http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 72, Number 4, Pages 1000–1018 (2023)
DOI:10.31801/cfsuasmas.1328691
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: July 17, 2023; Accepted: August 21, 2023

BMO ESTIMATE FOR THE HIGHER ORDER COMMUTATORS OF MARCINKIEWICZ INTEGRAL OPERATOR ON GRAND VARIABLE HERZ-MORREY SPACES

Babar SULTAN[1], Mehvish SULTAN[2] and Ferit GÜRBÜZ[3]

[1]Department of Mathematics, Quaid-I-Azam University 45320, Islamabad 44000, PAKISTAN
[2]Department of Mathematics , Capital University Of Science and Technology, Islamabad, PAKISTAN
[3]Department of Mathematics, Kırklareli University, Kırklareli 39100, TÜRKİYE

Abstract. Let $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$ with the normalized Lebesgue measure. Let $\Phi \in L^r(\mathbb{S}^{n-1})$ is a homogeneous function of degree zero and b is a locally integrable function on $\mathbb{R}^n$. In this paper we define the higher order commutators of Marcinkiewicz integral $[b, \mu_\Phi]^m$ and prove the boundedness of $[b, \mu_\Phi]^m$ under some proper assumptions on grand variable Herz-Morrey spaces $M\dot{K}^{\alpha(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n)$.

1. Introduction

Function spaces with variable exponents are an essential tools in harmonic analysis, operator theory and have gained significant attention in recent years, some instances of these works are in [3, 23]. The study of variable exponent function spaces is closely related to operator theory, which deals with linear operators acting on function spaces. In particular, the boundedness and compactness properties of operators in variable exponent spaces are of great interest. Understanding these properties is crucial for solving partial differential equations and analyzing various problems in applied mathematics.

The first generalization of Herz spaces with variable exponents, along with the proof of boundedness for sublinear operators in these spaces, was presented in [7].

2020 *Mathematics Subject Classification.* 46E30,47B38.
Keywords. BMO spaces, Marcinkiewicz integral operator, grand Herz-Morrey spaces, grand Herz spaces.

[1] babarsultan40@yahoo.com; 0000-0003-2833-4101
[2] mehvishsultanbaz@gmail.com; 0009-0005-2379-0476
[3] feritgurbuz@klu.edu.tr-Corresponding author; 0000-0003-3049-688X.

Herz-Morrey spaces, on the other hand, are further generalization of Herz spaces with variable exponents. The author in [6] introduced this class of function spaces. Continual Herz spaces with variable exponents were defined and studied in [14]. Understanding the boundedness of sublinear operators is of particular interest, its boundedness on continual Herz spaces can be seen in [14].

The concept of grand Morrey spaces, along with the boundedness of a class of integral operators in these spaces, were introduced in [10]. The author established the boundedness results for a specific class of integral operators, in newly defined grand Morrey spaces.

The idea of grand Herz spaces was introduced in [12]. This work expanded upon the classical Herz spaces by incorporating additional parameters. To explore the boundedness properties of other operators in grand variable Herz spaces, [2, 13, 18, 21, 22] can be consulted. These works likely provide insights into the boundedness of specific operators in the context of grand variable Herz spaces, enriching our understanding of the behavior of operators in this framework.

Subsequently, in the context of Herz-Morrey spaces with variable exponents, the concept of grand variable Herz-Morrey spaces were introduced in [17, 19]. These function spaces further extended the framework of Herz-Morrey spaces by incorporating the variable exponent setting. The authors demonstrated the boundedness of the Riesz potential operator in the newly defined grand variable Herz-Morrey spaces. Finally, in the article mentioned, the authors demonstrated the boundedness of higher-order commutators of the Marcinkiewicz integral operator in grand variable Herz-Morrey spaces. This result further explores the behavior of commutators in the context of grand variable Herz-Morrey spaces and contributes to the broader understanding of these function spaces.

Dividing the article into different sections helps to organize and present the material in a structured manner. Introduction provides an overview of the topic. A section presents the necessary mathematical background, definitions and relevant lemmas. Last section is dedicated to the main results of the article. It discusses the boundedness of the higher order commutators of the Marcinkiewicz integral operator in the context of grand variable Herz-Morrey spaces.

2. Preliminaries

It is worth noting that the Lebesgue space with variable exponent $L^{p(\cdot)}(H)$ inherits many properties from the classical Lebesgue spaces with constant exponents. We can define the Lebesgue space with variable exponent $L^{p(\cdot)}(H)$ as the set of all measurable functions f defined on a measurable set H such that the norm is finite. Consider a measurable set H in $\mathbb{R}^n$ and a measurable function $p(\cdot) : H \to [1, \infty)$.

Definition 1. *If H be a measurable set in $\mathbb{R}^n$ and $p(\cdot)$: $H \to [1, \infty)$ be a measurable function. We suppose that*

$$1 \le p_-(H) \le p(h) \le p_+(H) < \infty, \tag{1}$$

where $p_- := \operatorname{ess}\inf_{h\in H} p(h), \quad p_+ := \operatorname{ess}\sup_{h\in H} p(h).$

(a) Lebesgue space with variable exponent $L^{p(\cdot)}(H)$ *is defined as*

$$L^{p(\cdot)}(H) = \left\{ f \text{ measurable} : \int_H \left(\frac{|f(y)|}{\gamma}\right)^{p(y)} dy < \infty, \text{ where } \gamma \text{ is a constant} \right\}.$$

Norm in $L^{p(\cdot)}(H)$ *is defined as*

$$\|f\|_{L^{p(\cdot)}(H)} = \inf \left\{ \gamma > 0 : \int_H \left(\frac{|f(y)|}{\gamma}\right)^{p(y)} dy \leq 1 \right\}.$$

(b) The space $L^{p(\cdot)}_{\text{loc}}(H)$ *is defined as*

$$L^{p(\cdot)}_{\text{loc}}(H) := \left\{ f : f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset H \right\}.$$

In the sequel we use the well known log-condition

$$|p(x) - p(y)| \leq \frac{C(p)}{-\ln|x-y|}, \quad |x-y| \leq \frac{1}{2}, \quad x, y \in H, \tag{2}$$

where $C(p) > 0$. And the decay condition: there exists a number $p_\infty \in (1, \infty)$, such that

$$|p(h) - p_\infty| \leq \frac{C}{\ln(e + |h|)}, \tag{3}$$

and also decay condition

$$|p(h) - p_0| \leq \frac{C}{-\ln|h|}, |h| \leq \frac{1}{2}, \tag{4}$$

holds for some $p_0 \in (1, \infty)$. (Note that: $C > 0$ & $|h| \leq \frac{1}{2} \Rightarrow \frac{C}{\ln|h|} < 0$).
We use these notations in this article:

(i) The set $\mathcal{P}(H)$ consists of all measuable functions $p(\cdot)$ satisfying (1).
(ii) $\mathcal{P}^{\log} = \mathcal{P}^{\log}(H)$ consists of all functions $p \in \mathcal{P}(H)$ satisfying (1) and (2).
(iii) $\mathcal{P}_\infty(H)$ and $\mathcal{P}_{0,\infty}(H)$ are the subsets of $\mathcal{P}(H)$ and values of these subsets lies in $[1, \infty)$ which satisfy the condition (3) and both conditions (3) and (4) respectively.
(iv)
$$\chi_l = \chi_{R_l}, \ R_l = B_l \setminus B_{l-1}, \ B_l = B(0, 2^l) = \{x \in \mathbb{R}^n : |x| < 2^l\}$$
for all $l \in \mathbb{Z}$.

