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Surface applications of t-bezier curves

T-bezier eğrilerin yüzey uygulamaları

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Surface Applications of T-Bezier Curves

Highlights

- *In this article, rotational surfaces are created by selecting certain control points and using trigonometric Bezier basis functions to create the tube surface.*
- *Shape parameters play an important role in the surfaces formed by trigonometric Bezier basis functions with two shape parameters.*
- *Depending on the shape parameters, the geometrical structures of the surfaces also change, where Gaussian and mean curvature have an important place in the characterization analysis.*

Graphical Abstract

In this article, the variation of the rotational surfaces created with the selected shape parameters is seen in following figure.

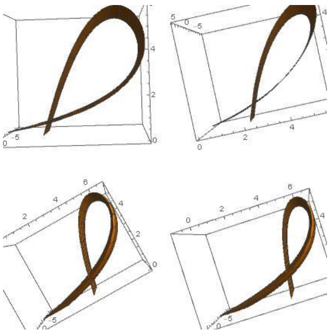

Figure. The changes of the rotational surfaces in different perspective with respect to same parameters

Aim

The aim of this study is to create a surface with the help of curves obtained as a result of basis functions and to provide engineers with new ideas in terms of design.

Design & Methodology

By using generalized trigonometric basis functions T-Bezier curves are created and new surfaces are designed by using T-Bezier curves.

Originality

Creating T-Bezier surfaces by rotating T-Bezier curve with two shape parameters around z-axis and characterizing the new surface by Gaussian and mean curvature are the paper's key originality.

Findings

Control points are selected to create the tube surface, T-Bezier curve is created with the selected control points, rotational surfaces are obtained with the help of shape parameters, and minimality and flatness of the obtained rotational surfaces are examined.

Conclusion

We examined the effects of shape parameters on surfaces independent of radius and height. In resulting surface design, unlike the Bezier curves, the curves in the trigonometric structure vary according to the value of the shape parameters which plays a major role in computer-based designs to be made in industry and engineering fields.

Declaration of Ethical Standards

The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

Surface Applications of T-Bezier Curves

Araştırma Makalesi/Research Article

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ABSTRACT

In controlling the shapes in the surface design, we create the rotational surfaces using T-Bezier curves with two shape parameters. Here, shape parameters play an important role in shape design. Finally, we give the characterizations of the mean and Gaussian curvatures for these rotational surfaces according to the shape parameters.

Keywords: T-Bezier basis functions, T-Bezier curves, rotational surfaces, shape parameter.

T-Bezier Eğrilerin Yüzey Uygulamaları

ÖZ

Yüzey tasarımında şekilleri kontrol etmede iki şekil parametresiyle T-Bezier eğrileri kullanılarak dönel yüzeyleri oluşturuyoruz. Burada şekil parametreleri şekil tasarımında önemli bir rol oynamaktadır. Son olarak, bu dönel yüzeyleri için ortalama ve Gauss eğriliklerinin şekil parametrelerine göre karakterizasyonlarını veriyoruz.

Anahtar Kelimeler: t-bezier baz fonksiyonları, t-bezier eğrileri, dönel yüzeyler, şekil parametresi

1. INTRODUCTION

Curve and surface geometry is one of the important fields in mathematics, although it has an important structure, especially in the field of industrial and engineering. Studies on curve and surface design have been done frequently in recent years. Computer-based mathematical modeling has become popular in recent years, and some of the pioneers of related studies by using curves as a major factor are Farin, Hoschek and Saxena (see [4],[5],[9]).

There are lots of curves used in surface design and the Bezier curve is one of the most important of these curves which is named by French engineer Pierre Bézier (1910–1999), who used the Bezier curve in the 1960s for designing curves for bodywork of Renault cars. De Casteljau's algorithm is a recursive method and has an important role to evaluate polynomials in Bernstein form or (for application and analysis about this curve, (see [2],[3], [4],[5],[8],[9]). Bezier curves are studied in different areas. As an example, Bezier curves were used with minimal jerk energy (see [15]) and also studied for different spaces (see [14]). Furthermore, Many studies have been done on Bézier curves in different spaces, characterization and investigations have been made on different surfaces (see [19],[20],[23-31]).

In addition, useful studies have been made in industrial field and examples have been made on the basis of curves (see [22]).

Surface geometry is an application of curves for engineering and industrial designs such as, rotational surfaces (see [1],[2]), tensor product surfaces (see [16]), rational Bezier surfaces, B-spline surfaces (see [11]), NURBS surfaces and etc (see [17],[18]). These surfaces have been studied by geometers and engineers generally in Euclidean space, Minkowski space (see [21]), Galilean space, pseudo-Galilean space and etc (see [1-8],[10-12]).

For example, Xi-An Hana, YiChen Maa, XiLi Huangc

have used T-basis functions with two shape parameters in their paper which is the specific type of generalized trigonometric Bezier (i.e. GT-Bezier) curve. So, we have been inspired by this work then we have made applications in cubic and higher order trigonometric Bezier (i.e. T-Bezier) curves and design surfaces by using rotational surfaces.

This study is divided into three steps:

1. By using generalized trigonometric basis functions creating T-Bezier curves,
2. Designing new surfaces by using T-Bezier curves,
3. Constructing new object models by using rotational matrix for engineering and industry.

As a result, the aim of this study is to create a surface with the help of curves obtained as a result of basis functions and to provide engineers with new ideas in terms of design.

2. PRELIMINARIES

2.1 Gaussian and Mean Curvature

We give the definition of Guassian and mean curvature which plays an important role of surface analysis.

