Iğdır Üniversitesi Fen Bilimleri Enstitüsü Dergisi, 13(4), 2915-2925, 2023
Journal of the Institute of Science and Technology, 13(4), 2915-2925, 2023
ISSN: 2146-0574, eISSN: 2536-4618
Mathematics **DOI: 10.21597/jist.1330915**
Research Article
Received: 21.07.2023 **Accepted: 22.08.2023**

To Cite: Öz, H. R. (2023). Extension of Pick's Theorem to Spherical Geometry using Girard's Theorem. *Journal of the Institute of Science and Technology*, 13(4), 2915-2925.

Extension of Pick's Theorem to Spherical Geometry using Girard's Theorem

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Highlights:
- Combination of Pick's and Girard's theorems for the area of spherical polygon
- Common statements of spherical excess and lattice points
- Extending to a more general theorem

Keywords:
- Pick's theorem
- Girard's theorem
- Spherical lattice
- Spherical geometry
- Planar geometry

ABSTRACT:

In this article, Pick's theorem is extended to three-dimensional bodies with two-dimensional surfaces, namely spherical geometry. The equation for the area of a polygon consisting of equilateral spherical triangles is obtained by combining Girard's theorem used to find area of any spherical triangle and Pick's theorem used to find area of a simple polygon with lattice point vertices in Euclidian geometry. Vertices of the polygon are represented by integer points. In this way, an equation to find area of a spherical polygon is presented. This equation could give an idea to be applied on cylindrical surfaces, hyperbolic geometry and more general surfaces. The theorem proposed in this article which is the extension of Pick's theorem using Girard's theorem seems to be a special case of a more general theorem.

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INTRODUCTION

This paper is about finding area of a spherical polygon. The polygon consists of equilateral spherical triangles. Areas of simple (containing no holes) polygons whose vertices are lattice points represented by integers in a plane can be calculated using Pick's theorem. The theorem considers the number of points at the boundary and interior of the polygon and it is valid for planes only. Although there are some studies for three-dimensional lattice points, the equation was not applied with general validity. The area of a single spherical triangle can be calculated using Girard's theorem. Although there are other methods for calculating area of any spherical polygo the method presented here is simple and more fundamental due to usage of Pick's theorem. This study unifies two different theorems and presents a way of calculating area of a simple spherical polygon consisting of equilateral triangles in terms of lattice points and excess angle. When the polygon is cut, the equation still holds due to nature of Pick's theorem and this property makes our proposal more general and usable for polygons containing arbitrary spherical triangles.

The pioneering work was done by Pick (1899). He proposed an equation for finding areas of planar polygons and intended to establish number theory on a geometric basis. He mentioned in its article that "…a more modest goal: an attempt is made to place the elements of number theory on a geometric basis from the outset (…bescheideneres ziel: es wird der Versuch gemacht, die elemente der Zahlenteorie von vorn herein auf geometrische basis zu stellen"). This claim could have been found exaggerous, but his theorem has been appreciated greatly and extensively due to its simplicity and beauty. The theorem was mainly used in plane polygons and its theory became even deeper.

Reeve (1958) tried to extend Pick's theorem into three dimensions. Honsberger (1970) proved the theorem using a primitive triangle having area of 1/2. Triangulation is a method used in geodesy studies since the past. Gaskell et al. (1976) mentioned that Pick's theorem is more than area concept and actually a combinatorial result in topology. According to them, the lattice points may have irregular intervals and mentioned that three arcs can bound a simple closed curve without crossing. Hadwiger and Wills (1976) gave a general formula and used it for translations. Liu (1979) showed that Pick's theorem is topologically equivalent to the formula of Euler (edges, vertices and faces). Varberg (1985) extended Pick's theorem for a general lattice polygon. Scott (1987) presented some formulas from the literature for regular polygons, convex polygons and nonsimple polygons using Pick's theorem. Grunbaum and Shephard (1993) presented an extension to general lattice polygons with multiple vertices using oriented polygons.