C is a positive constant, its value can change from line to line and is independent of main parameters involved.

Now we will define variable exponent Herz spaces.

Definition 2. *Let $u,v \in [1,\infty)$, $\alpha \in \mathbb{R}$, the norms of classical versions of non-homogeneous and homogeneous Herz spaces are given below,*

$$\|g\|_{K^{\alpha}_{u,v}(\mathbb{R}^n)} := \|g\|_{L^u(B(0,1))} + \left\{ \sum_{l\in\mathbb{N}} 2^{l\alpha v} \left(\int\limits_{R_{2^{l-1},2^l}} |g(y)|^u dy \right)^{\frac{v}{u}} \right\}^{\frac{1}{v}}, \tag{5}$$

where

$$\|g\|_{\dot{K}^{\alpha}_{u,v}(\mathbb{R}^n)} := \left\{ \sum_{l\in\mathbb{Z}} 2^{l\alpha v} \left(\int\limits_{R_{2^{l-1},2^l}} |g(y)|^u dy \right)^{\frac{v}{u}} \right\}^{\frac{1}{v}}, \tag{6}$$

respectively.

Definition 3. *Let $u \in [1,\infty)$, $v(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha \in \mathbb{R}$. The homogeneous version of variable exponent Herz space $\dot{K}^{\alpha,u}_{v(\cdot)}(\mathbb{R}^n)$ can be defined as*

$$\dot{K}^{\alpha,u}_{v(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{v(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{\dot{K}^{\alpha,u}_{v(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{7}$$

where

$$\|g\|_{\dot{K}^{\alpha,u}_{v(\cdot)}(\mathbb{R}^n)} = \left(\sum_{l=-\infty}^{\infty} \|2^{l\alpha} g\chi_l\|^u_{L^{v(\cdot)}} \right)^{\frac{1}{u}}.$$

Definition 4. *Let $u \in [1,\infty)$, $\alpha \in \mathbb{R}$ and $v(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The non-homogeneous version of variable exponent Herz space $K^{\alpha,u}_{v(\cdot)}(\mathbb{R}^n)$ can be defined as*

$$K^{\alpha,u}_{v(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{v(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{K^{\alpha,u}_{v(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{8}$$

where

$$\|g\|_{K^{\alpha,u}_{v(\cdot)}(\mathbb{R}^n)} = \|g\|_{L^{v(\cdot)}(B(0,1))} + \left(\sum_{k=-\infty}^{\infty} \|2^{k\alpha} g\chi_k\|^u_{L^{v(\cdot)}} \right)^{\frac{1}{u}}.$$

Definition 5. *Let $\alpha(\cdot) \in L^{\infty}(\mathbb{R}^n)$, $u \in [1,\infty)$, $q : \mathbb{R}^n \to [1,\infty)$, $\theta > 0$. A grand variable Herz spaces $\dot{K}^{\alpha(\cdot),u),\theta}_{q(\cdot)}$ are defined by,*

$$\dot{K}^{\alpha(\cdot),u),\theta}_{q(\cdot)} = \left\{ g \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{\dot{K}^{\alpha(\cdot),u),\theta}_{q(\cdot)}} < \infty \right\},$$

where

$$\|g\|_{\dot{K}^{\alpha(\cdot),u),\theta}_{q(\cdot)}} = \sup_{\epsilon>0} \left(\epsilon^{\theta} \sum_{k\in\mathbb{Z}} 2^{k\alpha(\cdot)u(1+\epsilon)} \|g\chi_k\|^{u(1+\epsilon)}_{L^{q(\cdot)}} \right)^{\frac{1}{u(1+\epsilon)}}.$$

Now we will define variable Herz-Morrey spaces.

Definition 6. *For $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$, $0<u<\infty$, $v(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and $0\le\beta<\infty$. A variable Herz-Morrey spaces $M\dot{K}^{\alpha(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n)$ are defined by,*

$$M\dot{K}^{\alpha(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n)=\left\{g\in L^{v(\cdot)}_{\text{loc}}(\mathbb{R}^n\setminus\{0\}):\|g\|_{M\dot{K}^{\alpha(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n)}<\infty\right\},$$

where

$$\|g\|_{M\dot{K}^{\alpha(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n)}=\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\sum_{t=-\infty}^{k_0}2^{k\alpha(\cdot)u}\|g\chi_k\|^u_{L^{v(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{u}}.$$

Definition 7. *To define homogeneous version of GVHM spaces, let $s:\mathbb{R}^n\to[1,\infty)$, $u\in[1,\infty)$, $\theta>0$, $0\le\lambda<\infty$, and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$. The GVHM spaces are given by:*

$$M\dot{K}^{\alpha(\cdot),u),\theta}_{\lambda,s(\cdot)}(\mathbb{R}^n)=\left\{g\in L^{s(\cdot)}_{\text{loc}}(\mathbb{R}^n\setminus\{0\}):\|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\lambda,s(\cdot)}(\mathbb{R}^n)}<\infty\right\},$$

where

$$\|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\lambda,s(\cdot)}(\mathbb{R}^n)}=\sup_{\epsilon>0}\sup_{l_o\in\mathbb{Z}}2^{-l_0\lambda}\left(\epsilon^\theta\sum_{k=-\infty}^{l_0}2^{k\alpha(\cdot)u(1+\epsilon)}\|g\chi_k\|^{u(1+\epsilon)}_{L^{s(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{u(1+\epsilon)}}.$$

Non-homogeneous version of GVHM spaces can be defined in the similar way. As grand variable Herz-Morrey spaces is the generalization of grand variable Herz spaces, $\lambda=0$, grand variable Herz-Morrey spaces become grand variable Herz spaces.

Definition 8 (BMO space)**.** *A BMO function is a locally integrable function u whose mean oscillation given by $\frac{1}{|B|}\int_B|u(y)-u_B|dy$ is bounded, i.e.*

$$\|u\|_{BMO}=\sup_B\frac{1}{|B|}\int\limits_B|u(y)-u_B|dy<\infty.$$

Lemma 1. *[14] Let $B>1$ and $p\in\mathcal{P}_{0,\infty}(\mathbb{R}^n)$. Then*

$$\frac{1}{t_0}s^{\frac{n}{p(0)}}\le\|\chi_{R_{s,B_s}}\|_{p(\cdot)}\le t_0s^{\frac{n}{p(0)}},\ \text{for } 0<s\le1 \tag{9}$$

and

$$\frac{1}{t_\infty}s^{\frac{n}{p_\infty}}\le\|\chi_{R_{s,B_s}}\|_{p(\cdot)}\le t_\infty s^{\frac{n}{p_\infty}},\ \text{for } s\ge1, \tag{10}$$

respectively, where $t_0\ge1$ and $t_\infty\ge1$ and depending on B but independent of s.

