Journal of Universal Mathematics
Vol.6 No.3-Supplement pp.1-15 (2023)
ISSN-2618-5660
DOI:10.33773/jum.1332081

MATRICES OF HYBRID NUMBERS

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Abstract. In this study, we investigate the matrices over the new extension of the real numbers in four dimensional space $\mathbb{E}_2^4$ called the hybrid numbers. Since the hybrid multiplication is noncommutative, this leads to finding a linear transformation on the complex field. Thus we characterize the hybrid matrices and examine their algebraic properties with respect to their complex adjoint matrices. Moreover, we define the co-determinant of hybrid matrices which plays an important role to construct the Lie groups.

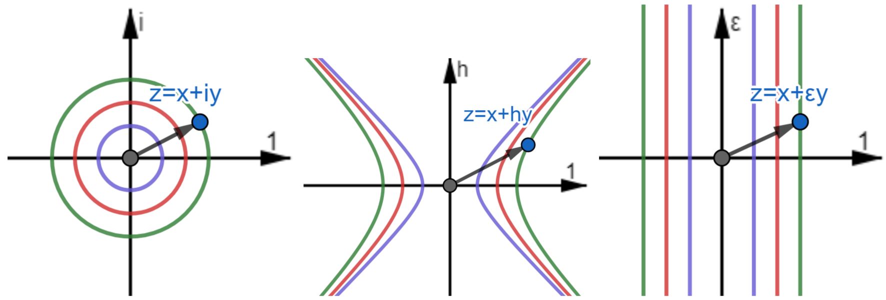
1. Introduction

The extension of the real number system raises by investigating the solutions of the quadratic equations given as follows:

$$x^2 + 1 = 0,\ x^2 - 1 = 0 \text{ and } x^2 = 0. \tag{1.1}$$

As a result, the new units called the imaginary $i^2 = -1$, the unipotent $h^2 = 1$ $(h \neq \mp 1)$ and the nilpotent $\varepsilon^2 = 0$ $(\varepsilon \neq 0)$ enter in the history of mathematics and yield the new number systems named by complex numbers, hyperbolic numbers and dual numbers, respectively [21, 23, 24]. All three number systems are two-dimensional vector spaces over the real numbers, this implies that the points of $\mathbb{R}^2$ can be identified by them with respect to their metric systems. These corresponding metrics yield two-dimensional Euclidean geometry, Lorentzian geometry and Galilean geometry, respectively. Then the identification of a point $A = (x, y)$ can be seen in the following planes with respect to the systems:

Date: **Received:** 2023-07-24; **Accepted:** 2023-10-08.
2000 *Mathematics Subject Classification.* 15A33, 53A35; 15A57, 53Z05.
Key words and phrases. Complex numbers, Dual numbers, Hyberbolic numbers, Hybrid numbers, Hypercomplex numbers, Complex matrices, Hybrid matrices.



(A) Euclidean plane (B) Lorentzian plane (C) Galilean plane

FIGURE 1. Coordinate planes of metric systems in two-dimensional space

Moreover, Clifford algebras can be studied on the vector spaces of complex numbers, dual numbers and hyperbolic numbers via elliptic, parabolic and hyperbolic bilinear forms, respectively. It is also known as EPH-classification of these number systems. The EPH-classification is closely linked with the elliptic, hyperbolic and parabolic analytic function theories [4, 6, 16].

The historical evolution of the ideas on how to manage the extension of numbers gives us the quaternions introduced by Hamilton [12] as the most-known generalization of complex numbers. The set of quaternions is generally represented in the form:

$$\mathbb{H} = \{q = q_0 + q_1 i + q_2 j + q_3 k : q_s \in \mathbb{R} \ , 0 \leq s \leq 3\} \tag{1.2}$$

where i, j, k are quaternionic units and hold $i^2 = j^2 = k^2 = ijk = -1$. Since the set $\mathbb{H}$ is a non-commutative associative algebra over the real numbers, the matrices of quaternions becomes one of the interesting topics in the matrix theory. A brief survey on the quaternionic matrices given by Zhang [26] presents some methods for some basic functions for matrices such as determinant, computing the eigenvalue. The method is based on finding the complex adjoint matrix of any quaternionic matrix. After that, various studies are born about the matrices of quaternions and their applications [5, 8, 10, 14, 25].