Now we present some information about Girard's theorem. Several equations and theorems on spherical geometry were summarized in the great work of Todhunter (1886). Miller (1943) published seven lessons in the history of mathematics mentioning that T. Harriot noted a theorem about area of a spherical triangle in 1603, but A. Girard proved it in 1629. He also gave information about some researchers like Menelaus (70-140 CE), Nasir Eddin (1201-1274), F. Vieta (1540-1603), F. Maurolico (1558), and A. M. Legendre (1752-1853). Bevis and Cambareri (1987) computed the area of a spherical polygon having arbitrary shape using summation equation and numbering its vertices in two directions. Brooks and Strantzen (2005) used a spherical triangle of area π to tile on a sphere and proved that an isosceles tetrahedron can be obtained. These equations directly used spherical geometric equations of Girard's theorem. There are much more studies to mention but to concentrate on what we aim, we avoid lengthy surveys and omit some details. However, spherical geometry is not only a subject of mathematics, astronomy, robotics, quantum mechanics or any close subjects, but also it is a subject of protein structure on viruses, etc. Finding orientation of various proteins on viruses or other structures might benefit from spherical geometry studies. Some examples: Sandeep et al. (2016) discussed the obese-tetrahedral having four pure quantum states. Spherical trigonometry and the theorem we proposed in this article can be helpful in quantum state studies, especially on Bloch sphere on which effective areas can be calculated. Zandi et al. (2004) presented minimum energy structures on viruses using Monte Carlo simulation. Sarkar and Rashid (2022) discussed the number of proteins on corona virus modelled by a sphere showing points and great circular arcs without considering any geometric equations. Again, the theorem we proposed can be helpful in locating the proteins under some constraints and calculate areas around them, which might be important for protein locations.

In this study, we state two theorems by combining Pick's theorem for planes and Girard's theorem for spherical geometry and a line theorem. Pick's theorem uses lattice points, their numbers and calculates the area of a planar polygon. It is simple, elegant and insightful. Girard's theorem uses summation of angles of a triangle on a sphere and subtracts π to calculate the area of any triangle. This equation can be extended to find any polygon area by dividing it into triangles. For a polygon formed by tessellation of single equilateral triangle on a unit sphere, we can easily find the area of the polygon by combining Pick's theorem and Girard's theorem. The new theorem uses both lattice point properties and primitive triangle thus its angle. Although we assumed equilateral spherical triangles, we hope that information can be used for arbitrary spherical triangles in further studies. Because equilateral triangle assumption limits slightly usability and range of the equation proposed in this article. The polygon can be cut and irregular polygons can be obtained, but the theorem still holds. Usage of only Girard's theorem still needs new angles formed by cut, but the new theorem just uses the lattice points once the polygon was formed at the beginning. This theorem can be extended to hyperbolic geometry and a more general theorem can be obtained. Since we combined two theorems and extended to spherical geometry, both Pick's theorem and Girard's theorem becomes special cases. That is why our theorem although it is more general, should be again a special case since there are other geometries and dimensions as discussed above.

MATERIALS AND METHODS

In this section, two old theorems will be repeated and two new theorems will be introduced.

Plane Geometry

Theorem 1: Pick's Theorem (Pick, 1899)

Area of a simple (no holes) polygon with lattice points (Fig. 1) in plane is (Eq. 1)

$$S = \left(i + \frac{b}{2} - 1\right) \quad (1)$$

where i and b are the number of points (vertices) in the interior and on the boundary of the polygon. In Fig 1. black dots denote lattice points, blue lines denote the polygon obtained. Here there are 12 points on the boundary and 3 points interior.

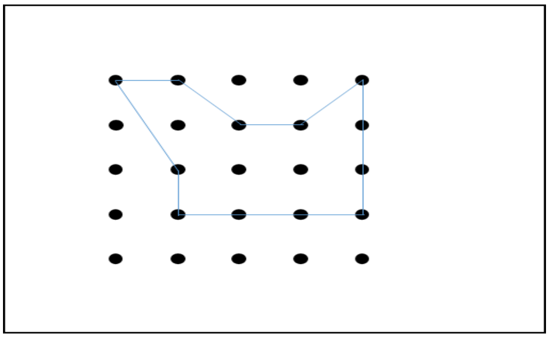

Figure 1. Pick's Theorem

Simple polygon means that it does not intersect itself and there are no holes in polygon. The coordinates of vertices are integers that are called lattice points that arise in different areas. This equation has been applied to planes only since it was established. There are several studies generalizing the equations for translations (Hadwiger and Wills, 1976), using oriented polygons (Grunbaum and Shephard, 1993), showing that Pick's theorem is topologically equivalent to the formula of Euler in terms of edges, vertices and faces (Liu, 1979). There has been several studies to extend Pick's theorem into volume of three-dimensional lattices but none of them has achieved.