Lemma 2. *[23][Generalized Hölder's inequality] Consider a measurable subset H such that $H\subseteq\mathbb{R}^n$, and $1\le p_-(H)\le p_+(H)\le\infty$. Then*

$$\|fg\|_{L^{r(\cdot)}(H)}\le\|f\|_{L^{p(\cdot)}(H)}\|g\|_{L^{q(\cdot)}(H)}$$

holds, where $f \in L^{p(\cdot)}(H)$*,* $g \in L^{q(\cdot)}(H)$ *and* $\frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}$ *for every* $x \in H$*.*

Lemma 3. *[8] Let* k *be a positive integer. Let* $b \in BMO(\mathbb{R}^n)$ *and choose* $w, l \in \mathbb{Z}$ *with* $l < w$*,*

$$\frac{1}{C}\|b\|_{BMO}^k \leq \sup_{B:ball} \frac{1}{\|\chi_B\|_{p(\cdot)}} \|(b - b_B)^k \chi_B\|_{p(\cdot)} \tag{11}$$

$$\leq C\|b\|_{BMO}^k, \tag{12}$$

$$||(b - b_{B_t})^k \chi_{B_w}||_{p(\cdot)} \leq C(w-t)^k \|b\|_{BMO}^k ||\chi_{B_w}||_{p(\cdot)}. \tag{13}$$

Let $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$ with the normalized Lebesgue measure. Let $\Phi \in L^r(\mathbb{S}^{n-1})$ is a homogeneous function of degree zero such that

$$\int_{\mathbb{S}^{n-1}} \Phi(y') d\sigma(y') = 0, \tag{14}$$

where $y' = y/|y|$ and y is not zero. The Marcinkiewicz fractional operator is introduced with Littlewood-Paley g-function as:

$$\mu_\Phi(f)(x) = \left(\int_0^\infty |F_{\Phi,s}(f)(x)|^2 \frac{ds}{s^3} \right)^{\frac{1}{2}},$$

where

$$F_{\Phi,s}(f)(x) = \int_{|x-y|\leq s} \frac{\Phi(x-y)}{|x-y|^{n-1}} f(y) dy.$$

Consider a locally integrable function b on $\mathbb{R}^n$, now we can define higher order commutators of Marcinkiewicz integral $[b, \mu_\Phi]^m$ by using μ_Φ and b

$$[b, \mu_\Phi]^m(f)(x) = \left(\int_0^\infty \left| \int_{|x-y|\leq s} \frac{\Phi(x-y)}{|x-y|^{n-1}} [b(x) - b(y)]^m f(y) dy \right|^2 \frac{ds}{s^3} \right)^{\frac{1}{2}}.$$

Lemma 4. *[11] Let* $a > 0$*,* $s \in [1, \infty]$*,* $0 < d \leq s$ *and* $-n + (n-1)\frac{d}{s} < u < \infty$*, then*

$$\left(\int_{|y|\leq a|x|} |y|^u |\Phi(x-y)|^d dy \right)^{1/d} \leq |x|^{(u+n)/d} \|\Phi\|_{L^s(\mathbb{S}^{n-1})}.$$

It is easy to see that for $m = 1$, we get $[b, \mu_\Phi]^m(f)(x) = [b, \mu_\Phi](f)(x)$ (commuu-tator of Marcinkiewicz integral operator defined in [25]). When $m = 0$, the higher order commutators of Marcinkiewicz integral operator will be simply Marcinkiewicz integrals operator.

It's worth noting that the specific details and advancements in these works can only be fully explored by referring to the papers [1, 4, 5, 9, 15, 16, 20, 24].

3. BMO Estimate for the Higher Order Commutators of Marcinkiewicz Integrals Operator

The main purpose of this paper is to establish the boundedness of higher order commutators of Marcinkiewicz fractional operator on grand variable Herz-Morrey spaces by using some properties of the variable exponent and BMO function. It is easy to see that our results generalize the main results of [13]. Now, we will show the boundedness of higher order commutators of Marcinkiewicz integrals operator on grand variable Herz-Morrey spaces.

Theorem 1. *Let $0 < v \leq 1$, $\alpha(\cdot), q(\cdot) \in \mathcal{P}_{0,\infty}(\mathbb{R}^n)$ with $1 < q^- \leq q^+ < \infty$, $m \in \mathbb{Z}$, $1 \leq u < \infty$, $0 \leq \beta < \infty$ and $b \in BMO(\mathbb{R}^n)$. Let Φ be a homogeneous of degree zero and $\Phi \in L^s(\mathbb{S}^{n-1})$, $s > q'^-$. Let α be such that :*

(i) $-\frac{n}{q(0)} - v - \frac{n}{s} < \alpha(0) < \frac{n}{q'(0)} - v - \frac{n}{s}$

(ii) $-\frac{n}{q_\infty} - v - \frac{n}{s} < \alpha_\infty < \frac{n}{q'_\infty} - v - \frac{n}{s}$,

then operator $[b, \mu_\Phi]^m$ will be bounded on $M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q(\cdot)}(\mathbb{R}^n)$.

Proof. Let $g \in M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q(\cdot)}(\mathbb{R}^n)$, and $g(x) = \sum_{l=-\infty}^{\infty} g(x)\chi_l(x) = \sum_{l=-\infty}^{\infty} g_l(x)$, for $k_0 > 0$ we have,

$$\begin{aligned}
&\left\|[b, \mu_\Phi]^m g\right\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q(\cdot)}(\mathbb{R}^n)} = \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \\
&\qquad \times \left(\epsilon^\theta \sum_{t=-\infty}^{k_0} 2^{t\alpha(\cdot)u(1+\epsilon)} \|\chi_t [b, \mu_\Phi]^m g\|_{L^{q(\cdot)}}^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
&\leq \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\epsilon^\theta \sum_{t=-\infty}^{k_0} 2^{t\alpha(\cdot)u(1+\epsilon)} \left(\sum_{l=-\infty}^{\infty} \|\chi_t [b, \mu_\Phi]^m g_l\|_{L^{q(\cdot)}}^{u(1+\epsilon)} \right) \right)^{\frac{1}{u(1+\epsilon)}} \\
&\leq \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\epsilon^\theta \sum_{t=-\infty}^{k_0} 2^{t\alpha(\cdot)u(1+\epsilon)} \left(\sum_{l=-\infty}^{t} \|\chi_t [b, \mu_\Phi]^m g_l\|_{L^{q(\cdot)}} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
&+ \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\epsilon^\theta \sum_{t=-\infty}^{k_0} 2^{t\alpha(\cdot)u(1+\epsilon)} \left(\sum_{l=t+1}^{\infty} \|\chi_t [b, \mu_\Phi]^m g_l\|_{L^{q(\cdot)}} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
&=: E_1 + E_2.
\end{aligned}$$

Apply Minkowski's inequality to split E_1.

$$
\begin{aligned}
E_1 \leq & \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\epsilon^{\theta} \sum_{t=-\infty}^{-1} 2^{t\alpha(\cdot)u(1+\epsilon)} \left(\sum_{l=-\infty}^{t} \|\chi_t [b,\mu_\Phi]^m g_l\|_{L^{q(\cdot)}} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
& + \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\epsilon^{\theta} \sum_{t=0}^{k_0} 2^{t\alpha(\cdot)u(1+\epsilon)} \left(\sum_{l=-\infty}^{t} \|\chi_t [b,\mu_\Phi]^m g_l\|_{L^{q(\cdot)}} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}} \\
:= & E_{11} + E_{12}.
\end{aligned}
$$