Another well-known member of non-commutative algebras is the set of split quaternions introduced by Cockle [7] as follows:

$$\widehat{\mathbb{H}} = \{\hat{q} = \hat{q}_0 + \hat{q}_1 i + \hat{q}_2 j + \hat{q}_3 k : \hat{q}_t \in \mathbb{R} \ , 0 \leq t \leq 3\} \tag{1.3}$$

where $i^2 = -1$ and $j^2 = k^2 = ijk = 1$. The difference between $\widehat{\mathbb{H}}$ and $\mathbb{H}$ is the existence of zero divisors, nilpotent elements and nontrivial idempotents in $\widehat{\mathbb{H}}$. After work of Zhang, the quaternionic matrices and their properties are studied over $\widehat{\mathbb{H}}$ by the compatible methods [1, 11, 15, 19, 20].

In the system $\mathbb{R}^4$, we meet the new phenomenon named as hybrid numbers and given in the following form:

$$\mathbb{K} = \{X = x_0 + x_1 i + x_2 \varepsilon + x_3 h : x_j \in \mathbb{R} \ , 0 \leq j \leq 3\} \tag{1.4}$$

where i, ε and h are the complex, dual and hyperbolic units, respectively [17]. There are considerable differences between $\mathbb{K}$ and the two sets previously describe, out of the noncommutativity. Under this view, hybrid numbers firstly give the blood relativity of two different classes of vectors:

$$\overbrace{\left\{\begin{array}{c} \textit{Timelike} \\ \textit{Spacelike} \\ \textit{Lightlike} \end{array}\right\}}^{\textbf{Characters}} \underset{\textit{Hybrid Numbers}}{\longleftrightarrow} \overbrace{\left\{\begin{array}{l} \textit{Elliptic} \\ \textit{Parabolic} \\ \textit{Hyperbolic} \end{array}\right\}}^{\textbf{EPH-classification}}$$

Secondly, there is the isomorphism between 2×2 real matrices and hybrid numbers and thus a classification of 2×2 real matrices and an algebraic method to find their roots are obtained by the hybrid numbers [18]. The short history of the hybrid numbers reveals us their advantages on real matrix algebra and the sequences of special numbers [9, 22].

In this study, we will examine the hybrid matrices by improving the Zhang's method over $\mathbb{K}$. In the second section, we give some basic notions and properties of hybrid numbers, and more importantly, we change the spelling of the hybrid numbers. They are rewritten in the form named as the $\mathbb{C}-$type which will be used to built a linear transformation between $\mathbb{K}$ and the set of 2×2 complex matrices. This correspondence yields the second relationship between eigenvalues and types of hybrid numbers as follows:

$$\overbrace{\left\{\begin{array}{c} \lambda_{1,2}\in\mathbb{C} \\ \lambda_1=\lambda_2 \\ \lambda_{1,2}\in\mathbb{R} \end{array}\right\}}^{\textbf{Eigenvalues}} \underset{\textit{Hybrid Numbers}}{\longleftrightarrow} \overbrace{\left\{\begin{array}{l} \textit{Elliptic} \\ \textit{Parabolic} \\ \textit{Hyperbolic} \end{array}\right\}}^{\textbf{EPH-classification}}$$

In the third section, the matrices of hybrid numbers are introduced and their properties are obtained. After that, in the fourth section, to prevent the disadvantages of the noncommutativity of hybrid numbers we define the complex adjoint of hybrid matrices. Hence the determinant of hybrid matrices could be characterized, and so they are analyzed in the theory of Lie groups.