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Definition 1: Let κ_1 and κ_2 be the principal curvatures of a surface patch $R(u,v)$. Then the Gaussian curvature of R is $K=\kappa_1\kappa_2$, and its mean curvature is $H=\frac{1}{2}(\kappa_1+\kappa_2)$. (see [13])

To compute K and H, we use the first and second fundamental forms of the surface:

$Edu^2+2Fdudv+Gdv^2$ and $Ldu^2+2Mdudv+Ndv^2$ with the matrix form

$$\mathcal{F}_1=\begin{pmatrix}E & F\\ F & G\end{pmatrix},\ \mathcal{F}_2=\begin{pmatrix}L & M\\ M & N\end{pmatrix}$$

The principal curvatures are the eigenvalues of $\mathcal{F}_1^{-1}\mathcal{F}_2$. Hence the determinant of this matrix is the product of κ_1 and κ_2, i.e. the Gaussian curvature K. So

$$K=\det(\mathcal{F}_1^{-1}\mathcal{F}_2)=\det(F_1)^{-1}\det(F_2)=\frac{LN-M^2}{EG-F^2}.$$

The trace of the matrix is the sum of its eigenvalues, thus, twice the mean curvature H. After some calculation, we obtain

$$H=\frac{1}{2}trace(\mathcal{F}_1^{-1}\mathcal{F}_2)=\frac{1}{2}\frac{LG-2MF+NE}{EG-F^2}.$$

2.2 Generalized trigonometric basis function

Now, we give the definition of generalized trigonometric basis functions by using recursive relation in this subsection.

Definition 2: For $-1\le\alpha,\beta\le 1$ and $0\le u\le 1$ the functions

$$\begin{aligned} B_{0,2}(u) &= (1-\sin(\tfrac{\pi}{2}u))(1-\alpha\sin(\tfrac{\pi}{2}u))\\ B_{1,2}(u) &= 1-B_{0,2}(u)-B_{2,2}(u) \qquad (1)\\ B_{2,2}(u) &= (1-\cos(\tfrac{\pi}{2}u))(1-\beta\cos(\tfrac{\pi}{2}u)) \end{aligned}$$

are known as 2nd-degree trigonometric basis functions. For general, $B_{i,m}(u)\ (i=0,1,\dots,m)$ described recursively for any integer $m\ge 3$ as

$$B_{i,m}(u)=(1-\sin(\tfrac{\pi}{2}u))B_{i,m-1}(u)+\sin(\tfrac{\pi}{2}u)B_{i-1,\ m-1}(u) \qquad (2)$$

m-th order of generalized trigonometric basis function. When $i=-1$ or $i>m$, $B_{i,m}(u)=0$. (see [3])

2.3 Properties of GT-basis functions

We give the properties of generalized trigonometric basis function.

The GT-basis functions have properties as follows:

1. Partition of unity: for $0\le u\le 1$, the sum of the basis functions is equal to 1, namely

$\sum_{i=1}^{m}B_{i,m}(u)=1.$

2. Nonnegativity: for $\alpha,\beta\in[-1,1]$,

$B_{i,\ m}(u)\ge 0\ \ (i=0,1,\dots,m)$

3. Terminal property:

$\forall i=0,1,2,3,\dots,m\ \ (m\ge 2),$

$B_{0,m}(0)=1,$

$B_{i,m}(0)=0\ \ (i=1,2,\dots,m),$

$B_{i,m}(1)=0\ \ (i=0,1,\dots,m-1),$ and

$B_{m,m}(1)=1.$

4. The first and second derivative of the basis functions with respect to u, for $u=0$ and $u=1$ the following equations are obtained as

$$\begin{aligned} B'_{0,m}(0)&=-\tfrac{\pi}{2}(m_1+\alpha)\\ B'_{1,m}(0)&=\tfrac{\pi}{2}(m_1+\alpha)\\ B'_{m-1,m}(1)&=-\tfrac{\pi}{2}(1+\beta)\\ B'_{m,m}(1)&=\tfrac{\pi}{2}(1+\beta)\\ B''_{0,m}(0)&=\tfrac{\pi^2}{4}(m_1m_2+2(m_2+1)\alpha)\\ B''_{1,m}(0)&=-\tfrac{\pi^2}{4}(2m_1m_2+2(m_2+1)\alpha\\ &\quad+(1-\beta))\\ B''_{2,m}(0)&=\tfrac{\pi^2}{4}\big(m_1m_2+2m_2\alpha+(1-\beta)\big)\\ B''_{m-2,m}(1)&=-\tfrac{\pi^2}{4}(\alpha-1)\\ B''_{m-1,m}(1)&=\tfrac{\pi^2}{4}\big(m_2-2\beta+(\alpha-1)\big)\\ B''_{m,m}(1)&=-\tfrac{\pi^2}{4}(m_2-2\beta) \end{aligned}$$

where $m_1=m-1,\ m_2=m-2.$

Now, using GT-basis functions, we can define cubic trigonometric basis functions for $m=3$ as follows:

For $m=3$, we have

$$\begin{aligned} B_{0,3}(u)&=(1-\sin\left(\tfrac{\pi}{2}u\right))B_{0,2}(u),\\ B_{1,3}(u)&=sin\left(\tfrac{\pi}{2}u\right)B_{0,2}(u)+(1-\\ &\quad sin\left(\tfrac{\pi}{2}u\right))B_{1,2}(u),\\ B_{2,3}(u)&=(1-sin\left(\tfrac{\pi}{2}u\right))B_{2,2}(u)+\\ &\quad\sin(\tfrac{\pi}{2}u)B_{1,2}(u),\\ B_{3,3}(u)&=\sin(\tfrac{\pi}{2}u)B_{2,2}(u). \qquad (3) \end{aligned}$$

In Figure 1, the graphs of 3rd-degree trigonometric functions for $\alpha,\beta=-1$ (blue dotted), $\alpha,\beta=-0.5$ (blue) and $\alpha,\beta=0.5$ (red dotted) are given.