We use the facts that, for each $t \in \mathbb{Z}$ and $l \leq t$ and a.e. $x \in R_t$, $y \in R_l$, we know that $|x-y| \approx |x| \approx 2^t$.

$$
\begin{aligned}
|\mu_\Phi(g\chi_l)(x)| \leq & \left(\int_{o}^{|x|} \left| \int_{|x-y|\leq t} \frac{\Phi(x-y)}{|x-y|^{n-1}} [b(x-b(y))]^m g_l(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
& + \left(\int_{|x|}^{\infty} \left| \int_{|x-y|\leq t} \frac{\Phi(x-y)}{|x-y|^{n-1}} [b(x-b(y))]^m g_l(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
=: & I_{11} + I_{12}.
\end{aligned}
$$

By the virtue of mean value theorem we obtain,

$$
\left| \frac{1}{|x-y|^2} - \frac{1}{|x|^2} \right| \leq \frac{|y|}{|x-y|^3}. \tag{15}
$$

For I_{11}, by using Minkowski's inequaltiy, generalized Hölder's inequality, and inequality 15 we have

$$
\begin{aligned}
I_{11} \leq & \int_{\mathbb{R}^n} \frac{|\Phi(x-y)|}{|x-y|^{n-1}} |b(x-b(y)|^m |g_l(y)| \left(\int_{|x-y|}^{|x|} \frac{dt}{t^3} \right)^{1/2} dy \\
\leq & \int_{\mathbb{R}^n} \frac{|\Phi(x-y)|}{|x-y|^{n-1}} |b(x-b(y)|^m |g_l(y)| \left| \frac{1}{|x-y|^2} - \frac{1}{|x|^2} \right|^{1/2} dy \\
\leq & \int_{\mathbb{R}^n} \frac{|\Phi(x-y)|}{|x-y|^{n-1}} |b(x-b(y)|^m |g_l(y)| \left| \frac{|y|}{|x-y|^3} \right|^{1/2} dy \\
\leq & \frac{2^{l/2}}{|x|^{n+1/2}} \int_{R_l} |\Phi(x-y)| |b(x-b(y)|^m |g(y)| dy
\end{aligned}
$$

$$\leq 2^{(l-t)/2}2^{-tn}\Bigg\{|b(x)-b_{B_l}|^m\int\limits_{R_l}|\Phi(x-y)||g_l(y)|dy$$
$$+\int\limits_{R_l}|b(y)-b_{B_l}|^m|\Phi(x-y)||g_l(y)|dy$$
$$\leq 2^{(l-t)/2}2^{-tn}\|g_l\|_{L^{q(\cdot)}}\Bigg\{|b(x)-b_{B_l}|^m\|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q'(\cdot)}}$$
$$+\|(b(\cdot)-b_{B_l})^m(\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q'(\cdot)}}\Bigg\}.$$

Similarly, we can consider I_{12}, we have

$$I_{12}\leq\int\limits_{\mathbb{R}^n}\frac{|\Phi(z-1-y)|}{|x-y|^{n-1}}|b(x-b(y)|^m|g_l(y)|\left(\int\limits_{|x|}^{\infty}\frac{dt}{t^3}\right)^{1/2}dy$$
$$\leq\int\limits_{\mathbb{R}^n}\frac{|\Phi(x-y)|}{|x-y|^n}|b(x)-b(y)|^m|g_l(y)|dy$$
$$\leq|x|^{-n}\int\limits_{R_l}|\Phi(x-y)||b(x)-b(y)|^m|g_l(y)|dy$$
$$\leq 2^{-tn}\|g_l\|_{L^{q(\cdot)}}\Bigg\{|b(x)-b_{B_l}|^m\|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q'(\cdot)}}$$
$$+\|(b(\cdot)-b_{B_l})^m(\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q'(\cdot)}}\Bigg\}.$$

So we have,

$$|\mu_\Phi(g\chi_l)(x)|$$
$$\leq 2^{-tn}\|g_l\|_{L^{q(\cdot)}}\Bigg\{|b(x)-b_{B_l}|^m\|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q'(\cdot)}}$$
$$+\|(b(\cdot)-b_{B_l})^m(\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q'(\cdot)}}\Bigg\}.$$

We define $q(\cdot)$ by the relation $\frac{1}{q'(x)}=\frac{1}{q(x)}+\frac{1}{s}$. By using Lemma (4) and generalized Hölder's inequality we have

$$\|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q'(\cdot)}}\leq\|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^s(\mathbb{R}^n)}\|\chi_l(\cdot)\|_{L^{q(\cdot)}}$$
$$\leq 2^{-lv}\left(\int\limits_{2^{l-1}<|y|<2^l}|\Phi(x-y)|^s|y|^{sv}dy\right)^{1/s}\|\chi_{B_l}\|_{L^{q(\cdot)}}$$
$$\leq 2^{-lv}2^{t(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_{B_l}\|_{L^{q(\cdot)}}.$$

Similarly, by using Lemma (3) we have

$$\begin{aligned}
&\|(b(\cdot)-b_{B_l})^m(\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q'(\cdot)}}\\
&\quad\le\|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^s(\mathbb{R}^n)}\|(b(\cdot)-b_{B_l})^m\chi_l(\cdot)\|_{L^{q(\cdot)}}\\
&\quad\le C\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_{B_l}\|_{L^{q(\cdot)}}\|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^s(\mathbb{R}^n)}\\
&\quad\le C\|b\|^m_{BMO(\mathbb{R}^n)}2^{-lv}2^{t(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_{B_l}\|_{L^{q(\cdot)}}.
\end{aligned}$$

As a result we get

$$\begin{aligned}
&\|[b,\mu_\Phi(g_l)]\chi_t\|_{L^{q(\cdot)}}\\
&\le C2^{-tn}\|g_l\|_{L^{q(\cdot)}}\Big\{\|(b(\cdot)-b_{B_l})^m\chi_t(\cdot)\|_{L^{q(\cdot)}}2^{-lv}2^{t(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_{B_l}\|_{L^{q(\cdot)}}\\
&+\|b\|^m_{BMO(\mathbb{R}^n)}2^{-lv}2^{t(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_{B_l}\|_{L^{q(\cdot)}}\|\chi_t\|_{L^{q(\cdot)}}\Big\}\\
&\le C2^{-tn}\|g_l\|_{L^{q(\cdot)}}\Big\{(t-l)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_{B_t}\|_{L^{q(\cdot)}}2^{-lv}2^{t(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_{B_l}\|_{L^{q(\cdot)}}\\
&+\|b\|^m_{BMO(\mathbb{R}^n)}2^{-lv}2^{t(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_{B_l}\|_{L^{q(\cdot)}}\|\chi_{B_t}\|_{L^{q(\cdot)}}\Big\}\\
&\le C2^{-tn}\|g_l\|_{L^{q(\cdot)}}(t-l)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_{B_t}\|_{L^{q(\cdot)}}2^{-lv}2^{t(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{B_l}\|_{L^{q(\cdot)}}\\
&\le C(t-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}2^{-tn}2^{-lv}2^{t(v+\frac{n}{s})}\|\chi_{B_t}\|_{L^{q(\cdot)}}\|\chi_{B_l}\|_{L^{q(\cdot)}}\|g_l\|_{L^{q(\cdot)}}.
\end{aligned}$$

