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*Research Article*

# On the eigenvalue-separation properties of real tridiagonal matrices

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ABSTRACT. In this paper, we give a simple sufficient condition for the eigenvalue-separation properties of real tridiagonal matrices $T$. This result is much more than the statement that the pertinent eigenvalues are distinct. Its derivation is based on recurrence formulae satisfied by the polynomials made up by the minors of the characteristic polynomial $\det(xE - T)$ that are proven to form a Sturm sequence. This is a new result, and it proves the simple spectrum property of a symmetric tridiagonal matrix studied in a Grünbaum paper. Two numerical examples underpin the theoretical findings. The style of the paper is expository in order to address a large readership.

**Keywords:** Characteristic polynomial, distinct eigenvalues, eigenvalue-separation properties, minors of determinant, Sturm sequence, tridiagonal matrix.

**2020 Mathematics Subject Classification:** 15A15, 15A18, 15B05, 65F15, 65F22.

## 1. INTRODUCTION

In [2, p.30], Grünbaum assumes that the roots of the characteristic polynomial of a special symmetric tridiagonal matrix are distinct. In this paper, we give a simple sufficient condition for this. More precisely, we are able to show that the minors of the characteristic polynomial of a real tridiagonal matrix satisfy a Sturm sequence provided that the products of corresponding entries above and below the diagonal are positive. In the case of a symmetric tridiagonal matrix, this condition means that all entries above and below the diagonal are different from zero. As a consequence, we obtain the eigenvalue-separation properties of Sturm sequences which is much more that the statement that the eigenvalues are distinct.

The paper is structured as follows. In Section 2, we take over a lemma on the Sturm sequence from [4] as the main tool to derive a sufficient condition for the eigenvalue-separation properties of tridiagonal matrices. Then, in Section 3, the lemma on Sturm sequences from Section 2 is applied to obtain the eigenvalue-separation properties of tridiagonal matrices provided that the products of corresponding entries above and below the diagonal are positive. Section 4 contains two numerical examples that underpin the theoretical findings, one with nonsymmetric and one with symmetric tridiagonal matrix. In Section 5, the conclusion is given. The non-cited references [1] and [3] are given because they also contain sections on Sturm sequences so that they may be of interest to the reader in the context of the treated subject.

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## 2. Preliminaries on Sturm Sequence

In [4, Section 10.3, pp. 194-200], it is shown that for every linear symmetric mapping $A : \mathbb{R}^n \to \mathbb{R}^n$ a basis of $n$ orthonormal vectors associated with Lanczos polynomials can be constructed such that the mapping $A$ in this basis is represented by a symmetric tridiagonal matrix $T$ and further that the minors of the characteristic polynomial $p(x) = det(x\,E - T)$ form a *Sturm sequence* having interesting eigenvalue-separation properties.

In this section, we take over the results derived [4] as the main tool to be used in Section 3. We start with a sequence of polynomials $p_0, p_1, \ldots, p_m$ with real $\beta_j, \gamma_j$ that fulfill the recursion formulae

(2.1) $$p_0(x) = 1,\; p_1(x) = x - \beta_0,\; p_{j+1}(x) = (x - \beta_j)\, p_j(x) - \gamma_j\, p_{j-1}(x),\; \gamma_j > 0,$$

$j = 1, \ldots, m-1$ for $x \in \mathbb{R}$. These polynomials then form a *Sturm sequence*, also called *Sturm chain*, allowing far-reaching assertions about the position and separation properties of their zeros as stated in the following lemma.

**Lemma 2.1** (Lemma on Sturm sequence)**.** *For every $j = 1, \ldots, m$, all the zeros of the polynomial $p_j$ in (2.1) are real as well as pairwise distinct and can be arranged according to*

(P1) $$-\infty < \lambda_{jj} < \lambda_{j,j-1} < \cdots < \lambda_{j2} < \lambda_{j1} < +\infty.$$

*For every $j = 1, \ldots, m-1$ and every zero $\lambda$ of $p_j$, the relations*

(P2) $$p_j(\lambda) = 0, \quad p_{j+1}(\lambda)\, p_{j-1}(\lambda) < 0$$

*hold. If one sets formally $\lambda_{j,j+1} = -\infty$ and $\lambda_{j0} = +\infty$ (that are the left and right boundaries in (P1)), then*

(P3) $$(-1)^k\, p_j(x) > 0, \quad \lambda_{j,k+1} < x < \lambda_{jk},\; k = 0, \ldots, j,\; j = 0, \ldots, m$$

*and, for every j=0,1, …,m-1, there is just one zero $\lambda_{j+1,k}$ of $p_{j+1}$ in the interval*

(P4) $$\lambda_{jk} < \lambda_{j+1,k} < \lambda_{j,k-1},\; k = 1, \ldots, j+1.$$

*Proof.* See [4, Section 10.3, Formula (27), p.200], where the text was taken over with minor changes for the reason of clarity. □

We note that the necessity of the condition $\gamma_j > 0$ in the definition of the Sturm sequence is not obvious, but is seen during the proof of Lemma 2.1 in the above-cited book. We remark further that the proof does not depend on the assumption that $p_j(x)$ is a minor of $p_m(x) = det(x\,E - T)$ with some matrix $T$. But, in Section 3, we will use polynomials $p_j(x)$ that are minors of $p_m(x) = det(x\,E - T)$, where $T$ is a real tridiagonal matrix.