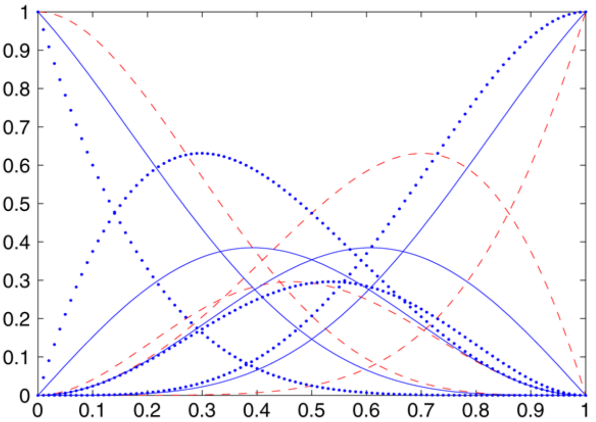


Figure 1. T-cubic basis functions

2.4 T-Bezier curves

Firstly, we give definition of T-Bezier curves. Afterwards give the properties of 3-rd degree T-Bezier curves.

Definition 3. For any given control points $P_i \in \mathbb{R}^2$ or $\mathbb{R}^3$ $(i = 0,1,\dots,m)$, the generalized T-Bezier curve can be obtained as

$r(t) = \sum_{i=0}^{m} P_i B_{i,m}(u), \quad 0 \le u \le 1$

where $B_{i,m}(u)$ are GT-basis functions. Using trigonometric cubic basis functions, trigonometric cubic (i.e. T-Cubic) Bezier curve is as follows:

$r(u) = P_0 B_{0,3}(u) + P_1 B_{1,3}(u) + P_2 B_{2,3}(u) + P_3 B_{3,3}(u)$

$r(u) = P_0(1-\sin(\frac{\pi}{2}u))^2\,(1-\alpha\sin\left(\frac{\pi}{2}u\right) +$
$P_1\ \sin(\frac{\pi}{2}\,u)(1-\sin\,(\frac{\pi}{2}u))(2+\alpha-\alpha\sin\,\left(\frac{\pi}{2}\,u\right)) +$
$P_2\cos(\frac{\pi}{2}\,u)(1\,-\,\cos\left(\frac{\pi}{2}u\right))(2+\,\beta-\beta\cos(\frac{\pi}{2}u)) +$
$P_3\left(1\,-\,\cos\left(\frac{\pi}{2}\,u\right)\right)^2(1-\,\beta\,\,cos\left(\frac{\pi}{2}u\right)).$
(4)

Properties of cubic T-Bezier curves are given as follows:

a) Terminal properties:
$$r\,(0) \,=\, P_0, r\,(1) \,=\, P_3,$$
$$r'\,(0) = \frac{\pi}{2}\,(2+\alpha)\,(P_1\,-\,P_0),$$
$$r'\,(1) = \frac{\pi}{2}\,(2+\beta)(P_3\,-\,P_2)$$

b) Convex hull property: The bezier curve created with the selected control points is located in the convex region and does not go out of the region in any way..

c) Geometric invariance: The shape of the trigonometric curve is independent of the chosen coordinates, there is no change in its shape when rotation and translation are applied.

Definition 4. For $u \in (0,1)$,

$r(u) = \sum_{i=0}^{3} P_i w_i(u) + \alpha\, sin(\frac{\pi}{2}u) +$
$(1-sin(\frac{\pi}{2}u))^2\,(P_1 - P_0) +$
$\beta\, cos\left(\frac{\pi}{2}u\right)\left(1-cos\left(\frac{\pi}{2}u\right)\right)^2(P_3 - P_2)$

where

$w_0(u) = (1-sin(\frac{\pi}{2}u))^2,.$
$w_1(u) = 2\sin(\frac{\pi}{2}u)(1-sin(\frac{\pi}{2}u)),$
$w_2\,(u) = 2\,\cos(\frac{\pi}{2}\,u)(1-\cos\left(\frac{\pi}{2}\,u\right),$
$w_3(u)\,=\,(1-\cos\,(\frac{\pi}{2}\,u))^{\,2}.$

As can be seen, the shape of the trigonometric Bezier curve depends on the shape parameters α, β on the control edges $(P_1 - P_0)$ and $(P_3 - P_2)$ respectively. In addition, the behavior of the curves are given as follows:

i) when the value of α increases, the curve gets closer to the corner control point $P_1 - P_0$,

ii) when the value of α decreases, the curve moves further away from the corner control point $P_1 - P_0$,

iii) when the value of β increases, the curve gets closer to the corner control point $P_3 - P_2$,

iv) when the value of β decreases, the curve moves further away from the corner control point $P_3 - P_2$.

3. ROTATIONAL SURFACES GENERATED BY T-BEZIER CURVES

In (see [2]), rotational surfaces of cubic Bezier curves were created using the tube surface generated with the help of selected control points. Now, we will construct this surface for cubic T-Bezier curves.

Representation of a tube surface for bezier curve is given in (see [2]).

The first thing we need to do is to determine a center point on the base of the tube to be built. So the center point is $(x_1, y_1, z_1) = (0,0,0)$.

Given a tube of radius r_1 and r_2, where $r_1, r_2 \in [a,b]$, h is tube's height in the interval $[c,d]$. In the previous study (see [2]) the shape of geometric design changed by the values of tube's radius and tube's height. Now, shape parameters α, β aim to differences in the shape components of geometric design.