Spherical geometry

Now one can consider a spherical arbitrary triangle as shown in Figure 2 and apply Girard's theorem to find its area (Eq. 2).

Theorem 2: Girard's Theorem

Area of any spherical triangle is given by in terms of spherical excess and radius of the sphere using the following formula

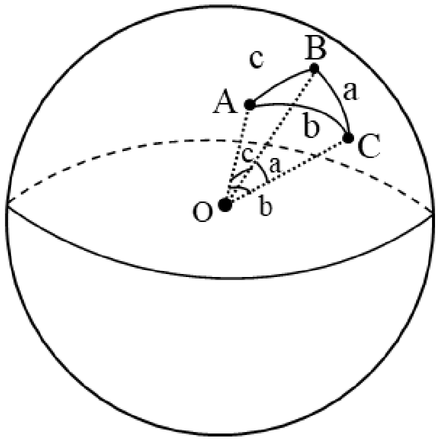

Figure 2. Girard's Theorem

$$S = (A + B + C - \pi)R^2 \tag{2}$$

where A, B and C are the internal angles of the spherical triangle and R is the radius, and O is the center. Lowercase letters are arc lengths of the triangle. If we consider a unit sphere then $R = 1$ and the area is directly equal to $A + B + C - \pi$ which is named as spherical excess E, all angles are measured in radians. This theorem seems to have been noted for the first time by T. Harriot (1603) and was proved by A. Girard in 1629 (Miller, 1943). Arc lengths a, b and c are also in radians since the radius is 1. The spherical excess can also be found from L'Huilier's theorem using arc lengths (Todhunter, 1886, Art. 101-103). Then one can use spherical trigonometric relations to find the unknown quantities.

Another equation for calculating polygon area composed of arbitrary angle angles, if the angles are given is (Todhunter, 1886, Art. 99).

$$S = \sum_{n=1}^{N} \theta_n - (N - 2)\pi \tag{3}$$

where θ_n is the angle interior to the polygon and N is the number of edges. For arbitrary radius, the equation should be multiplied by R^2. The number of spherical triangles is always $N - 2$ (Bevis & Cambareri, 1987). Eq. 3 is true as long as the polygon can be decomposed into triangles having each angle less than π (Todhunter, 1886, Art. 99).

Now let us assume a polygon on a sphere (Fig. 3), and let a polygon was formed by equilateral triangles. This assumption is needed for lattice point approach so that the vertices on the sphere behave as integers. This is a spherical lattice. Throughout the study, the sphere will be assumed as a unit sphere.

Theorem 3: (Girard-Pick-Öz, 2023).

The area of equilateral triangular polygon with lattice points on a sphere (spherical lattice) is (Fig. 3)

$$S = 2E\left(i + \frac{b}{2} - 1\right)R^2 \tag{4}$$

where E is the spherical excess $(= 3A - \pi)$, i and b are the number of interior and boundary points, respectively. The term arc was discussed in Gaskell et al. (1976) just for the planar case to define a simple curve using three non-intersecting arcs emphasizing the triangulation that is applied to the spherical surface in our study. Therefore, the important thing is the primitive triangle (Gaskell et al, 1976; Honsberger, 1970) and then the rest can be formed by copying it. In a plane, it is easy, since the plane in infinite. On a sphere, it is not easy due to finite area. Tessellation limits should be kept in mind as discussed in the following section.

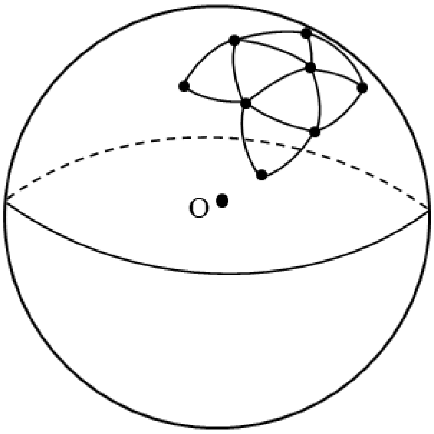

Figure 3. A Polygon Consisting of Equilateral Triangles on a Sphere

There is an important question. Why do we have $2E$ in front of Eq. 4? Twice the spherical excess shows an area of a shape of four sides. The shape can be formed by two adjacent equilateral triangles. That means, two adjacent equilateral triangles with a common side on a sphere is the projection of a parallelogram tangent to the sphere at the middle, so that the projected area is $2E$. A square or a rectangle cannot be projected such that all sides are equal, if the diagonals are also equal. Let us call it the 'parallelogram projected onto a sphere'. Even though all arcs will intersect each other, it is possible to form a shape having four arcs. For our case, all arcs have equal length, for the unit sphere equal angles (Fig.4).