2. Basic Concepts of Hybrid Numbers

In this section, we initially introduce hybrid numbers with fundamental features. Then we establish a new form called $\mathbb{C}-$type and give the properties of hybrid numbers in the new form.

A hybrid number occurs in the combination form of the three types of number systems, complex, dual and hyperbolic numbers, as the following:

$$X = x_0 + x_1 i + x_2\varepsilon + x_3 h \tag{2.1}$$

where $x_j \in \mathbb{R}$ $, 0 \le j \le 3$ and the basis elements $\{1, i, \varepsilon, h\}$ are satisfying the multiplication rules given in the following table.

(2.2)

$\cdot$	1	i	ε	h
1	1	i	ε	h
i	i	-1	$1-h$	$i+\varepsilon$
ε	ε	$1+h$	0	$-\varepsilon$
h	h	$-i-\varepsilon$	ε	1

By the compotentwise addition and scalar multiplication, the set of hybrid numbers denoted by $\mathbb{K}$ becomes a 4-dimensional vector space over the real numbers.

Furthermore, the hybrid number algebra is an associative, noncommutative ring with respect to the addition and multiplication operations.

The hybrid number X is composed of the scalar part $S(X) = x_0$ and the vector part $V(X) = x_1 i + x_2 \varepsilon + x_3 h$. The conjugate of X is the hybrid number defined by $\overline{X} = S(X) - V(X)$. If $x_2 = x_3 = 0$, then the conjugate of hybrid number means the conjugate of complex number, and vice versa. Moreover, there are two kinds of vectorial representation of X given by $\mathcal{V}(X) = (x_0, x_1 - x_2, x_2, x_3)$ and $\mathcal{V}_h(X) = (x_1 - x_2, x_2, x_3)$ which is specifically called the hybrid vector of X. Thus, there exist the following functions:

$$\begin{aligned} \mathcal{C}(X) &= x_0^2 + (x_1 - x_2)^2 - x_2^2 - x_3^2 \\ \mathcal{C}_h(X) &= -(x_1 - x_2)^2 + x_2^2 + x_3^2 \end{aligned} \tag{2.3}$$

where $\mathcal{C}(X) = -\langle \mathcal{V}(X), \mathcal{V}(X) \rangle$ and $\mathcal{C}_h(X) = \langle \mathcal{V}_h(X), \mathcal{V}_h(X) \rangle$ are equipped with the signature $(-,-,+,+)$ of $\mathbb{E}_2^4$ the four dimensional Minkowski space and the subspace $\mathbb{E}_1^3$, respectively. These functions yield the following classifications of the hybrid number X with respect to the corresponding Minkowski metrics:

A hybrid number $X \in \mathbb{K}$ is

- Spacelike if $\mathcal{C}(X) < 0$ or $X = 0$,
- Timelike if $\mathcal{C}(X) > 0$,
- Lightlike (null) if $\mathcal{C}(X) = 0$ and $X \neq 0$,

which are called the characters of the hybrid number X.

The types of the hybrid number X are given by

- If $\mathcal{C}_h(X) < 0$, X is elliptic,
- If $\mathcal{C}_h(X) > 0$, X is hyperbolic,
- If $\mathcal{C}_h(X) = 0$, X is parabolic.

Consequently, the following table is set to show the relation between the two characterizations of hybrid numbers.

(2.4)

Classification by Types	**Classification by Characters**
Elliptic	Timelike
Hyperbolic	Spacelike, Timelike, Lightlike
Parabolic	Timelike, Lightlike

Until now, we summarize briefly the basic algebraic properties of the noncommutative ring $\mathbb{K}$ for more details the reader is referred to [17].