$P_0 = (x_1 + r_1\cos v\ ,\ y_1\ +\ r_1\ \sin v\ ,\ z_1),$
$P_1\ =\ (\ x_1,\ 0,\ z_1),$
$P_2\ =\ (x_2,\ 0,\ z_2),$
$P_3\ = (x_1\ +\ r_2\ \cos\ v\,,\ y_1\ +\ r_2\ \sin\ v\,,\ z_1)$

are control points of cubic Bezier curve. Further by using control points,

$r(u) = P_0 B_{0,3}(u) + P_1 B_{1,3}(u) + P_2 B_{2,3}(u) + P_3 B_{3,3}(u),$

$r(u) = \Big(r_1 B_{0,3}(u) + x_1 B_{1,3}(u) + x_2 B_{2,3}(u) + r_2 B_{3,3}(u), 0, z_1 B_{1,3}(u) + z_2 B_{2,3}(u) + h B_{3,3}(u)\Big),$
$0 \le u \le 1$ (5)

With the trigonometric cubic basis functions, from equation (5) we have $r(u)$ as trigonometric curve as follows:

$$r(u) = (\zeta_1(u), 0, \zeta_2(u))$$

where

$\zeta_1(u) = r_1(1-sin(\frac{\pi}{2}u))^2\,(1-\alpha\sin(\frac{\pi}{2}u)) +$
$x_1\sin\left(\frac{\pi}{2}\,u\right)\,(1-\sin(\frac{\pi}{2}\,u))\,(2+\alpha-\alpha\sin\,(\frac{\pi}{2}\,u)) +$
$x_2\cos\,\left(\frac{\pi}{2}\,u\right)(1-\cos(\frac{\pi}{2}\,u))\,(2+\beta-\beta\cos(\frac{\pi}{2}\,u)) +$
$r_2\left(1-cos\left(\frac{\pi}{2}u\right)\right)^2\left(1-\beta\, cos\left(\frac{\pi}{2}u\right)\right)$

and $\zeta_2(u)$ is

$z_1\, sin(\frac{\pi}{2}u)(1 - sin(\frac{\pi}{2}u))(2 + \alpha - \alpha\, sin(\frac{\pi}{2}u)) +$
$z_2 \cos(\frac{\pi}{2}u)(1 - \cos(\frac{\pi}{2}u))(2 + \beta - \beta \cos(\frac{\pi}{2}u)) +$
$h\left(1 - cos\left(\frac{\pi}{2}u\right)\right)^2 \left(1 - \beta\, cos\left(\frac{\pi}{2}u\right)\right).$

In Figure 2, for $r_1 = \alpha = \beta = 1, r_2 = 2, x_1 = 10, x_2 = 5, z_1 = 25, z_2 = 10$, $h = 10$, the trigonometric bezier curve is given.

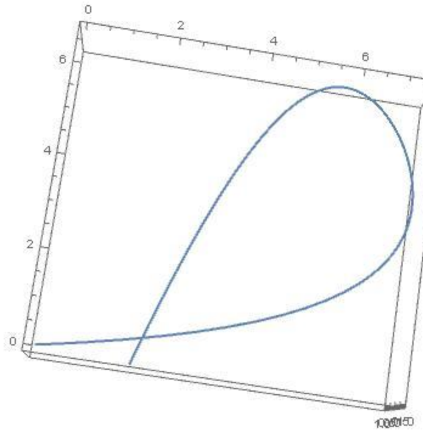


Figure 2. T-Bezier curve

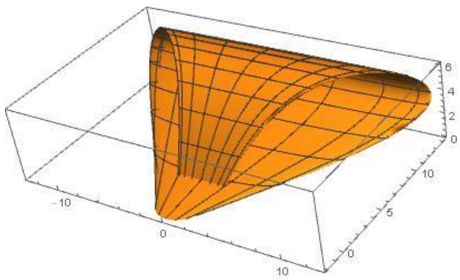
3.1 Characterization of T-Cubic bezier rotational surfaces according to the values of shape parameters

This is an important subsection which includes rotation of cubic T-Bezier curve around z-axis, we can draw surfaces by using different shape parameters.

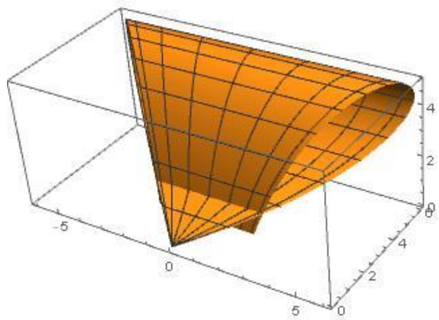
By rotating the T-Bezier curve (5) around z-axis, we get $R(u, v)$ that is defined as rotational surface

$$\begin{aligned}
R\,(u, v) = (&[r_1(1 - sin(\tfrac{\pi}{2}u))^2\,(1 - \alpha\, sin(\tfrac{\pi}{2}u)) + \\
&x_1\, sin\left(\tfrac{\pi}{2}\, u\right)(1 - sin(\tfrac{\pi}{2}\, u))\,(2 + \alpha - \alpha\, sin(\tfrac{\pi}{2}\, u)) + \\
&x_2\, cos\left(\tfrac{\pi}{2}\, u\right)(1 - cos(\tfrac{\pi}{2}\, u))\,(2 + \beta - \beta\, cos(\tfrac{\pi}{2}\, u)) + \\
&r_2\left(1 - cos\left(\tfrac{\pi}{2}\, u\right)\right)^2\left(1 - \beta\, cos\left(\tfrac{\pi}{2}\, u\right)\right)]\cos\, v, \\
&[r_1(1 - sin(\tfrac{\pi}{2}u))^2\,(1 - \alpha\, sin(\tfrac{\pi}{2}u)) + \\
&x_1\, sin(\tfrac{\pi}{2}u)(1 - sin(\tfrac{\pi}{2}u))(2 + \alpha - \alpha\, sin(\tfrac{\pi}{2}u)) + \\
&x_2\, cos(\tfrac{\pi}{2}u)(1 - cos(\tfrac{\pi}{2}u))(2 + \beta - \beta\, cos(\tfrac{\pi}{2}u)) + \\
&r_2\left(1 - cos\left(\tfrac{\pi}{2}\, u\right)\right)^2\left(1 - \beta\, cos\left(\tfrac{\pi}{2}\, u\right)\right)]\sin\, v, \\
&z_1\, sin\left(\tfrac{\pi}{2}\, u\right)(1 - sin(\tfrac{\pi}{2}\, u))\,(2 + \alpha - \alpha\, sin(\tfrac{\pi}{2}\, u)) + \\
&z_2 \cos\left(\tfrac{\pi}{2}\, u\right)(1 - \cos(\tfrac{\pi}{2}\, u))\,(2 + \beta - \beta \cos(\tfrac{\pi}{2}\, u)) + \\
&h\left(1 - cos\left(\tfrac{\pi}{2}u\right)\right)^2\left(1 - \beta\, cos\left(\tfrac{\pi}{2}u\right)\right)).
\end{aligned} \tag{6}$$