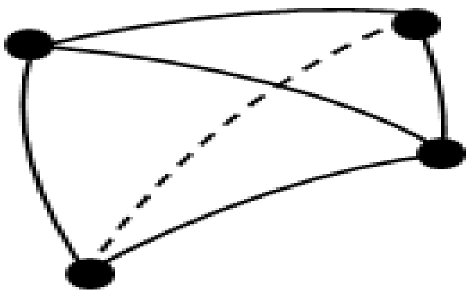

Figure 4. A Parallelogram Consisting of Equilateral Triangles Projected Onto a Sphere

In Fig. 4, all edges are formed by corresponding great circles. All arcs have equal lengths (or angles). All internal angles are equal. This is the spherical excess used in Theorem 3. It tessellates on the sphere for even numbers of triangles completely and changes its orientation. What exactly is the shape of the original parallelogram? This can be calculated using projective geometry backwards. It is for more advanced researchers than the author of this article. However, a simple idea can be given by just doing preliminary analysis. The planar parallelogram must have equal sides and diagonals such that its projection onto sphere has again equal arcs including its diagonals. Alternatively, just an equilateral planar triangle is projected onto sphere giving a spherical triangle just having $1E$. The lengths of the original equilateral planar triangle or planar parallelogram corresponding to the projected one can be calculated for each case using projective geometry tools. Therefore, we place a lattice on a sphere such that the arc angle (length for the unit sphere) between any adjacent vertices is constant. This is achieved by equilateral triangles on the sphere. That guarantees lattice assumption. Lattice area equation comes from Pick's theorem and curvature effect comes from Girard's theorem. Since the smallest lattice is a spherical triangle, we multiply by 2.

We have two equations giving the same answers. Final operation will be against both of them. Equating Eq. 3 and 4 each other, for the unit sphere we get

$$\sum_{n=1}^{N} \theta_n - (N-2)\pi = 2E\left(i + \frac{b}{2} - 1\right) \tag{5}$$

and in open form as follows

$$\sum_{n=1}^{N} \theta_n - (N-2)\pi = 2(3A-\pi)\left(i + \frac{b}{2} - 1\right) \tag{6}$$

θ_n and A can be the same for some cases, but not in general. The number of edges is equal to number of boundary points. Keeping spherical excess in short, the equation is

$$\sum_{n=1}^{b} \theta_n - (b-2)\pi = 2E\left(i + \frac{b}{2} - 1\right) \tag{7}$$

We cannot proceed further with this equation but some researchers might do.

Arbitrary geometry

As mentioned above, up to now there is no analog of Pick's theorem to calculate the volume of three or higher dimensional bodies. Moving forward is locked for the time being, but moving backward is possible. Now comes the last theorem. It is very simple, but necessary to close the gap.

Theorem 4: (Pick-Öz, 2023).

The length of a curve with lattice points in space is equal to (Fig. 3)

$$L = \begin{cases} i + b - 1 & \text{for an open curve} \\ i + b & \text{for a closed curve} \end{cases} \tag{8}$$

where i and b are the number of interior and boundary (or end) points, respectively. Eq. 8 seems to be almost trivial but its resemblance to Pick's theorem for planes is attractive. Extending Eqs. 1 and 8 into three dimensions is possible but some attempts made during this study were unsuccessful similar to the fate of previous studies. Eq. 8 can be used for curves and lines provided that, the arc length or distance between adjacent lattice points is constant. Curvature can be used to arrange lattices on space curve.