Our first aim in the present paper is to find a linear transformation between hybrid numbers and express them via the matrix of the transformation thus we could explore the properties of hybrid numbers in another convenient way. For this inherent reason, the multiplication rule of the unit ε in (2.2) allows us to observe the hybrid numbers in terms of the basis $\{i, h\}$. Thus we can explain the hybrid number $X = x_0 + x_1 i + x_2 \varepsilon + x_3 h$ as follows:

$$X = z_1 + z_2 h, \quad z_1, z_2 \in \mathbb{C} \tag{2.5}$$

where $z_1 = x_0 + (x_1 - x_2)\, i$, $z_2 = x_3 + x_2 i$. Since this appears, at first sight, to be a complex hyperbolic number, we call (2.5) as the $\mathbb{C}-$type of hybrid number X in order to avoid the confusion. Then we can obviously conclude the following.

Theorem 2.1. *Every hybrid number can be uniquely expressed in the form of* $\mathbb{C}-$*type.*

Corollary 2.2. *The $\mathbb{C}-$type of a hybrid number become equivalent to its open form if and only if the hybrid number is a complex number.*

The fundamental functions on the set of hybrid numbers of the $\mathbb{C}-$type are given as follows:
$i)$ Addition:

$$X+Y=(z_1+w_1)+(z_2+w_2h),$$

$ii)$ Multiplication:

$$XY=z_1w_1+z_2\overline{w_2}+(z_1w_2+z_2\overline{w_1})h,$$

$iii)$ The hybrid conjugate:

$$\overline{X}=\overline{z_1}-z_2h,$$

$iv)$ Functions of characteristics:

$$\mathcal{C}(X)=|z_1|^2-|z_2|^2 \text{ and } \mathcal{C}_h(X)=-V(z_1)^2+|z_2|^2,$$

$v)$ The inverse of a hybrid number:

$$X^{-1}=\frac{\overline{z_1}}{\mathcal{C}(X)}-\frac{z_2}{\mathcal{C}(X)}h$$

$vi)$ The two kind norms of a hybrid number:

$$\| X \|=\sqrt{|\mathcal{C}(X)|} \text{ and } \| X \|_h=\sqrt{|\mathcal{C}_h(X)|}$$

where $X=z_1+z_2h$, $Y=w_1+w_2h\in\mathbb{K}$ and $V(z_1)$ is the imaginary part of z_1.
The next theorem summarizes the properties of the hybrid conjugate.

Theorem 2.3. *For the hybrid numbers $X=z_1+z_2h$ and $Y=w_1+w_2h$, the properties listed below are true.*
i. $X=\overline{(\overline{X})}$,
ii. $X\overline{X}=\overline{X}X=z_1\overline{z_1}-z_2\overline{z_2}$,
iii. $\overline{X+Y}=\overline{X}+\overline{Y}$,
iv. $\overline{XY}=\overline{Y}\ \overline{X}$,
v. $\mathcal{C}(X)=\mathcal{C}(\overline{X})$ *and* $\mathcal{C}_h(X)=\mathcal{C}_h(\overline{X})$,
vi. $\overline{(X^{-1})}=\left(\overline{X}\right)^{-1}$,
vii. $X=\overline{X}$ *if and only if X is a real number,*
viii. $hz=\overline{z}h$ *or* $hzh=\overline{z}$ *for any complex number z.*

Proof. In general, the properties can be proved easily. Let's at least have confidence in the accuracy of (iv) and (vi).
The proof for (iv);

$$\begin{aligned}\overline{XY} &= \overline{z_1w_1+z_2\overline{w_2}-(z_1w_2+z_2\overline{w_1})\,h} \\ &= \overline{w_1}\ \overline{z_1}+w_2\overline{z_2}-(\overline{w_1}z_2+w_2z_1)\,h \\ &= \overline{Y}\ \overline{X}.\end{aligned}$$

The proof for (vi);

$$\begin{aligned}
\left(\overline{X}\right)^{-1} &= \frac{\overline{(\overline{z_1})}}{\mathcal{C}(\overline{X})} - \frac{(-z_2)}{\mathcal{C}(\overline{X})}h \\
&= \frac{z_1}{\mathcal{C}(X)} + \frac{z_2}{\mathcal{C}(X)}h \\
&= \overline{(X^{-1})}.
\end{aligned}$$