We mention that with the definitions $\lambda_{j,j+1} = -\infty$ and $\lambda_{j0} = +\infty$, property (P4) reads for $k = j+1$ as follows

$$(P4)_{k=j+1} \qquad -\infty < \lambda_{j+1,j+1} < \lambda_{jj}$$

and for $k = 1$ as follows

$$(P4)_{k=1} \qquad \lambda_{j1} < \lambda_{j+1,j} < +\infty.$$

We want to point out that if we would not have introduced the definitions $\lambda_{j,j+1} = -\infty$ and $\lambda_{j0} = +\infty$, then instead of $(P4)_{k=j+1}$, we would have

$$(P4)'_{k=j+1} \qquad \lambda_{j+1,j+1} < \lambda_{jj}$$

which is equivalent to $(P4)_{k=j+1}$ since we have trivially $-\infty < \lambda_{j-1,j+1}$; and instead of $(P4)_{k=1}$, we would have

$$(P4)'_{k=1} \qquad \lambda_{j1} < \lambda_{j+1,j}$$

which is equivalent to $(P4)_{k=1}$ since we have trivially $\lambda_{j-1,j} < +\infty$.

## 3. Application to Real Tridiagonal Matrices

This section is the core of the present paper. Its results are obtained by applying Lemma 2.1 on Sturm sequences to real tridiagonal matrices. We start with the real tridiagonal matrix

$$T = tridiag[a_{i-1}, c_i, d_i]_{i=1,\ldots,N+1} \in \mathbb{R}^{(N+1)\times(N+1)}$$

with

$$\begin{array}{ll} a_0 = a_{N+1} = 0, & a_i \in \mathbb{R},\ i = 1,\ldots,N, \\ d_0 = d_{N+1} = 0, & d_i \in \mathbb{R},\ i = 1,\ldots,N, \end{array} \quad c_i \in \mathbb{R},\ i = 1,\ldots,N+1,$$

or, written in full,

$$T = \begin{bmatrix} c_1 & d_1 & & & & \\ a_1 & c_2 & d_2 & & & \\ & a_2 & c_3 & d_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{N-1} & c_N & d_N \\ & & & & a_N & c_{N+1} \end{bmatrix}. \tag{3.2}$$

Such a matrix for the symmetric case $d_i = a_i,\ i = 1,\ldots,N$ with the special diagonal entries $c_i = b_i - a_i - a_{i-1},\ i = 1,\ldots,N+1$, where $b_i$ are real elements is studied in [2, p.27].

Here, we want to apply Lemma 2.1 on the Sturm sequence. For this, we change the denotations of the entries of matrix $T$ as follows. With

$$\gamma_0' = \gamma_m' = 0,$$

we set

$$T = tridiag[\gamma_{i-1}', \beta_{i-1}, \delta_i']_{i=1,\ldots,m}$$

or, written out,

$$T = \begin{bmatrix} \beta_0 & \delta_1' & & & & \\ \gamma_1' & \beta_1 & \delta_2' & & & \\ & \gamma_2' & \beta_2 & \delta_3' & & \\ & & \ddots & \ddots & \ddots & \\ & & & \gamma_{m-2}' & \beta_{m-2} & \delta_{m-1}' \\ & & & & \gamma_{m-1}' & \beta_{m-1} \end{bmatrix}. \tag{3.3}$$

Comparison of (3.2) and (3.3) leads to

$$\beta_{i-1} = c_i,\ i = 1,\ldots,m(= N+1), \tag{3.4}$$

$$\begin{array}{lcl} \gamma_{i-1}' & = & a_{i-1},\ i = 2,\ldots,m(= N+1), \\ \delta_{i-1}' & = & d_{i-1},\ i = 2,\ldots,m(= N+1). \end{array} \tag{3.5}$$

Now, we introduce the condition

$$\begin{array}{l} a_j \neq 0,\ j = 1,\ldots,N(= m-1), \\ d_j \neq 0,\ j = 1,\ldots,N(= m-1). \end{array}$$