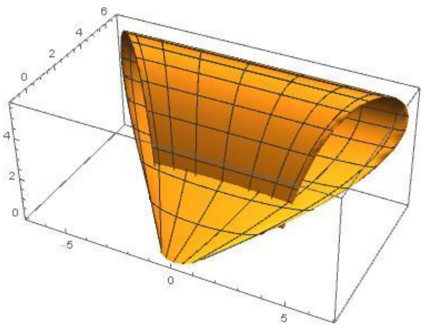
In Figure 3, we can change the shape parameters α and β from 1 to -1. When shape parameters $\alpha = \beta$ and decreases, the shape getting closer to the bottom of the surface. For $r_1 = 2, r_2 = 5, x_1 = 20, x_2 = 2, z_1 = 10, z_2 = 2$ and $h = 0.1$:



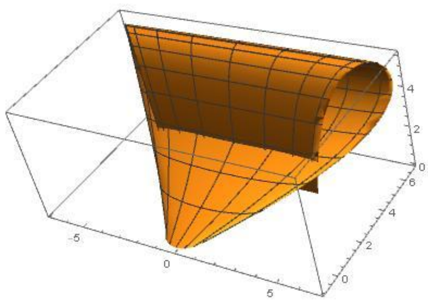
(A) $\alpha = \beta = 1$



(B) $\alpha = \beta = 0.5$



(C) $\alpha = \beta = -0.5$



(D) $\alpha = \beta = -1$

Figure 3. Variation of the T-Bezier surface with respect to same parameters

In Figure 4, the shape parameters of T-Bezier surfaces are $\alpha=\beta=1$ (top left), $\alpha=\beta=0.5$ (top right), $\alpha=\beta=-0.5$ (bottom left), $\alpha=\beta=-1$ (bottom right), respectively.

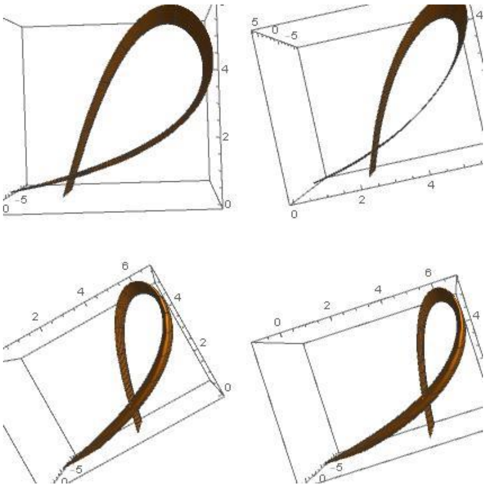
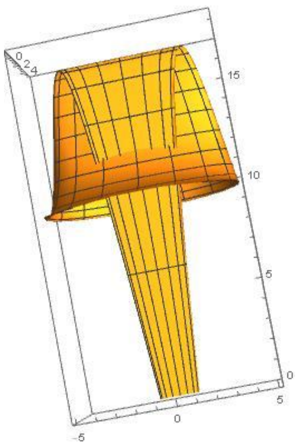
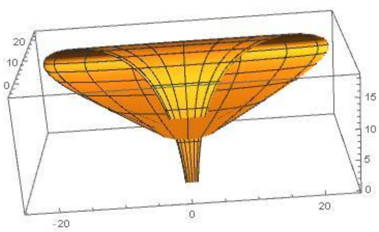


Figure 4. The changes of the rotational surfaces in different perspective with respect to same parameters

In Figure 5, we take distinct shape parameters ($\alpha\neq\beta$) for the same $r_1, r_2, x_1, x_2, z_1, z_2, h$ values.



(A) $\alpha=-0.5,\ \beta=-1$



(B) $\alpha=0.5,\ \beta=1$

Figure 5. Variation of the T-Bezier surface with respect to distinct parameters

In Figure 5, when shape parameters $\alpha\neq\beta$ and decreases, the shape getting closer to the bottom of the surface.

