}{[z+2]}\right)\right]^{\frac{\beta}{2}}\right\}\\ &= D\left\{\left(\Phi_{\alpha,z}(\tau,q)\right)^{\frac{\beta}{2}}\right\}\end{aligned}$$

so that the proof of Theorem 2 is finished. ■

4. Conclusion

Our goal in this article was to establish a new kind of *q*-Stancu-Durrmeyer operators (*q*-SD operators) which are completely different than the previously defined ones. By using the concept of the quantum analysis (in other words *q*-calculus), we derived a new kind of *q*-SD operators, we gave formulae for the 1st and the 2nd order moments, $U_z^{q,\alpha}(t^j,\tau)$ for $j=0,1,2$ and we found an estimation for the second order central moment, $U_z^{q,\alpha}((t-\tau)^2,\tau)$. Moreover, by using the 2nd modulus of continuity $\omega_2(\zeta,\delta)$, we examined approximation properties of the *q*-SD operators in the local sense and by using β order Lipschitz kind maximal function, we established a direct estimation for the *q*-SD operators again in the local sense.

Statement of Conflict of Interest

The author has declared no conflict of interest.

Author's Contributions

The contribution of the author is 100%.

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