This entails

$$\begin{array}{l} \gamma_j' = a_j \neq 0,\ j = 1,\ldots,m-1(= N), \\ \delta_j' = d_j \neq 0,\ j = 1,\ldots,m-1(= N). \end{array}$$

Next, we define

$$\gamma_j := \gamma_j'\,\delta_j',\ j = 1,\ldots,m-1(= N). \tag{3.6}$$

Further, we make the restriction

$$\gamma_j > 0,\ j = 1, \ldots, m-1. \tag{3.7}$$

We choose these $\gamma_j$ in the Sturm sequence (2.1). Further, more generally than in [4, Section 10.3, Formula (25), p. 199], we define

$$p_j(x) = \det \begin{bmatrix} x-\beta_0 & -\delta_1' & & & & \\ -\gamma_1' & x-\beta_1 & -\delta_2' & & & \\ & -\gamma_2' & x-\beta_2 & -\delta_3' & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\gamma_{j-2}' & x-\beta_{j-2} & -\delta_{j-1}' \\ & & & & -\gamma_{j-1}' & x-\beta_{j-1} \end{bmatrix}, \tag{3.8}$$

$j = 1, \ldots, m$ so that $p_j(x)$ for $j = 1, \ldots, m$ are the minors of

$$p_m(x) = det(x\,E - T)$$

with the identity matrix $E$ and matrix $T$ in (3.3).

**Theorem 3.1** (Theorem on Eigenvalue-Separation Properties)**.** *Let the real tridiagonal matrix $T$ in* (3.2) *be given. Further, let this matrix be rewritten in the form* (3.3) *with the entries defined in* (3.4), (3.5), *and* (3.6)*. Then, if the condition* (3.7) *is satisfied, the polynomials defined by* (3.8) *fulfill the recursion formulae* (2.1).

*Proof.* The proof is done by mathematical induction.
Base case ($j=1$): For $j = 1$, we have

$$\begin{aligned} p_{1+1}(x) = p_2(x) &= \det \begin{bmatrix} x-\beta_0 & -\delta_1' \\ -\gamma_1' & x-\beta_1 \end{bmatrix} = \begin{vmatrix} x-\beta_0 & -\delta_1' \\ -\gamma_1' & x-\beta_1 \end{vmatrix} \\ &= (x-\beta_0)\,(x-\beta_1) - \gamma_1'\,\delta_1' \\ &= (x-\beta_1)\,\underbrace{(x-\beta_0)}_{=p_1(x)} - \gamma_1\,\underbrace{1}_{=p_0(x)} \\ &= (x-\beta_1)\,p_1(x) - \gamma_1\,p_0(x) \end{aligned}$$

so that the recursion formula $p_{j+1}(x) = (x-\beta_j)\,p_j(x) - \gamma_j\,p_{j-1}(x)$ in (2.1) is proven for $j = 1$.

Before we continue with the induction step, we add two further base steps in order to obtain more insight into the process and to prepare the induction step.
Additional base case ($j=2$): For $j = 2$, we get

$$p_{2+1}(x) = p_3(x) = \begin{vmatrix} x-\beta_0 & -\delta_1' & 0 \\ -\gamma_1' & x-\beta_1 & -\delta_2' \\ 0 & -\gamma_2' & x-\beta_2 \end{vmatrix}.$$

Expansion of this determinant along the last row leads to

$$\begin{aligned} p_{2+1}(x) &= (-\gamma_2')\,(-1)^{3+2} \begin{vmatrix} x-\beta_0 & 0 \\ -\gamma_1' & -\delta_2' \end{vmatrix} + (x-\beta_2)\,(-1)^{3+3} \begin{vmatrix} x-\beta_0 & -\delta_1' \\ -\gamma_1' & x-\beta_1 \end{vmatrix} \\ &= \gamma_2'\,[(x-\beta_0)\,(-\delta_2')] + (x-\beta_2)\,p_2(x) \\ &= (x-\beta_2)\,p_2(x) - \gamma_2'\,\delta_2'\,(x-\beta_0) \\ &= (x-\beta_2)\,p_2(x) - \gamma_2\,(x-\beta_1) \\ &= (x-\beta_2)\,p_2(x) - \gamma_2\,(x-\beta_{2-1}) \end{aligned}$$

so that the recursion formula $p_{j+1}(x) = (x-\beta_j)\,p_j(x) - \gamma_j\,p_{j-1}(x)$ in (2.1) is proven for $j=2$.
Additional base case ($j=3$): For $j=3$, we obtain

$$p_{3+1}(x) = p_4(x) = \begin{vmatrix} x-\beta_0 & -\delta_1' & 0 & 0 \\ -\gamma_1' & x-\beta_1 & -\delta_2' & 0 \\ 0 & -\gamma_2' & x-\beta_2 & -\delta_3' \\ 0 & 0 & -\gamma_3' & x-\beta_3 \end{vmatrix}.$$