Differentiating $R(u,v)$ surface according to u and v, we can obtain the following equations

$$R_u(u,v)=\big(\left[r_1B'_{0,3}(u)+x_1B'_{1,3}(u)+x_2B'_{2,3}(u)+r_2B'_{3,3}(u)\right]\cos v,$$
$$\left[r_1B'_{0,3}(u)+x_1B'_{1,3}(u)+x_2B'_{2,3}(u)+r_2B'_{3,3}(u)\right]\sin v,$$
$$\left[z_1B'_{1,3}(u)+z_2B'_{2,3}(u)+hB'_{3,3}(u)\right]\big)$$

$$R_v(u,v)=(-\sigma\sin v,\sigma\cos v,0)$$

where $\sigma=r_1B_{0,3}(u)+x_1B_{1,3}(u)+x_2B_{2,3}(u)+r_2B_{3,3}(u),\ 0\le u\le 1.$

To compute gaussian and mean curvature, we need to use the first and second fundamental forms of the $R(u,v)$ surface. Firstly, the coefficients of first fundamental form of $R(u,v)$ surface are calculated as

$$E=\gamma^2+\delta^2,\ \ F=0,\ \ G=\sigma^2, \tag{7}$$

where

$\gamma=r_1B'_{0,3}(u)+x_1B'_{1,3}(u)+x_2B'_{2,3}(u)+r_2B'_{3,3}(u),$

$\delta=z_1B'_{1,3}(u)+z_2B'_{2,3}(u)+hB'_{3,3}(u)$ with

$$B'_{0,3}(u)=\frac{-\pi\cos(\frac{\pi}{2}u)(\sin(\frac{\pi}{2}u)-1)(3\alpha\sin(\frac{\pi}{2}u)-\alpha-2)}{2},$$

$$B'_{1,3}(u)=\frac{\pi\cos(\frac{\pi}{2}u)(3\alpha\sin^2(\frac{\pi}{2}u)+(-4\alpha-4)\sin(\frac{\pi}{2}u)+\alpha+2)}{2},$$

$$B'_{2,3}(u)=\frac{-\pi(3\beta\cos^2(\frac{\pi}{2}u)+(-4\beta-4)\cos(\frac{\pi}{2}u)+\beta+2).\sin(\frac{\pi}{2}u)}{2},$$

$$B'_{3,3}(u)=\frac{\pi(\cos(\frac{\pi}{2}u)-1)(3\beta\cos(\frac{\pi}{2}u)-\beta-2)\sin(\frac{\pi}{2}u)}{2}.$$

In addition, the unit normal of $R(u,v)$ can be obtained as

$$N(u,v)=\frac{1}{\sqrt{\delta^2+\gamma^2}}(-\delta\cos v,-\delta\sin v,\gamma).$$

The second derivatives of $R(u,v)$ are obtained as

$$R_{uu}(u,v)=\big(\left[r_1B''_{0,3}(u)+x_1B''_{1,3}(u)+x_2B''_{2,3}(u)+r_2B''_{3,3}(u)\right]\cos v,$$
$$\left[r_1B''_{0,3}(u)+x_1B''_{1,3}(u)+x_2B''_{2,3}(u)+r_2B''_{3,3}(u)\right]\sin v,$$
$$\left[z_1B''_{1,3}(u)+z_2B''_{2,3}(u)+hB''_{3,3}(u)\right]\big),$$

$$R_{uv}(u,v)=(-\gamma\sin v,\gamma\cos v,0),$$

$$R_{vv}(u,v)=(-\sigma\cos v,-\sigma\sin v,0).$$

Secondly, the coefficients of second fundamental form of $R(u,v)$ surface are calculated as

$$L=\frac{-\phi\delta+\psi\gamma}{\sqrt{\delta^2+\gamma^2}},\quad M=0,\quad N=\frac{\sigma\delta}{\sqrt{\delta^2+\gamma^2}}, \tag{8}$$

respectively.

Here,

$\phi = r_1 B''_{0,3}(u) + x_1 B''_{1,3}(u) + x_2 B''_{2,3}(u) + r_2 B''_{3,3}(u)$ and $\psi = z_1 B''_{1,3}(u) + z_2 B''_{2,3}(u) + h B''_{3,3}(u)$ with

$$B''_{0,3}(u) = \frac{\pi^2}{4}\left[3\alpha sin^3\left(\frac{\pi}{2}u\right) + (-4\alpha - 2)sin^2\left(\frac{\pi}{2}u\right) + \left(-6\alpha cos^2\left(\frac{\pi}{2}u\right) + \alpha + 2\right)\sin\left(\frac{\pi}{2}u\right) + (4\alpha + 2)cos^2\left(\frac{\pi}{2}u\right)\right],$$

$$B''_{1,3}(u) = \frac{-\pi^2}{4}\left[3\alpha sin^3\left(\frac{\pi}{2}u\right) + (-4\alpha - 4)sin^2\left(\frac{\pi}{2}u\right) + \left(-6\alpha cos^2\left(\frac{\pi}{2}u\right) + \alpha + 2\right)sin\left(\frac{\pi}{2}u\right) + (4\alpha + 4)cos^2\left(\frac{\pi}{2}u\right)\right],$$

$$B''_{2,3}(u) = \frac{\pi^2}{4}\left[\left(6\beta\cos\left(\frac{\pi}{2}u\right) - 4\beta - 4\right)sin^2\left(\frac{\pi}{2}u\right) - 3\beta cos^3\left(\frac{\pi}{2}u\right) + (4\beta + 4)cos^2\left(\frac{\pi}{2}u\right) + (-\beta - 2)\cos\left(\frac{\pi}{2}u\right)\right],$$

$$B''_{3,3}(u) = \frac{-\pi^2}{4}\left[\left(6\beta\cos\left(\frac{\pi}{2}u\right) - 4\beta - 2\right)sin^2\left(\frac{\pi}{2}u\right) - 3\beta cos^3\left(\frac{\pi}{2}u\right) + (4\beta + 2)cos^2\left(\frac{\pi}{2}u\right) + (-\beta - 2)\cos\left(\frac{\pi}{2}u\right)\right].$$

Hence, based on our results, we write the following theorem.