Comparison of Eqs. 3 and 4 gives an idea about similarities and differences between them. Eq. 4 is used for any simple polygon on a sphere if the angles and number of edges are known. Eq. 3 can be used for any simple polygon too, but since the triangles are defined using lattice structure, it can be cut and areas of newly obtained arbitrary polygonal sections can be easily calculated. Usage of Eq. 4 needs determination of newly formed angles. That is one of the advantages of Eq. 3 over Eq. 4. One can easily find the area of any polygon using this equation. If one wants to cut the polygon and find its area, newly formed angles should be measured or calculated (Eq. 4). Its advantage is to deal with arbitrary triangles. However, Eq. 3 does not care for the newly formed angles; it can directly calculate area of the newly formed polygon by simply counting boundary and interior points. Having unique primitive triangle is the bad side of this method compared to Eq. 4, but counting which is one of the most fundamental human operations compensates it. Ability of making arbitrary cuts is also an advantage. Before giving some numerical examples one needs to guarantee that, the vectors connecting origin and the points on the sphere should be linearly independent of each other. This can be done by finding Gram determinant of the vectors (Gantmacher, 1960). Gram determinant and spherical excess are related to each other through Cagnoli's theorem (Todhunter, 1886, Art. 132; Porta et al. 2018). Gram determinant (Eq. 9) is given as follows.

$$G\,(1,2,3)=\begin{vmatrix}\langle \mathbf{p}_1,\mathbf{p}_1\rangle & \langle \mathbf{p}_1,\mathbf{p}_2\rangle & \langle \mathbf{p}_1,\mathbf{p}_3\rangle \\ \langle \mathbf{p}_2,\mathbf{p}_1\rangle & \langle \mathbf{p}_2,\mathbf{p}_2\rangle & \langle \mathbf{p}_2,\mathbf{p}_3\rangle \\ \langle \mathbf{p}_3,\mathbf{p}_1\rangle & \langle \mathbf{p}_3,\mathbf{p}_2\rangle & \langle \mathbf{p}_3,\mathbf{p}_3\rangle\end{vmatrix} \tag{9}$$

where $\langle \mathbf{p}_i, \mathbf{p}_j\rangle$ is the standard Euclidean inner product. This can be easily calculated and seen that it is larger than zero. Gramians are zero precisely when the coordinate vectors are dependent and positive otherwise (Gantmacher, 1984). Therefore, the points on the unit sphere are at different locations.

Numerical examples of these theorems will be given in the next section.

RESULTS AND DISCUSSION

In this section, some numerical examples will be given and various possibilities will be discussed.

Starting with Pick's theorem (Pick, 1899), for the planar polygon in Fig. 1, $i = 3$, and $b = 12$. Substituting them into Eq.1 gives $S = 8\ units$. This is the area enclosed by blue lines passing through lattice points.

It is possible to tile different triangles on a sphere. Here we are going to discuss two cases with equilateral triangles on a sphere. In the first one, we are going to tabulate various angles and number of triangles than can tile the sphere. In the second one, we are going to tabulate a special case in which around every point (vertex) there are the same number of triangles, so that some numerical examples will be given comfortably.

In Table 1, some of the possible polygons formed by equilateral spherical triangles are listed. By using the angles, the sphere is completely tiled, e.g. the polygon is simple. For example, if the number of triangles is 10, the angle must be 84^o. Then one can use Eq. 4 to calculate its area. For other angles corresponding to non-integer triangle numbers such as 110^o or 115^o, the equation still holds but the polygon is not simple on the sphere, e.g. some holes on the sphere. If one makes sure that the boundary of the polygon obtained with desired cuts do not include any holes, then Eq. 4 can be used safely. Although usage of different cuts changes the angles along the cut, since the lattice is formed by equilateral triangles, there will be no difficulty in using Eq. 4.

Table 1. Number of Equilateral Triangles Tessellating A Sphere and Corresponding Angles

# of ELT's	Angle (deg)
1	300
2	180
3	140
4	120
5	108
10	84
100	62.4
1000	60.24
10000	60.024

*ELT: Equilateral triangle

In this paper, we have no means of presenting a proof that they are going to tile the sphere without any gap even though the numbers of triangles are integers. That is why we need to look at a special case in which we have equilateral triangles that tessellate the sphere such that about any vertex there are the same number of triangles. In this case, one should find possible numbers of triangles (Table 2). An equilateral triangle on a plane has angle that measures 60^o so that sum of the angles is 180^o. As the first condition, in spherical trigonometry the angle and the total angle should be larger than 60^o and 180^o, respectively. The second condition is that, an equilateral triangle should tessellate without leaving hole or gap, etc. This can be satisfied by dividing 360^o (angle about a vertex) by integers so that the angle for each triangle is larger than 60^o.