Expansion of this determinant along the last row leads to

$$\begin{aligned} p_{3+1}(x) &= (-\gamma_3')\,(-1)^{4+3} \begin{vmatrix} x-\beta_0 & -\delta_1' & 0 \\ -\gamma_1' & x-\beta_1 & 0 \\ 0 & -\gamma_2' & -\delta_3' \end{vmatrix} \\ &\quad + (x-\beta_3)\,(-1)^{4+4} \underbrace{\begin{vmatrix} x-\beta_0 & -\delta_1' & 0 \\ -\gamma_1' & x-\beta_1 & -\delta_2' \\ 0 & -\gamma_2' & x-\beta_2 \end{vmatrix}}_{=p_3(x)} \\ &= \gamma_3' \left\{ (-\gamma_2')\,(-1)^{3+1} \begin{vmatrix} x-\beta_0 & 0 \\ -\gamma_1' & 0 \end{vmatrix} + (-\delta_3')\,(-1)^{3+3} \begin{vmatrix} x-\beta_0 & -\delta_1' \\ -\gamma_1' & x-\beta_1 \end{vmatrix} \right\} \\ &\quad + (x-\beta_3)\,p_3(x) \\ &= \gamma_3'\,[\,0 + (-\delta_3')\,p_2(x)\,] + (x-\beta_3)\,p_3(x) = (x-\beta_3)\,p_3(x) - \gamma_3'\,\delta_3'\,p_2(x) \\ &= (x-\beta_3)\,p_3(x) - \gamma_3\,(x-\beta_{3-1}) \end{aligned}$$

so that the recursion formula $p_{j+1}(x) = (x-\beta_j)\,p_j(x) - \gamma_j\,p_{j-1}(x)$ in (2.1) is proven for $j=3$.
Induction step: Assume that the recursion formula

$$p_{j+1}(x) = (x-\beta_j)\,p_j(x) - \gamma_j\,p_{j-1}(x)$$

is proven for an index $j \geq 4$ (for $j \in \{1,2,3\}$, we have already checked the validity). Then, we have to show

$$p_{j+2}(x) = (x-\beta_{j+1})\,p_{j+1}(x) - \gamma_{j+1}\,p_j(x).$$

Now, according to (3.8), we have

$$p_{j+2}(x) = \begin{vmatrix} x-\beta_0 & -\delta_1' & & & & & 0 \\ -\gamma_1' & x-\beta_1 & -\delta_2' & & & & 0 \\ & -\gamma_2' & x-\beta_2 & -\delta_3' & & & 0 \\ & & \ddots & \ddots & \ddots & & \vdots \\ & & & -\gamma_{j-1}' & x-\beta_{j-1} & -\delta_j' & 0 \\ & & & & -\gamma_j' & x-\beta_j & -\delta_{j+1}' \\ 0 & 0 & 0 & \cdots & 0 & -\gamma_{j+1}' & x-\beta_{j+1} \end{vmatrix}.$$

Expansion along the last row gives

$$
rclp_{j+2}(x) = (-\gamma'_{j+1})\,(-1)^{(j+2)+(j+1)} \begin{vmatrix} x-\beta_0 & -\delta'_1 & & & 0 \\ -\gamma'_1 & x-\beta_1 & -\delta'_2 & & 0 \\ & -\gamma'_2 & x-\beta_2 & -\delta'_3 & 0 \\ & \ddots & \ddots & \ddots & \\ & & -\gamma'_{j-1} & x-\beta_{j-1} & 0 \\ 0 & 0 & 0 & -\gamma'_j & -\delta'_{j+1} \end{vmatrix}
$$

$$
+ (x-\beta_{j+1})\,(-1)^{(j+2)+(j+2)} \underbrace{\begin{vmatrix} x-\beta_0 & -\delta'_1 & & & \\ -\gamma'_1 & x-\beta_1 & -\delta'_2 & & \\ & -\gamma'_2 & x-\beta_2 & -\delta'_3 & \\ & \ddots & \ddots & \ddots & \\ & & -\gamma'_{j-1} & x-\beta_{j-1} & -\delta'_j \\ & & 0 & -\gamma'_j & x-\beta_j \end{vmatrix}}_{=p_{j+1}(x)}
$$

$$
= \gamma'_{j+1} \left\{ (-\gamma'_j)\,(-1)^{(j+1)+j} \begin{vmatrix} x-\beta_0 & -\delta'_1 & 0 & 0 & 0 \\ -\gamma'_1 & x-\beta_1 & -\delta'_2 & 0 & 0 \\ & -\gamma'_2 & x-\beta_2 & -\delta'_3 & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & -\delta'_{j-1} & 0 \end{vmatrix} \right.
$$