Applying results to E_{11} we can get

$$\begin{aligned}
E_{11}\le C\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\Bigg[\epsilon^\theta\sum_{t=-\infty}^{-1}2^{t\alpha(0)u(1+\epsilon)}\Bigg(\sum_{l=-\infty}^{t}(t-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}\\
\times\,2^{(l-t)(n/q'(0)-v-\frac{n}{s})}\|g_l\|_{L^{q(\cdot)}}\Bigg)^{u(1+\epsilon)}\Bigg]^{\frac{1}{u(1+\epsilon)}}
\end{aligned}$$

$$\begin{aligned}
&\le C\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\quad\times\Bigg[\epsilon^\theta\sum_{t=-\infty}^{-1}\Bigg(\sum_{l=-\infty}^{t}2^{\alpha(0)l}\|g_l\|_{L^{q(\cdot)}}2^{b(l-t)}(t-l)^m\Bigg)^{u(1+\epsilon)}\Bigg]^{\frac{1}{u(1+\epsilon)}}.
\end{aligned}$$

Let $b=\frac{n}{q_1'(0)}-v-\frac{n}{s}-\alpha(0)>0$, applying Hölder's inequality, Fubini's theorem for series and $2^{-u(1+\epsilon)}<2^{-u}$ we get,

$$E_{11}\le C\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\Bigg[\epsilon^\theta\sum_{t=-\infty}^{-1}\Bigg(\sum_{l=-\infty}^{t}2^{\alpha(0)u(1+\epsilon)l}\|g_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}}$$

$$\times\ 2^{bu(1+\epsilon)(l-t)/2} \sum_{l=-\infty}^{t} 2^{b(u(1+\epsilon))'(l-t)/2}(t-l)^{m(u(1+\epsilon))'}\Bigg)^{\frac{u(1+\epsilon)}{(u(1+\epsilon))'}}\Bigg]^{\frac{1}{u(1+\epsilon)}}$$

$$\leq C\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}$$

$$\times\left(\epsilon^\theta \sum_{t=-\infty}^{-1}\sum_{l=-\infty}^{t} 2^{\alpha(0)u(1+\epsilon)l}\|g_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}} 2^{bu(1+\epsilon)(l-t)/2}\right)^{\frac{1}{u(1+\epsilon)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}$$

$$\times\left(\epsilon^\theta \sum_{l=-\infty}^{-1} 2^{\alpha(0)u(1+\epsilon)l}\|g_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}} \sum_{t=l+2}^{-1} 2^{bu(1+\epsilon)(l-t)/2}\right)^{\frac{1}{u(1+\epsilon)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}$$

$$\times\left(\epsilon^\theta \sum_{l=-\infty}^{-1} 2^{\alpha(0)u(1+\epsilon)l}\|g_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}} \sum_{t=l+2}^{-1} 2^{bu(1+\epsilon)(l-t)/2}\right)^{\frac{1}{u(1+\epsilon)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\epsilon^\theta \sum_{l=-\infty}^{-1} 2^{\alpha(0)u(1+\epsilon)l}\|g_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}}\right)^{\frac{1}{u(1+\epsilon)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\epsilon^\theta \sum_{l=-\infty}^{k_0} 2^{\alpha(\cdot)u(1+\epsilon)l}\|g_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}}\right)^{\frac{1}{u(1+\epsilon)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q(\cdot)}(\mathbb{R}^n)}.$$

Now apply Minkowski's inequality to split E_{12}, we have

$$E_{12} \leq \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\epsilon^\theta \sum_{t=0}^{k_0} 2^{t\alpha(\cdot)u(1+\epsilon)}\left(\sum_{l=-\infty}^{-1}\|\chi_t[b,\mu_\Phi]^m g_l\|_{L^{q(\cdot)}}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}$$

$$+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\epsilon^\theta \sum_{t=0}^{k_0} 2^{t\alpha(\cdot)u(1+\epsilon)}\left(\sum_{l=0}^{t-2}\|\chi_t[b,\mu_\Phi]^m g_l\|_{L^{q(\cdot)}}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}$$

$:= A_1 + A_2.$

To obtain the estimate for A_2, we can follow a similar approach as for E_{11}, but with some modifications. We will replace $q'(0)$ with q'_∞ and use the fact that $b = \frac{n}{q'_\infty} - v - \frac{n}{s} - \alpha_\infty > 0$.

For A_1 we have

$$\begin{aligned}&\|[b,\mu_\Phi]_\beta(g\chi_l)\chi_t\|_{L^{q(\cdot)}}\\ \le& C(t-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}2^{-tn}2^{-lv}2^{t(v+\frac{n}{s})}\|\chi_{B_t}\|_{L^{q(\cdot)}}\|\chi_{B_l}\|_{L^{q(\cdot)}}\|g_l\|_{L^{q(\cdot)}}\\ \le& C(t-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}2^{l(\frac{n}{q(0)}-v)}2^{t(v+\frac{n}{s}-\frac{n}{q'_\infty})}\|g_l\|_{L^{q(\cdot)}}.\end{aligned}$$

Now by using the fact $-\frac{n}{q'_\infty} + v + \frac{n}{s} + \alpha_\infty < 0$ we have,

$$\begin{aligned}A_1 \le& \sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\epsilon^\theta\sum_{t=0}^{k_0}2^{t\alpha_\infty u(1+\epsilon)}\left(\sum_{l=-\infty}^{-1}\|\chi_t[b,\mu_\Phi]^m g_l\|_{L^{q(\cdot)}}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&\le C\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times\left[\epsilon^\theta\sum_{t=0}^{k_0}2^{t\alpha_\infty u(1+\epsilon)}\left(\sum_{l=-\infty}^{-1}2^{l(\frac{n}{q(0)}-v)}2^{t(v+\frac{n}{s}-\frac{n}{q'_\infty})}(t-l)^m\|g_l\|_{L^{q(\cdot)}}\right)^{u(1+\epsilon)}\right]^{\frac{1}{u(1+\epsilon)}}\\
&\le C\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left[\epsilon^\theta\sum_{t=0}^{k_0}2^{t(\alpha_\infty+v+\frac{n}{s}-\frac{n}{q'_\infty}u(1+\epsilon)}\right.\\
&\qquad\left.\times(t-l)^{mu(1+\epsilon)}\left(\sum_{l=-\infty}^{-1}2^{l(\frac{n}{q(0)}-v)}\|g_l\|_{L^{q(\cdot)}}\right)^{u(1+\epsilon)}\right]^{\frac{1}{u(1+\epsilon)}}\\
&\le C\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times\left(\epsilon^\theta\left(\sum_{l=-\infty}^{-1}2^{l(\frac{n}{q(0)}-v)}\|g_l\|_{L^{q(\cdot)}}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&\le C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times\left(\epsilon^\theta\left(\sum_{l=-\infty}^{-1}2^{l\alpha(0)}\|g\chi_l\|_{L^{q(\cdot)}}2^{l(\frac{n}{q(0)}-v-\alpha(0))}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}.\end{aligned}$$