Table 2. Angle and Number of Equilateral Triangles Tessellating A Sphere with Point Symmetry

# of ELT about a vertex	Angle (deg)	Total # of triangles	Total # of points
1	360	1	1
2	180	2	2
3	120	4	4
4	90	8	6
5	72	20	12

*ELT: Equilateral triangle

These are the only angles larger than 60^o. The third column represents the number of triangles on the sphere. The fourth column represents the total number of vertices on the sphere. For example, if there are five equilateral triangles with a common point, then there will be 20 triangles on the sphere, each having 72^o. Total number of points will be 12 on the sphere. This configuration is relatively easier to visualize and apply the equation to find the area compared to the case given in Table 1. Given these angles, all the equilateral triangles tessellate on the sphere. The first number is for single triangle, namely sphere, the second one is half sphere, the third one is quarter of a sphere but not regular one, the fourth one is the quarter of half sphere (familiar one) and the last one is as it is. The sphere cannot be divided into more than 20 parts if the polygon about a vertex formed by equilateral triangles has the same number of common points. As seen below, it is possible to cut the triangles and obtain arbitrary polygons.

Equipped with the information listed above, now we can proceed in calculating areas of polygons formed by equilateral triangles, even containing arbitrary triangles obtained by cuts.

Here are some examples using $S = 2E\left(i + \frac{b}{2} - 1\right)$:

A polygon composed of 120^o degree equilateral triangles

This is a projection of a tetrahedron radially onto a unit sphere. The total number of points and triangles are both four. Area of each triangle $S = 4\pi/4 = \pi$. The spherical excess $E = \pi$.

For $i = 0$, $b = 3$, the area $S = \pi$ which is the area of a single triangle. This result confirms a theorem saying that on a unit sphere, for any spherical triangle of area π, four congruent copies of it tile the sphere (Brooks & Strantzen, 2005) and it is also similar to quantum states (Sandeep et al., 2016).

For $i = 0, b = 4,$ the area $S = 2\pi$ which is the area of two adjacent triangles.

For $i = 1, b = 3,$ the area $S = 3\pi$ which is the area of three adjacent triangles.

For $i = 1, b = 4,$ the area $S = 4\pi$ which is the area of four adjacent triangles (full sphere). Here one of the points is counted twice, one for the boundary and one for the interior. That point is arbitrary. What we obtained in a different way is the same as Brooks and Strantzen (2005).

For this case, only polygon areas given above can be calculated. There is no additional polygon possible. No triangle can be cut into two for this case. This partially hides beauty of Eq. 2 since the results are obvious even though they are obtained in an elegant way. This is in accordance with Michael Atiyah's saying that, "… always to suspect an impressive sounding theorem if it does not have a special case which is both simple and non-trivial."

A polygon formed by 72^o degree equilateral triangles:

Now let us proceed with the maximum number of equilateral triangles having maximum number of common points (vertices). The polygon is composed of 72^o degree equilateral triangles (Fig. 4): The total number of points and triangles are 12 and 20 respectively. This shape is similar to the shape of corona virus given by Sarkar and Rashid (2022). They drew proteins at the vertices and intermediate locations using just geometry without introducing any spherical trigonometric equations. The cuts given below can pass through intermediate proteins and necessary areas can be calculated using Eq. 4. Area of each triangle $S = 4\pi/20 = \pi/5$. The spherical excess $E = \pi/5$. There are several possibilities for this tessellation. All intermediate cuts between vertices are still arcs of corresponding great circle.

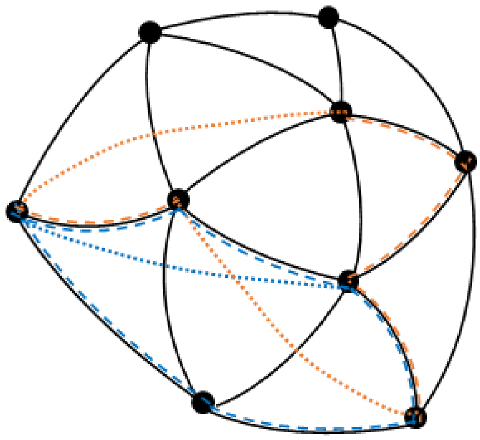

Figure 4. A Polygon Composed of 72^o Degree Equilateral Triangles

For $i = 3,\ b = 6,$ the area $S = 2\pi$ which is the area of a hemisphere composed of triangular tessellations.