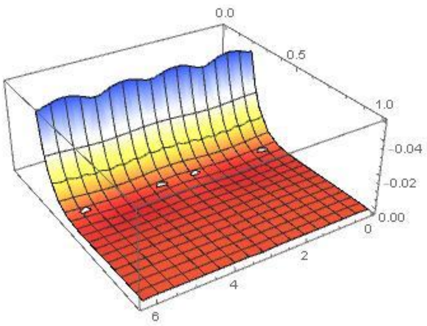
Theorem 1. The mean curvature and Gaussian curvature of $R(u,v)$ surface which is a surface formed by the rotation of the curve about the z-axis are

$$H = \frac{\sigma(\psi\gamma - \phi\delta) + \delta(\gamma^2 + \delta^2)}{2\sigma\sqrt{(\gamma^2 + \delta^2)^3}} \text{ and}$$

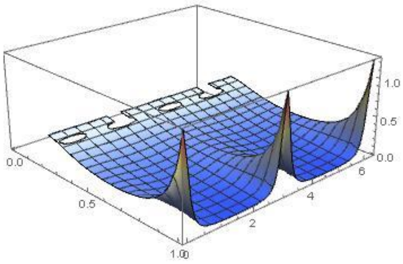
$$K = \frac{\delta(\psi\gamma - \phi\delta)}{\sigma(\delta^2 + \gamma^2)^2}, \qquad (9)$$

respectively.

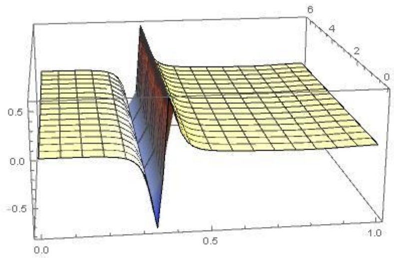
In Figure 6 A,B,C and D show the graphics of $R(u,v)$ surfaces' Gaussian and mean curvature functions, and also variation of shape parameters to curvatures for given control points $r_1 = 1$, $r_2 = 2$, $x_1 = 5$, $x_2 = 10$, $z_1 = 25$, $z_2 = 10$ and $h = 10$.



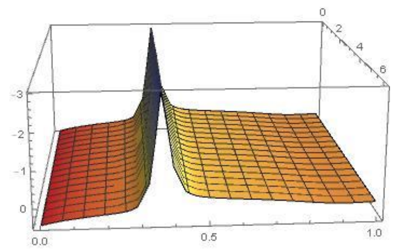
(A) Gaussian curvature for $\alpha, \beta < 0$



(B) Mean curvature for $\alpha, \beta < 0$



(C) Gaussian curvature for $0 < \alpha, \beta \leq 1$



(D) Mean curvature for $0 < \alpha, \beta \leq 1$

Figure 6. Gaussian and mean curvature functions' graphics of T-Bezier surfaces

Corollary 1. $R(u,v)$ is the surface formed by the rotation of the T-bezier curve around the z- axis, the following results are obtained regarding the curvatures according to the selection of the shape parameter and the compatibility of the control points.

i. If P_1 is on the x-axis and $0 < \alpha, \beta \leq 1$, then the mean curvature of $R(u,v)$ disappears at the initial point of $r(u)$.

ii. If P_1 is on the origin and $-1 \leq \alpha, \beta < 0$, then the mean curvature of $R(u,v)$ can not disappear at the initial point of $r(u)$.

iii. If P_2 and P_3 are on the x-axis and $0 < \alpha, \beta \leq 1$, then the Gaussian curvature of $R(u,v)$ disappears at the initial point of $r(u)$.

iv. If P_2 and P_3 are on the origin and $-1 \leq \alpha, \beta < 0$, then the Gaussian curvature of $R(u,v)$ can not disappear at the initial point of $r(u)$.

Theorem 2. Let $R(u,v)$ is the surface formed by the rotation of the T-bezier curve around the z- axis. For $f_1(u)=\sin\left(\frac{\pi}{2}u\right), f_2(u)=\cos\left(\frac{\pi}{2}u\right), 0\le u\le 1,$ the surface is flat if and only if

$f_1(u)=\frac{2\alpha+2\pm\sqrt{\alpha^2+2\alpha+4}}{3\alpha}, f_2(u)=\frac{2\beta+2\pm\sqrt{\beta^2+2\beta+4}}{3\beta}$ or $f_2(u)=\frac{\beta+2}{3\beta}$ where α,β shape parameters of cubic T-basis functions.

Proof. (⇒)Since $K=\frac{\delta(\psi\gamma-\phi\delta)}{\sigma(\delta^2+\gamma^2)^2}$, then it is sufficient that $\delta=0$ where

$\delta=z_1B'_{1,3}(u)+z_2B'_{2,3}(u)+hB'_{3,3}(u)$.

So,

$$3\alpha sin^2\left(\frac{\pi}{2}u\right)+(-4\alpha-4)\sin\left(\frac{\pi}{2}u\right)+\alpha+2=0, \quad (10)$$

$$3\beta cos^2\left(\frac{\pi}{2}u\right)+(-4\beta-4)\cos\left(\frac{\pi}{2}u\right)+\beta+2=0, \quad (11)$$

and

$$3\beta\cos\left(\frac{\pi}{2}u\right)-\beta-2=0. \quad (12)$$

Solving the equations (10),(11) and (12), we have $f_1(u)=\frac{2\alpha+2\pm\sqrt{\alpha^2+2\alpha+4}}{3\alpha}, f_2(u)=\frac{2\beta+2\pm\sqrt{\beta^2+2\beta+4}}{3\beta}$ and $f_2(u)=\frac{\beta+2}{3\beta}$ respectively.