$$
\left. + (-\delta'_{j+1})\,(-1)^{(j+1)+(j+1)} \begin{vmatrix} x-\beta_0 & -\delta'_1 & & & 0 \\ -\gamma'_1 & x-\beta_1 & -\delta'_2 & & 0 \\ & -\gamma'_2 & x-\beta_2 & -\delta'_3 & 0 \\ & \ddots & \ddots & \ddots & \\ & & -\gamma'_{j-2} & x-\beta_{j-2} & -\delta'_{j-1} \\ & & 0 & -\gamma'_{j-1} & x-\beta_{j-1} \end{vmatrix} \right\}
$$

$$
\begin{aligned}
&+ (x-\beta_{j+1})\,p_{j+1}(x) \\
&= \gamma'_{j+1}\,\{\,\gamma'_j \cdot 0 - \delta'_{j+1}\,p_j(x)\,\} + (x-\beta_{j+1})\,p_{j+1(x)} \\
&= (x-\beta_{j+1})\,p_{j+1}(x) - \gamma_{j+1}\,p_j(x)
\end{aligned}
$$

so that, indeed, the recursion formula $p_{j+2}(x) = (x-\beta_{j+1})\,p_{j+1}(x) - \gamma_{j+1}\,p_j(x)$ in (2.1) is proven, which was to be shown. This ends the proof of Theorem 3.1. □

From Theorem 3.1, we obtain the following important consequences.

**Consequence 3.1** (Simple sufficient condition for eigenvalue-separation properties of real tridiagonal matrices). *Let the real tridiagonal matrix $T$ be given by (3.2) with elements $a_i$ satisfying $a_0 = a_{N+1} = 0$ and real $a_i,\ i = 1,\dots,N$ as well as $d_0 = d_{N+1} = 0$ and real elements $d_i,\ i = 1,\dots,N$ with the condition that $a_i\,d_i > 0,\ i = 1,\dots,N$. Since the polynomials defined in (3.8) fulfill the recursion formulae in (2.1), the eigenvalues of matrix $T$ are real as well as pairwise distinct, and they can be arranged such that the properties (P1) - (P4) in Lemma 2.1 on Sturm sequences are valid.*

*Proof.* We need only mention that, in this special case, $\gamma_j = \gamma_j'\,\delta_j' = a_i\,d_j > 0,\ j = 0,\ldots,m-1 = N$. ☐

**Consequence 3.2** (Simple sufficient condition for eigenvalue-separation properties of symmetric tridiagonal matrices). *Let the symmetric tridiagonal matrix $T$ be given by (3.2) with $a_0 = a_{N+1} = 0$ satisfying elements $d_i = a_i \neq 0,\ j = 1,\ldots,N$. Since the polynomials defined in (3.8) fulfill the recursion formulae in (2.1), the eigenvalues of matrix $T$ are real as well as pairwise distinct, and they can be arranges such that the properties (P1) - (P4) in Lemma 2.1 on Sturm sequences are valid.*

*Proof.* We need only mention that, in this case, $\gamma_j = {\gamma_j'}^2 = a_i^2 > 0,\ j = 0,\ldots,m-1 = N$. ☐

**Remark 3.1.**

*(R1) This is much more than the statement that the eigenvalues of $T$ are distinct.*

*(R2) The obtained result holds, in particular, in the case when $d_i = a_i,\ i = 1,\ldots,N$ and $c_i = b_i - a_i - a_{i-1},\ i = 1,\ldots,N+1$ with real elements $b_i$ studied in* [2, p.27]*. Whereas Grünbaum assumes that the pertinent eigenvalues are distinct, in this paper, under mild conditions it could be proven that they are distinct.*

*(R3) We remind the reader that the eigenvectors of symmetric matrices with distinct eigenvalues are pairwise orthogonal.*

## 4. Numerical Example

In this section, we present two numerical examples. The first one treats a real nonsymmetric tridiagonal matrix and the second one a symmetric tridiagonal matrix.

4.1. **Numerical Example 1.** As the first numerical example, we choose

$$T = T_4 = \begin{bmatrix} 0 & -2 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -1 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

so that $m = 4$ as well as $a_i = -1,\ i = 1,2,3 (= N)$ and $d_i = -2,\ i = 1,2,3 (= N)$ or $\gamma_j' = -1,\ j = 1,2,3 (= m-1)$ and $\delta_j' = -2,\ j = 1,2,3 (= m-1)$ as well as $c_i = 0,\ i = 1,2,3,4 (= N+1)$. Further, $\beta_j = 0,\ j = 0,1,2,3 (= m-1)$, $\gamma_j = \gamma_j'\,\delta_j' = 2 > 0,\ j = 1,2,3 (= m-1)$. Let $E_i \in \mathbb{R}^{i\times i}$ be the $i \times i$-identity matrix for $i = 2,3,4$. Herewith,