□

Now let us define the following bijective map,

$$\begin{aligned}
\psi_X : \quad & \mathbb{K} && \to \mathbb{K} \\
& Y && \to \psi(Y) = YX
\end{aligned} \tag{2.6}$$

where as a consequence of the ring structure of hybrid numbers we could see that ψ_X is a linear map. It is well known that every linear map can be represented by a matrix, so we get

$$\begin{aligned}
\psi_X(1) &= z_1 + z_2 h, \\
\psi_X(h) &= \overline{z_2} + \overline{z_1} h,
\end{aligned}$$

and then the matrix of the transformation ψ with respect to the standard bases is given as follows:

$$[\psi_X] = \begin{bmatrix} z_1 & z_2 \\ \overline{z_2} & \overline{z_1} \end{bmatrix}.$$

where $X = z_1 + z_2 h$.

Consequently, the following theorem is stated.

Theorem 2.4. *Every hybrid number can be represented by a* 2×2 *complex matrices.*

Notice that the subset of the matrix ring $M_2(\mathbb{C})$ given such as

$$\mathcal{K} = \left\{ A = \begin{bmatrix} z_1 & z_2 \\ \overline{z_2} & \overline{z_1} \end{bmatrix} : z_1, z_2 \in \mathbb{C} \right\} \tag{2.7}$$

actually represents the set of hybrid numbers $\mathbb{K}$. Since the transformation between $\mathcal{K}$ and $\mathbb{K}$ is bijective and linear, then the operations are preserved. Moreover, let the corresponding matrix of $X = z_1 + z_2 h$ be $A = \begin{bmatrix} z_1 & z_2 \\ \overline{z_2} & \overline{z_1} \end{bmatrix}$, we have

$$detA = \mathcal{C}(X),\ trA = 2\ funcRe(z_1) \text{ and } \lambda_{1,2} = \frac{trA}{2} \mp \sqrt{\mathcal{C}_h(X)} \tag{2.8}$$

where λ_1 and λ_2 are the eigenvalues of A.

Corollary 2.5. *The inverse of a hybrid number exists if and only if the determinant of the corresponding complex matrix of the hybrid number is different from zero.*

Definition 2.6. The characters of $A \in \mathcal{K}$ can be defined as
i. A is spacelike, if $\det A < 0$,
ii. A is timelike, if $\det A > 0$,
iii. A is lightlike, if $\det A = 0$.

Definition 2.7. The types of $A \in \mathcal{K}$ can be given in terms of its eigenvalues $\lambda_{1,2}$ as follows:
i. A is elliptic, if $\lambda_{1,2} \in \mathbb{C}$,
ii. A is hyperbolic, if $\lambda_{1,2} \in \mathbb{R}$,
iii. A is parabolic, if $\lambda_1 = \lambda_2$.

Corollary 2.8. *If $A \in \mathcal{K}$ is a Hermitian matrix, then its corresponding hybrid number must be hyperbolic or parabolic.*

Now, we observe the matrices of hybrid numbers according to the three different concepts of complex matrix theory which are unitary, Hermitian and skew-Hermitian matrices. Let $A \in \mathcal{K}$ be the corresponding complex matrix of the hybrid number $X = z_1 + z_2 h$, we can give the following statements.

- If A is the unitary matrix, then $A\overline{A}^T = I_2$ which yields

$$z_1\overline{z_1} + z_2\overline{z_2} = 1 \text{ and } z_1 z_2 = 0. \tag{2.9}$$