(⇐) for $0\le u\le 1$, we have the following equation

$$\delta=z_1\frac{\pi\cos(\frac{\pi}{2}u)(3\alpha f_1^2(u)+(-4\alpha-4)f_1(u)+\alpha+2)}{2}+z_2\frac{-\pi(3\beta f_2^2(u)+(-4\beta-4)f_2(u)+\beta+2).f_1(u)}{2}+h\frac{\pi(f_2(u)-1)(3\beta f_2(u)-\beta-2)f_1(u)}{2}.$$

Since $z_1\neq 0$, $z_2\neq 0$ and $h\neq 0$, then

$3\alpha f_1^2(u)+(-4\alpha-4)f_1(u)+\alpha+2=0,$

$3\beta f_2^2(u)+(-4\beta-4)f_2(u)+\beta+2=0,$

$3\beta f_2(u)-\beta-2=0$. For $f_1(u)=\frac{2\alpha+2\pm\sqrt{\alpha^2+2\alpha+4}}{3\alpha}, f_2(u)=\frac{2\beta+2\pm\sqrt{\beta^2+2\beta+4}}{3\beta}$ and $f_2(u)=\frac{\beta+2}{3\beta}$, δ is zero. So the numerator of K becomes zero. Since K is zero, then the surface is flat.

Theorem 3. Let $R(u,v)$ is the surface formed by the rotation of the T-bezier curve around the z- axis. Then, for $f_1(u)=\sin\left(\frac{\pi}{2}u\right), f_2(u)=\cos\left(\frac{\pi}{2}u\right), 0\le u\le 1,$ the surface is minimal if and only if $f_1(u)=\frac{\alpha+4}{3\alpha}$, $f_2(u)=\frac{\beta+4}{3\beta}$

where α,β shape parameters of cubic T- basis functions.

Proof. (⇒)Since $H=\frac{\sigma(\psi\gamma-\phi\delta)+\delta(\gamma^2+\delta^2)}{2\sigma\sqrt{(\gamma^2+\delta^2)^3}}$, then it is sufficient that $\delta=\gamma=0$ where

$\delta=z_1B'_{1,3}(u)+z_2B'_{2,3}(u)+hB'_{3,3}(u),$

$\gamma=r_1B'_{0,3}(u)+x_1B'_{1,3}(u)+x_2B'_{2,3}(u)+r_2B'_{3,3}(u).$

Since $\gamma=0$, we also need to solve

$$3\alpha\sin\left(\frac{\pi}{2}u\right)-\alpha-2=0 \quad (13)$$

Solving the equations (10-11) together we have

$f_1(u)=\frac{\alpha+4}{3\alpha}$,

and (12-13),

$f_2(u)=\frac{\beta+4}{3\beta}$

respectively.

(⇐) for $0\le u\le 1$, we have the following equations

$$\delta=z_1\frac{\pi\cos(\frac{\pi}{2}u)(3\alpha f_1^2(u)+(-4\alpha-4)f_1(u)+\alpha+2)}{2}+z_2\frac{-\pi(3\beta f_2^2(u)+(-4\beta-4)f_2(u)+\beta+2).f_1(u)}{2}+h\frac{\pi(f_2(u)-1)(3\beta f_2(u)-\beta-2)f_1(u)}{2}$$ and

$$\gamma=r_1\frac{-\pi f_2(u)(f_1(u)-1)(3\alpha f_1(u)-\alpha-2)}{2}+x_1\frac{\pi\cos(\frac{\pi}{2}u)(3\alpha f_1^2(u)+(-4\alpha-4)f_1(u)+\alpha+2)}{2}+x_2\frac{-\pi(3\beta f_2^2(u)+(-4\beta-4)f_2(u)+\beta+2).f_1(u)}{2}+r_2\frac{\pi(f_2(u)-1)(3\beta f_2(u)-\beta-2)f_1(u)}{2}.$$

Since $r_1\neq 0$, $r_2\neq 0$, $x_1\neq 0$, $x_2\neq 0$, $z_1\neq 0$, $z_2\neq 0$ and $h\neq 0$, then

$3\alpha f_1(u)-\alpha-2=0,$

$3\alpha f_1^2(u)+(-4\alpha-4)f_1(u)+\alpha+2=0,$

$3\beta f_2^2(u)+(-4\beta-4)f_2(u)+\beta+2=0,$

$3\beta f_2(u)-\beta-2=0$. For $f_1(u)=\frac{\alpha+4}{3\alpha}$, $f_2(u)=\frac{\beta+4}{3\beta}$, δ and γ becomes zero. So the numerator of H becomes zero. Since $H$is zero, then the surface is minimal.

4. CONCLUSION

In this article, by using generalized trigonometric basis functions, we created a surface by rotating the curve we obtained with the help of selected control points to create the tube deformation around the z-axis. We examined the effects of shape parameters on surfaces independent of radius and height. In the resulting surface design, unlike the Bezier curves, the curves in the trigonometric structure vary according to the value of the shape parameters. It plays a major role in computer-based designs to be made in industry and engineering fields.

The fact that the design change of the rotating surfaces depends on the shape parameters gives satisfactory results. As a result of surface characterization, it is a geometrically important result that we know where the surface is flat and where it is minimal by calculating the curvatures.

Hereby our methodology was as follows. First, we created a curve from the basis functions and a surface from the curve. Afterwards, we observed the effect of the change of shape parameters on the surface. Geometrically, we found important results and made characterizations related to the surface.

It is hoped that this study will significantly benefit industrial design when considering T-Bezier curves in different spaces and also when different basis functions are used.

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DECLARATION OF ETHICAL STANDARDS

The authors of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

AUTHORS CONTRIBUTIONS

Hakan GÜNDÜZ: The design of the rotational surface, evaluation of the results and writing of the article.

Hacı Bayram KARADAĞ: The plan for this study, assisting in topic selection, and helping to analyze the surface design.

CONFLICT OF INTEREST

There is no conflict of interest in this study.

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