$$p_4(x) = (x\,E_4 - T_4).$$

Further, let

$$T_3 = \begin{bmatrix} 0 & -2 & 0 \\ -1 & 0 & -2 \\ 0 & -1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

and

$$T_2 = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Then,

$$p_j(x) = (x\,E_j - T_j), \quad j = 4,3,2.$$

For the eigenvalues

$$\lambda_{j,k},\ k = 1,\ldots,j$$

of $T_j$ for $j = (m = 4), 3, 2$, we obtain

$$\lambda_4 := \begin{bmatrix} \lambda_{4,1} \\ \lambda_{4,2} \\ \lambda_{4,3} \\ \lambda_{4,4} \end{bmatrix} = \begin{bmatrix} 2.2882 \\ 0.8740 \\ -0.8740 \\ -2.2882 \end{bmatrix}, \quad \lambda_3 := \begin{bmatrix} \lambda_{3,1} \\ \lambda_{3,2} \\ \lambda_{3,3} \end{bmatrix} = \begin{bmatrix} 2.0000 \\ -0.0000 \\ -2.0000 \end{bmatrix}$$

and

$$\lambda_2 := \begin{bmatrix} \lambda_{2,1} \\ \lambda_{2,2} \end{bmatrix} = \begin{bmatrix} 1.4142 \\ -1.4142 \end{bmatrix},$$

where the numbering of the vector components is such that

$$-\infty < \lambda_{j,j} < \lambda_{j,j-1} < \cdots < \lambda_{j,2} < \lambda_{j,1} < +\infty$$

for $j = 4, 3, 2$. Therefore, (P1) is satisfied.

Further, we check (P2). The cases $j = 4$ and $j = 1$ are left to the reader. For $j = 3$, we obtain

$$(P2)_{k=1}: \quad p_3(\lambda_{3,1}) = 0\,, \quad p_4(\lambda_{3,1})\,p_2(\lambda_{3,1}) = -8 < 0,$$

$$(P2)_{k=2}: \quad p_3(\lambda_{3,2}) = 0\,, \quad p_4(\lambda_{3,2})\,p_2(\lambda_{3,2}) = -8 < 0,$$

$$(P2)_{k=3}: \quad p_3(\lambda_{3,3}) = 0\,, \quad p_4(\lambda_{3,3})\,p_2(\lambda_{3,3}) = -8 < 0.$$

Moreover, for $j = 2$,

$$(P2)_{k=1}: \quad p_2(\lambda_{2,1}) = 0, \quad p_3(\lambda_{2,1})\,p_1(\lambda_{2,1}) = -4 < 0,$$

$$(P2)_{k=2}: \quad p_2(\lambda_{2,2}) = 0, \quad p_3(\lambda_{2,2})\,p_1(\lambda_{2,2}) = -4 < 0.$$

Thus, (P2) is numerically underpinned. Next, we check (P3). The cases $j = 4$ and $j = 1$ are left to the reader. For $j = 3$, we obtain

$$(P3)_{k=1}: \quad (-1)^k\,p_3(x) = \underbrace{(-1)^k}_{<0}\,\underbrace{(x - \lambda_{3,1})}_{<0}\,\underbrace{(x - \lambda_{3,2})}_{>0}\,\underbrace{(x - \lambda_{3,3})}_{>0} = 3 > 0$$

for

$$x = \frac{\lambda_{3,2} + \lambda_{3,1}}{2} \doteq 1$$

so that

$$\lambda_{3,2} < x < \lambda_{3,1}.$$

Further,

$$(P3)_{k=2}: \quad (-1)^k\,p_3(x) = \underbrace{(-1)^k}_{>0}\,\underbrace{(x - \lambda_{3,1})}_{<0}\,\underbrace{(x - \lambda_{3,2})}_{<0}\,\underbrace{(x - \lambda_{3,3})}_{>0} = 3 > 0$$

for

$$x = \frac{\lambda_{3,3} + \lambda_{3,2}}{2} = -1$$

so that

$$\lambda_{3,3} < x < \lambda_{3,2}.$$

Moreover, for $j = 2$,

$$(P3)_{k=1}: \quad (-1)^k\,p_2(x) = \underbrace{(-)^k}_{<0}\,\underbrace{(x - \lambda_{2,1})}_{<0}\,\underbrace{(x - \lambda_{2,2})}_{>0} = 2 > 0$$

for

$$x = \frac{\lambda_{2,2} + \lambda_{2,1}}{2} = 0$$

so that

$$\lambda_{2,2} < x < \lambda_{2,1}.$$

On the whole, (P3) is numerically underpinned. Finally, we check (P4). The cases $j = 1$ and $j = 4$ are left to the reader. For $j = 3$, we obtain