Now we can apply the Hölder's inequality and we can also use the fact that $\frac{n}{q'(0)} - \frac{n}{s} - v - \alpha(0) > 0$. Here is

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left[\epsilon^\theta \sum_{l=-\infty}^{-1} 2^{\alpha(0)lu(1+\epsilon)} \|g_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}} \times \left(\sum_{l=-\infty}^{-1} 2^{l(\frac{n}{q'(0)} - v - \alpha(0))(u(1+\epsilon))'} \right)^{\frac{u(1+\epsilon)}{(u(1+\epsilon))'}} \right]^{\frac{1}{u(1+\epsilon)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\epsilon^\theta \left(\sum_{l=-\infty}^{k_0} 2^{\alpha(\cdot)lu(1+\epsilon)} \|g_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}} \right) \right)^{\frac{1}{u(1+\epsilon)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q(\cdot)}(\mathbb{R}^n)}.$$

Now we will find the estimate for E_2. For each $t \in \mathbb{Z}$ and $l \geq t+1$ and a.e. $x \in R_t$, $y \in R_l$, we know that $|x-y| \approx |y| \approx 2^l$, we consider

$$|\mu_\Phi(g\chi_l)(x)| \leq \left(\int_o^{|y|} \left| \int_{|x-y|\leq t} \frac{\Phi(x-y)}{|x-y|^{n-1}} g_l(y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2} + \left(\int_{|y|}^{\infty} \left| \int_{|x-y|\leq t} \frac{\Phi(x-y)}{|x-y|^{n-1}} f_l(y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2}$$

$$=: I_{21} + I_{22}.$$

By using similar arguments as used in estimating I_{11}, we obtain

$$I_{21} \leq 2^{(t-l)/2} 2^{-\ln} \|g_l\|_{L^{q(\cdot)}} \left\{ |b(x) - b_{B_l}|^m \|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q'(\cdot)}} \|b(\cdot) - (b_{B_l})^m (\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q'(\cdot)}} \right\}.$$

Similar to the arguments of I_{12}, we have

$$I_{22} \leq 2^{-\ln} \|g_l\|_{L^{q(\cdot)}} \left\{ |b(x) - b_{B_l}|^m \|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q'(\cdot)}} + \|b(\cdot) - (b_{B_l})^m (\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q'(\cdot)}} \right\}.$$

So, we have

$$|[b,\mu_\Phi] - (f_l)(x)| \leq 2^{-\ln} \|g_l\|_{L^{q(\cdot)}}$$

$$\left\{|b(x)-b_{B_l}|^m\|\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q'(\cdot)}}+\|b(\cdot)-(b_{B_l})^m(\Phi(x-\cdot)\chi_l(\cdot)\|_{L^{q'(\cdot)}}\right\}.$$

Consequently we will get

$$\begin{aligned}
&\|[b,\mu_\Phi(g_l)]\chi_t\|_{L^{q(\cdot)}}\\
&\le C2^{-\ln}\|g_l\|_{L^{q(\cdot)}}\Bigg\{\|(b(\cdot)-b_{B_l})^m\chi_t(\cdot)\|_{L^{q(\cdot)}}2^{-lv}2^{t(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_{B_l}\|_{L^{q(\cdot)}}\\
&+\|b\|^m_{BMO(\mathbb{R}^n)}2^{-lv}2^{t(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_{B_l}\|_{L^{q(\cdot)}}\|\chi_t\|_{L^{q(\cdot)}}\Bigg\}\\
&\le C2^{-\ln}\|g_l\|_{L^{q(\cdot)}}\Bigg\{(t-l)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_{B_t}\|_{L^{q(\cdot)}}2^{-lv}2^{t(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_{B_l}\|_{L^{q(\cdot)}}\\
&+\|b\|^m_{BMO(\mathbb{R}^n)}2^{-lv}2^{t(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_{B_l}\|_{L^{q(\cdot)}}\|\chi_{B_t}\|_{L^{q(\cdot)}}\Bigg\}\\
&\le C2^{-\ln}\|g_l\|_{L^{q(\cdot)}}(t-l)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_{B_t}\|_{L^{q(\cdot)}}2^{-lv}2^{t(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{B_l}\|_{L^{q(\cdot)}}\\
&\le C(t-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}2^{-\ln}2^{-lv}2^{t(v+\frac{n}{s})}\|\chi_{B_t}\|_{L^{q(\cdot)}}\|\chi_{B_l}\|_{L^{q(\cdot)}}\|g_l\|_{L^{q(\cdot)}}\\
&\le C(t-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}2^{-\ln}2^{-lv}2^{t(v+\frac{n}{s})}2^{ln/q_\infty}2^{tn/q_\infty}\|g_l\|_{L^{q(\cdot)}}\\
&\le C(t-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}2^{(t-l)^m(v+\frac{n}{s}+\frac{n}{q_\infty})}\|g_l\|_{L^{q(\cdot)}}.
\end{aligned}$$

Now splitting E_2 we have

$$\begin{aligned}
E_2\le&\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^\theta\sum_{t=-\infty}^{k_0}2^{t\alpha(\cdot)u(1+\epsilon)}\left(\sum_{l=t+2}^{\infty}\|\chi_t[b,\mu_\Phi]^m g_l\|_{L^{q(\cdot)}}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
\le&\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^\theta\sum_{t=-\infty}^{-1}2^{t\alpha(\cdot)u(1+\epsilon)}\left(\sum_{l=t+2}^{\infty}\|\chi_t[b,\mu_\Phi]^m g_l\|_{L^{q(\cdot)}}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
&+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^\theta\sum_{t=0}^{k_0}2^{t\alpha(\cdot)u(1+\epsilon)}\left(\sum_{l=t+2}^{\infty}\|\chi_t[b,\mu_\Phi]^m g_l\|_{L^{q(\cdot)}}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
:=&E_{21}+E_{22}.
\end{aligned}$$

For E_{22} we have

$$\begin{aligned}
E_{22}\le& C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\\
&\times\left(\epsilon^\theta\sum_{t=0}^{k_0}2^{t\alpha_\infty u(1+\epsilon)}\left(\sum_{l=t+2}^{\infty}2^{(t-l)(v+\frac{n}{s}+\frac{n}{q_\infty})}(t-l)^m\|g_l\|_{L^{q(\cdot)}}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}
\end{aligned}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}$$
$$\times \left(\epsilon^\theta \sum_{t=0}^{k_0} \left(\sum_{l=t+2}^{\infty} 2^{l\alpha_\infty} \|g_l\|_{L^{q(\cdot)}} 2^{(t-l)(v+\frac{n}{s}+\frac{n}{q_\infty})} (t-l)^m \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}},$$

for $d = \frac{n}{q_\infty} + v + \frac{n}{s} > 0$. Then we can apply the Hölder's inequality for series and $2^{-u(1+\epsilon)} < 2^{-u}$ to get