For $i = 0, b = 4,$ the area $S = 2\pi/5$ which is the area of two adjacent triangles.

For $i = 1,\ b = 5,$ the area $S = \pi$ which is the area of five adjacent triangles whose one of the vertices is common. It is also equal to the area of a unit circle.

For $i = 0,\ b = 5,$ the area $S = 3\pi/5$ which is the area of three triangles which are aligned one after another.

For $i = 0,\ b = 4,$ the area $S = 2\pi/5$ which is the area of three adjacent triangles cut by an arc (blue dashed line in Fig. 4, lower part, dotted line is the intermediate cut). This area is also equal to the area of two adjacent triangles. This is a stunning result. The other part (upper) of the tessellation has area $\pi/5$ for $i = 0$ and $b = 3$. The newly formed angles by the cut between two vertices are calculated using spherical trigonometric equations (Todhunter, 1886, Art. 47) and the areas are in perfect agreement with the result of Eq. 4. This shows us that there is no need to calculate newly formed angles by cuts. The new triangle has angles $144^{o}, 36^{o}, 36^{o}$. Its area is equal to the area of primitive equilateral triangle. Thus, Theorem 3 can be effectively used even there is a cut similar to planar polygons using Pick's theorem.

For $i = 0$, $b = 6$, the area $S = 4\pi/5$ (brown dashed line in Fig. 4, dotted line is the intermediate cut). This result is stunning since arbitrary polygonal areas can be calculated using cuts made by intermediate great circular arcs.

Similar results can be obtained for various tessellations since this angle has rich combinations although the exact number of possible tessellations was not calculated here. One must be careful about where the great circle should pass when it is desired to find an arbitrary tessellation not to make mistake about the number of boundary and interior points.

CONCLUSION

In this article, we presented two new theorems. The first one is for calculating a polygonal area formed by tessellating equilateral triangles on a sphere. The second one is for calculating length of a line or space curve. This gives very basic and almost trivial information, but it is necessary to close the gap in the completeness of the history and theorems. The first theorem is an extension of Pick's theorem to spherical geometry by combining with Girard's theorem. Pick's theorem is for calculating areas of planar polygons having lattice points and Girard's theorem is for calculating areas of spherical triangles. Girard's theorem was also used in the literature to calculate spherical polygonal areas by a small modification.

Theorem 3 proposed here combines the lattice area equation (Pick's theorem) and the spherical excess (Girard's theorem), and multiplies by 2. Doubling the spherical excess is a clue that it is a projection of a parallelogram onto the sphere. Any polygon can be formed by a spherical equilateral triangle tessellation on the sphere and its area can be calculated using Theorem 3. Arbitrary cuts can be made between any vertices so that arbitrary polygons are formed and the theorem still holds. The triangulation or triangular tessellation used on a sphere could be more fundamental than original square lattices offered by Pick (1899). The formula proposed in our study depends on a primitive triangle. This can be thought as a shift by half distance in consecutive rows horizontally in the planar lattice structure even though it is not compulsory. This phenomenon is inevitable in spherical geometry. The points has to shift so that triangles tessellate on the sphere completely.

After this point, Pick's theorem cannot be thought as stuck to planar geometry, instead it can be used in spherical geometry using Eq. 4. Our problem is still two-dimensional since it is the surface of a three dimensional sphere. Extending Pick's theorem using Girard's theorem brings to light such a possibility that they are special cases of a generality. As Hilbert said, "The art of doing mathematics consists in finding that special case which contains all the germs of generality". We can conclude that not only Pick's theorem and Girard's theorem but also the theorem presented in our article should be a special case of a more general theorem. The general case could be a theorem in Euclidean, Spherical and Hypebolic geometry combined or topology. Any two-dimensional shape in three or more dimensions might be treated using a more general theorem.

Further studies and open questions:

The original shape of the triangle or parallelogram before projection onto sphere.

Triangular approach for volume calculations.

The area equation in hyperbolic lattices

Unification of area equation for lattices in three major geometries.

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