$$(P4)_{k=1}: \quad \lambda_{3,1} \doteq 2.0000 < 2.2882 \doteq \lambda_{4,1} < \lambda_{3,0} = +\infty,$$

and $\lambda_{4,1}$ is the only component of $\lambda_4$ satisfying the above inequality. Further,

$$(P4)_{k=2}: \quad \lambda_{3,2} \doteq -0.0000 < 0.8740 \doteq \lambda_{4,2} < \lambda_{3,1} \doteq 2.0000,$$

and $\lambda_{4,2}$ is the only component of $\lambda_4$ satisfying the above inequality. Finally,

$$(P4)_{k=3}: \quad \lambda_{3,3} \doteq -2.0000 < -0.8740 \doteq \lambda_{4,3} < \lambda_{3,2} \doteq -0.0000,$$

and $\lambda_{4,3}$ is the only component of $\lambda_4$ satisfying the above inequality.

For $j = 2$, we obtain

$$(P4)_{k=1}: \quad \lambda_{2,1} \doteq 1.4142 < 2.0000 \doteq \lambda_{3,1} < \lambda_{2,0} = +\infty,$$

and $\lambda_{3,1}$ is the only component of $\lambda_3$ satisfying the above inequality. Further,

$$(P4)_{k=2}: \quad \lambda_{2,2} \doteq -1.4142 < -0.0000 \doteq \lambda_{3,2} < \lambda_{2,1} \doteq 1.4142,$$

and $\lambda_{3,2}$ is the only component of $\lambda_3$ satisfying the above inequality.

Therefore, (P4) is numerically underpinned.

**Remark 4.2.** *The computations of the eigenvalues were done by the Matlab routine eig.m.*

4.2. **Numerical Example 2.** As the second numerical example, we choose

$$T = T_4 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

so that $m = 4$ as well as $a_i = -1 \neq 0,\ i = 1,2,3(= N)$ or $\gamma_j' = -1,\ j = 1,2,3(= m-1)$ and $c_i = 0,\ i = 1,2,3,4(= N+1)$. Further, $\beta_j = 0,\ j = 0,1,2,3(= m-1)$, $\gamma_j = {\gamma_j'}^2 = 1 > 0,\ j = 1,2,3(= m-1)$. Let $E_i \in \mathbb{R}^{i\times i}$ be the $i \times i$-identity matrix for $i = 2,3,4$. Herewith,

$$p_4(x) = (x\,E_4 - T_4).$$

Further, let

$$T_3 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

and

$$T_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Then,

$$p_j(x) = (x\,E_j - T_j), \quad j = 4,3,2.$$

For the eigenvalues

$$\lambda_{j,k},\ k = 1,\ldots,j$$

of $T_j$ for $j = (m = 4), 3, 2$, we obtain

$$\lambda_4 := \begin{bmatrix} \lambda_{4,1} \\ \lambda_{4,2} \\ \lambda_{4,3} \\ \lambda_{4,4} \end{bmatrix} = \begin{bmatrix} 1.6180 \\ 0.6180 \\ -0.6180 \\ -1.6180 \end{bmatrix}, \quad \lambda_3 := \begin{bmatrix} \lambda_{3,1} \\ \lambda_{3,2} \\ \lambda_{3,3} \end{bmatrix} = \begin{bmatrix} 1.4142 \\ 0.0000 \\ -1.4142 \end{bmatrix}$$

and

$$\lambda_2 := \begin{bmatrix} \lambda_{2,1} \\ \lambda_{2,2} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

where the numbering of the vector components is such that

$$-\infty < \lambda_{j,j} < \lambda_{j,j-1} < \cdots < \lambda_{j,2} < \lambda_{j,1} < +\infty$$

for $j = 4, 3, 2$. Therefore, (P1) is satisfied. Further, we check (P2). The cases $j = 4$ and $j = 1$ are left to the reader. For $j = 3$, we obtain

$$(P2)_{k=1}: \quad p_3(\lambda_{3,1}) = 0\,, \quad p_4(\lambda_{3,1})\, p_2(\lambda_{3,1}) = -1 < 0,$$

$$(P2)_{k=2}: \quad p_3(\lambda_{3,2}) = 0\,, \quad p_4(\lambda_{3,2})\, p_2(\lambda_{3,2}) = -1 < 0,$$

$$(P2)_{k=3}: \quad p_3(\lambda_{3,3}) = 0\,, \quad p_4(\lambda_{3,3})\, p_2(\lambda_{3,3}) = -1 < 0.$$