$$E_{22} \leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left[\epsilon^\theta \sum_{t=0}^{k_0} \left(\sum_{l=t+2}^{\infty} 2^{l\alpha_\infty u(1+\epsilon)} \|g_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}} 2^{du(1+\epsilon)(t-l)/2} \right) \right.$$
$$\left. \times \left(\sum_{l=t+2}^{\infty} 2^{d(u(1+\epsilon))'(t-l)/2} (t-l)^{m(u(1+\epsilon))'} \right)^{\frac{u(1+\epsilon)}{(u(1+\epsilon))'}} \right]^{\frac{1}{u(1+\epsilon)}}$$
$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}$$
$$\times \left(\epsilon^\theta \sum_{t=0}^{k_0} \sum_{l=t+2}^{\infty} 2^{l\alpha_\infty u(1+\epsilon)} \|g_l\|^{u(1+\epsilon)}_{L^{q(\cdot)}} 2^{du(1+\epsilon)(t-l)/2} \right)^{\frac{1}{u(1+\epsilon)}}$$
$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}$$
$$\times \left(\epsilon^\theta \sum_{t=0}^{k_0} \sum_{l=t+2}^{\infty} \sum_{j=-\infty}^{l} 2^{j\alpha_\infty u(1+\epsilon)} \|g_j\|^{u(1+\epsilon)}_{L^{q(\cdot)}} 2^{du(1+\epsilon)(t-l)/2} \right)^{\frac{1}{u(1+\epsilon)}}$$
$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\epsilon^\theta \sum_{t=0}^{k_0} \sum_{l=t+2}^{\infty} 2^{du(1+\epsilon)(t-l)/2} \right)^{\frac{1}{u(1+\epsilon)}}$$
$$\times \|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q(\cdot)}(\mathbb{R}^n)}$$
$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)} \|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q(\cdot)}(\mathbb{R}^n)}.$$

Now for E_{21} using Minkowski's inequality we have

$$E_{21} \leq \sup_{\epsilon>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta} \left(\epsilon^\theta \sum_{t=-\infty}^{-1} 2^{t\alpha(\cdot)u(1+\epsilon)} \left(\sum_{l=t+2}^{-1} \|\chi_t [b,\mu_\Phi]^m g_l\|_{L^{q(\cdot)}} \right)^{u(1+\epsilon)} \right)^{\frac{1}{u(1+\epsilon)}}$$

$$+\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\beta}\left(\epsilon^{\theta}\sum_{t=-\infty}^{-1}2^{t\alpha(\cdot)u(1+\epsilon)}\left(\sum_{l=0}^{\infty}\|\chi_t[b,\mu_\Phi]^m g_l\|_{L^{q(\cdot)}}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}$$
$$:=B_1+B_2.$$

The estimate for B_1 follows in a similar manner to E_{22} with q_∞ replaced by $q(0)$ and using the fact that $\frac{n}{q(0)}+v+\frac{n}{s}+\alpha(0)>0$. For B_2 we have

$$\begin{aligned}
&\|[b,\mu_\Phi](g_l)\chi_t\|_{L^{q(\cdot)}}\\
&\le C2^{-\ln}\|g_l\|_{L^{q(\cdot)}}\Big\{\|(b(\cdot)-b_{B_l})^m\chi_t(\cdot)\|_{L^{q(\cdot)}}2^{-lv}2^{t(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_{B_l}\|_{L^{q(\cdot)}}\\
&+\|b\|^m_{BMO(\mathbb{R}^n)}2^{-lv}2^{t(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_{B_l}\|_{L^{q(\cdot)}}\|\chi_t\|_{L^{q(\cdot)}}\Big\}\\
&\le C2^{-\ln}\|g_l\|_{L^{q(\cdot)}}\Big\{(t-l)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_{B_t}\|_{L^{q(\cdot)}}2^{-lv}2^{t(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_{B_l}\|_{L^{q(\cdot)}}\\
&+\|b\|^m_{BMO(\mathbb{R}^n)}2^{-lv}2^{t(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\|\chi_{B_l}\|_{L^{q(\cdot)}}\|\chi_{B_t}\|_{L^{q(\cdot)}}\Big\}\\
&\le C2^{-\ln}\|g_l\|_{L^{q(\cdot)}}(t-l)^m\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_{B_t}\|_{L^{q(\cdot)}}2^{-lv}2^{t(v+\frac{n}{s})}\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{B_l}\|_{L^{q(\cdot)}}\\
&\le C(t-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}2^{-\ln}2^{-lv}2^{t(v+\frac{n}{s})}\|\chi_{B_t}\|_{L^{q(\cdot)}}\|\chi_{B_l}\|_{L^{q(\cdot)}}\|g_l\|_{L^{q(\cdot)}}\\
&\le C(t-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}2^{-ln}2^{-lv}2^{t(v+\frac{n}{s})}\|g_l\|_{L^{q(\cdot)}}2^{ln/q_\infty}2^{tn/q(0)}\\
&\le C(t-l)^m\|\Phi\|_{L^s(\mathbb{S}^{n-1})}\|b\|^m_{BMO(\mathbb{R}^n)}2^{-l(v+\frac{n}{s}+\frac{n}{q_\infty})}2^{t(v+\frac{n}{q(0)}+\frac{n}{s})}\|g_l\|_{L^{q(\cdot)}}.
\end{aligned}$$

$$\begin{aligned}
B_2\le&\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}2^{t\alpha(0)u(1+\epsilon)}\left(\sum_{l=0}^{\infty}\|\chi_t[b,\mu_\Phi]^m g_l\|_{L^{q(\cdot)}}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
\le&C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}2^{t\alpha(0)u(1+\epsilon)}\right.\\
&\times\left.\left(\sum_{l=0}^{\infty}2^{-l(v+\frac{n}{s}+\frac{n}{q_\infty})}2^{t(v+\frac{n}{q(0)}+\frac{n}{s})}\|g_l\|_{L^{q(\cdot)}}(t-l)^m\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}\\
\le&C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^{\theta}\sum_{k=-\infty}^{-1}2^{t(v+\frac{n}{q(0)}+\frac{n}{s}+\alpha(0))u(1+\epsilon)}(t-l)^m\right)
\end{aligned}$$

$$\times\left(\left(\sum_{l=0}^{\infty}2^{-l(v+\frac{n}{s}+\frac{n}{q_\infty})}\|g_l\|_{L^{q(\cdot)}}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}$$

$$\times\left(\epsilon^\theta\left(\sum_{l=0}^{\infty}2^{-l(v+\frac{n}{s}+\frac{n}{q_\infty})}\|g_l\|_{L^{q(\cdot)}}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}$$

$$\times\left(\epsilon^\theta\left(\sum_{l=0}^{\infty}2^{\alpha_\infty l}\|g_l\|_{L^{q(\cdot)}}2^{-l(v+\frac{n}{s}+\frac{n}{q_\infty}+\alpha_\infty)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}$$

$$\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}$$

$$\times\left(\epsilon^\theta\left(\sum_{l=0}^{\infty}\sum_{j=-\infty}^{l}2^{\alpha_\infty j}\|g_j\|_{L^{q(\cdot)}}2^{-l(v+\frac{n}{s}+\frac{n}{q_\infty}+\alpha_\infty)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}.$$