Moreover, for $j = 2$,

$$(P2)_{k=1}: \quad p_2(\lambda_{2,1}) = 0, \quad p_3(\lambda_{2,1})\, p_1(\lambda_{2,1}) = -1 < 0,$$

$$(P2)_{k=2}: \quad p_2(\lambda_{2,2}) = 0, \quad p_3(\lambda_{2,2})\, p_1(\lambda_{2,2}) = -1 < 0.$$

Thus, (P2) is numerically underpinned. Next, we check (P3). The cases $j = 4$ and $j = 1$ are left to the reader. For $j = 3$, we obtain

$$(P3)_{k=1}: \quad (-1)^k\, p_3(x) = \underbrace{(-1)^k}_{<0}\, \underbrace{(x - \lambda_{3,1})}_{<0}\, \underbrace{(x - \lambda_{3,2})}_{>0}\, \underbrace{(x - \lambda_{3,3})}_{>0} \doteq 1.0607 > 0$$

for

$$x = \frac{\lambda_{3,2} + \lambda_{3,1}}{2} \doteq 0.7071$$

so that

$$\lambda_{3,2} < x < \lambda_{3,1}.$$

Further,

$$(P3)_{k=2}: \quad (-1)^k\, p_3(x) = \underbrace{(-1)^k}_{>0}\, \underbrace{(x - \lambda_{3,1})}_{<0}\, \underbrace{(x - \lambda_{3,2})}_{<0}\, \underbrace{(x - \lambda_{3,3})}_{>0} \doteq 1.0607 > 0$$

for

$$x = \frac{\lambda_{3,3} + \lambda_{3,2}}{2} \doteq -0.7071$$

so that

$$\lambda_{3,3} < x < \lambda_{3,2}.$$

Moreover, for $j = 2$,

$$(P3)_{k=1}: \quad (-1)^k\, p_2(x) = \underbrace{(-)^k}_{<0}\, \underbrace{(x - \lambda_{2,1})}_{<0}\, \underbrace{(x - \lambda_{2,2})}_{>0} = 1 > 0$$

for

$$x = \frac{\lambda_{2,2} + \lambda_{2,1}}{2} = 0$$

so that

$$\lambda_{2,2} < x < \lambda_{2,1}.$$

On the whole, (P3) is numerically underpinned. Finally, we check (P4). The cases $j = 1$ and $j = 4$ are left to the reader. For $j = 3$, we obtain

$$(P4)_{k=1}: \quad \lambda_{3,1} \doteq 1.4142 < 1.6180 \doteq \lambda_{4,1} < \lambda_{3,0} = +\infty,$$

and $\lambda_{4,1}$ is the only component of $\lambda_4$ satisfying the above inequality. Further,

$$(P4)_{k=2}: \quad \lambda_{3,2} \doteq 0.0000 < 0.6180 \doteq \lambda_{4,2} < \lambda_{3,1} \doteq 1.4142,$$

and $\lambda_{4,2}$ is the only component of $\lambda_4$ satisfying the above inequality. Finally,

$$(P4)_{k=3}: \quad \lambda_{3,3} \doteq -1.4142 < -0.6180 \doteq \lambda_{4,3} < \lambda_{3,2} \doteq 0.0000,$$

and $\lambda_{4,3}$ is the only component of $\lambda_4$ satisfying the above inequality.

For $j = 2$, we obtain

$$(P4)_{k=1}: \quad \lambda_{2,1} = 1 < 1.4142 \doteq \lambda_{3,1} < \lambda_{2,0} = +\infty,$$

and $\lambda_{3,1}$ is the only component of $\lambda_3$ satisfying the above inequality. Further,

$$(P4)_{k=2}: \quad \lambda_{2,2} = -1 < 0.0000 \doteq \lambda_{3,2} < \lambda_{2,1} = 1,$$

and $\lambda_{3,2}$ is the only component of $\lambda_3$ satisfying the above inequality. Therefore, (P4) is numerically underpinned.

**Remark 4.3.** *Again, the computations of the eigenvalues were done by the Matlab routine eig.m.*

## 5. Conclusion

In this paper, as the main new result, we could show that the eigenvalues of a real tridiagonal matrix have the eigenvalue-separation properties (P1) - (P4) of Lemma 2.1 provided that the products of corresponding entries above and below the diagonal are positive. In the special case of a symmetric tridiagonal matrix, this turns into the simple sufficient condition that all entries above and below the diagonal are different from zero. This applies, in particular, to the special matrix studied by Grünbaum who assumed that the eigenvalues are distinct whereas here this could be proven. The eigenvalue-separation properties are much more than the property that its eigenvalues are just distinct. A further interesting point is that the elements $\gamma_j$ in (2.1) are independent of the diagonal entries $\beta_j$ so that the sufficient condition $\gamma_j > 0$ depends only on the entries under and above the diagonal, not on the diagonal entries.

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