Now by applying Hölder's inequality and using the fact that $\frac{n}{q(\infty)}+v+\frac{n}{s}+\alpha(\infty)>0$ we have

$$B_2\leq C\|b\|^m_{BMO(\mathbb{R}^n)}\sup_{\epsilon>0}\sup_{k_0\in\mathbb{Z}}2^{-k_0\beta}\left(\epsilon^\theta\left(\sum_{l=0}^{\infty}2^{-l(v+\frac{n}{s}+\frac{n}{q_\infty}+\alpha_\infty)}\right)^{u(1+\epsilon)}\right)^{\frac{1}{u(1+\epsilon)}}$$

$$\times\|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q(\cdot)}(\mathbb{R}^n)}$$

$$\leq\|b\|^m_{BMO(\mathbb{R}^n)}\|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q(\cdot)}(\mathbb{R}^n)}.$$

Combining the estimates for E_1 and E_2 yields
$\left\|[b,\mu_\Phi]^m(g)\right\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q(\cdot)}(\mathbb{R}^n)}\leq\|b\|^m_{BMO(\mathbb{R}^n)}\|g\|_{M\dot{K}^{\alpha(\cdot),u),\theta}_{\beta,q(\cdot)}(\mathbb{R}^n)}$,
which completes the proof. □

Author Contribution Statements The authors contributed equally to this work. All authors read and approved the final copy of this paper.

Declaration of Competing Interests The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement The authors are thankful to the referees for making valuable suggestions leading to the better presentations of this paper.

References

[1] Asim, M., Hussain, A., Weighted variable Morrey-Herz estimates for fractional Hardy operators, *J. Inequal. Appl.,* 2(2022) (2022) 12pp. https://doi.org/10.1186/s13660-021-02739-z

[2] Bashir, S., Sultan, B., Hussain, A., Khan, A., Abdeljawad, T., A note on the boundedness of Hardy operators in grand Herz spaces with variable exponent, *AIMS Mathematics,* 8(9) (2023), 22178–22191. https://doi.org/10.3934/math.20231130

[3] Diening, L., Harjuletho, P., Hastö, P., Ruzicka, M., Lebesgue and Sobolev Spaces with Variable Exponents, Springer, 2011. https://doi.org/10.1007/978-3-642-18363-8

[4] Hussain, A., Asim, M., Aslam, M., Jarad, F., Commutators of the fractional Hardy operator on weighted variable Herz-Morrey spaces, *J. Funct. Spaces.*, ID 9705250 (2021), 10 pages. https://doi.org/10.1155/2021/9705250

[5] Hussain, A., Asim, M., Jarad, F., Variable λ-Central Morrey space estimates for the fractional Hardy operators and commutators, *Journal of Mathematics*, ID 5855068 (2022), 12 pp. https://doi.org/10.1155/2022/5855068

[6] Izuki, M., Boundedness of vector-valued sublinear operators on Herz-Morrey spaces with variable exponents, *Math. Sci. J.*, 13(10) (2009), 243–253.

[7] Izuki, M., Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization, *Anal. Math.*, 36(1) (2010), 33–50.

[8] Izuki, M., Boundedness of commutators on Herz spaces with variable exponent, *Rendiconti del Circolo Matematico di Palermo.*, 59 (2010), 199–213. https://doi.org/10.1007/s12215-010-0015-1

[9] Kováčik, O., Rákosník, J., On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslov. Math. J.*, 41(4) (1991), 592–618.

[10] Meskhi, A., Maximal functions, potentials and singular integrals in grand Morrey spaces, *Complex Variables and Elliptic Equations,* 56(10-11) (2011), 1003-1019. https://doi.org/10.1080/17476933.2010.534793

[11] Muckenhoupt, B., Wheeden, R. L., Weighted norm inequalities for singular and fractional integrals, *Trans. Am. Maths Soc.*, 161 (1971), 249–258.

[12] Nafis, H., Rafeiro, H., Zaighum, M. A., A note on the boundedness of sublinear operators on grand variable Herz spaces, *J. Inequal Appl.*, 2020(1) (2020), 1–13. https://doi.org/10.1186/s13660-019-2265-6

[13] Nafis, H., Rafeiro, H., Zaighum, M. A., Boundedness of the Marcinkiewicz integral on grand Herz spaces, *J. Math. Ineq.*, 15(2) (2021), 739–753. https://doi.org/10.7153/jmi-2021-15-52

[14] Samko, S., Variable exponent Herz spaces, *Mediterr. J. Math.* 10(4) (2013), 2007–2025. https://doi.org/10.1007/s00009-013-0335-4

[15] Sultan, B., Sultan, M., Mehmood, M., Azmi, F., Alghafli, M.A., Mlaiki, N., Boundedness of fractional integrals on grand weighted Herz spaces with variable exponent, *AIMS Mathematics,* 8(1) (2023), 752–764. https://doi.org/10.3934/math.2023036

[16] Sultan, B, Azmi, F.M., Sultan, M., Mahmood, T., Mlaiki, N., Souayah, N., Boundedness of fractional integrals on grand weighted Herz-Morrey spaces with variable exponent, *Fractal and Fractional*, 6(11) (2022), 660. https://doi.org/10.3390/fractalfract6110660

[17] Sultan, B., Azmi, F., Sultan, M., Mehmood, M., Mlaiki, N., Boundedness of Riesz potential operator on grand Herz-Morrey spaces, *Axioms*, 11(11) (2022), 583. https://doi.org/10.3390/axioms11110583

[18] Sultan, M., Sultan, B., Aloqaily, A., Mlaiki, N., Boundedness of some operators on grand Herz spaces with variable exponent, *AIMS Mathematics*, 8(6) (2023), 12964–12985. https://doi.org/10.3934/math.2023653

[19] Sultan, B., Sultan, M., Khan, A., Abdeljawad, T., Boundedness of Marcinkiewicz integral operator of variable order in grand Herz-Morrey spaces, *AIMS Mathematics*, 8(9) (2023), 22338–22353. https://doi.org/10.3934/math.20231139

[20] Sultan, B., Sultan, M., Zhang, Q. Q., Mlaiki, N., Boundedness of Hardy operators on grand variable weighted Herz spaces, *AIMS Mathematics*, 8(10) (2023), 24515–24527.

[21] Sultan, B., Sultan, M., Khan, I., On Sobolev theorem for higher commutators of fractional integrals in grand variable Herz spaces, *Commun. Nonlinear Sci. Numer. Simul.*, 126 (2023). DOI 10.1016/j.cnsns.2023.107464

[22] Sultan, B., Sultan, M., Boundedness of commutators of rough Hardy operators on grand variable Herz spaces, *Forum Mathematicum,* 2023. https://doi.org/10.1515/forum-2023-0152

[23] Uribe, D. C., Fiorenza, A., Variable Lebesgue spaces, Foundations and Harmonic Analysis, Appl. Numer. Harmon. Anal., Birkhäuser, Heidelberg, 2013.

[24] Uribe, D. C., et al., The boundedness of classical operators on variable L^p spaces, *Acad. Sci. Fenn. Math.*, 31(1) (2006), 239–264.

[25] Wang, H., Commutators of Marcinkiewicz integrals on Herz spaces with variable exponent, *Czech Math J.*, 66 (2016), 251–269.