  For the case $z_1 = 0$, we have $z_2\overline{z_2} = 1$ means that $z_2 = \cos\theta + \sin\theta i$. Then the $\mathbb{C}-$type of X is

$$X = e^{i\theta} h \tag{2.10}$$

  where X is a spacelike hyperbolic hybrid number. On the other hand, if $z_2 = 0$, the $\mathbb{C}-$type forms of X meets the open form of it means that X is a complex number such that

$$X = e^{i\theta}. \tag{2.11}$$

- If A is the Hermitian matrix, then we obtain $z_1 = \overline{z_1}$ and $\mathcal{C}_h(X) = z_2\overline{z_2}$. From *Corollary 4*, we can distinguish two cases:
  i. If X is parabolic hybrid number, then $z_2 = 0$ and $X \in \mathbb{R}\backslash\{0\}$,
  ii. If X is hyperbolic hybrid number, X could have the three kinds of characters. In addition to, the null case will be appeared as the Pythagorean condition and therefore the components can be expressed as follows:

$$z_1 = w(u^2 + v^2) \text{ and } z_2 = w\left[2uv + \left(u^2 - v^2\right) i\right] \tag{2.12}$$

  where w is constant, u and v are relatively prime.
- A is the skew-Hermitian matrix if and only if X is a pure complex number, namely $X = xi$, $x \in \mathbb{R}$.

3. Introduction to Hybrid Matrices

In this section, our first results concern the matrices of hybrid numbers. After that we explain them in terms of the complex matrices by using the $\mathbb{C}-$type form of hybrid numbers. Hence we could analyze the properties of hybrid number matrices by using the algorithms of the complex matrix theory.

Let us introduce the set of $m \times n$ type matrices with the hybrid number components, denoted by $M_{m,n}(\mathbb{K})$. If $m = n$, then we briefly use the notation $M_n(\mathbb{K})$. With the ordinary matrix addition and multiplication the set $M_n(\mathbb{K})$ is going to become a noncommutative ring where the unit is I_n. If the equation $AB = BA = I_n$ exists for $B \in M_n(\mathbb{K})$, we call A is invertible and denote $B = A^{-1}$.

Moreover, the propositions of left vector space endowed by [26] can be satisfied by the following scalar multiplication:

$$XA = [Xa_{\alpha\beta}] \tag{3.1}$$

where $X \in \mathbb{K}$ and $A = [a_{\alpha\beta}] \in M_{m,n}(\mathbb{K})$. Hence we know that $M_{m,n}(\mathbb{K})$ is a left vector space over $\mathbb{K}$. Similarly, the definition of scalar multiplication $AX = [a_{\alpha\beta}X]$ yields the right vector space over $\mathbb{K}$.

If we use the $\mathbb{C}-$types of the components of the hybrid matrix $A = [a_{\alpha\beta}] \in M_{m,n}(\mathbb{K})$, then the components are written as $a_{\alpha\beta} = a^1_{\alpha\beta} + a^2_{\alpha\beta}h \in \mathbb{K}$ and we get

$$A = A_1 + A_2 h \tag{3.2}$$

where $A_1 = [a^1_{\alpha\beta}]$, $A_2 = [a^2_{\alpha\beta}] \in M_{m,n}(\mathbb{C})$. The transpose of A is $A^T = [a_{\beta\alpha}] = A_1^T + A_2^T h$ and the conjugate of A is $\overline{A} = [\overline{a_{\alpha\beta}}] = \overline{A_1} - A_2 h$.

Definition 3.1. The conjugate transpose of a hybrid matrix A, denoted by A^*, is

$$A^* = \overline{A}^T$$

where the entries of $\overline{A}$ are the hybrid conjugates of the corresponding entries of A.

Ideally, we shall consider specific square hybrid matrices in terms of the conjugate transpose as follows:

- $A = A^*$, A is Hermitian,
- $A = -A^*$, A is skew-Hermitian,
- $A^{-1} = A^*$, A is unitary,
- $AA^* = A^*A$, A is normal.

Definition 3.2. Let $\lambda \in \mathbb{K}$ and $A \in M_n(\mathbb{K})$. If λ satisfies the following equation

$$Ax = \lambda x, \tag{3.3}$$

then λ is called the left eigenvalue of A for some non-zero $x \in M_{n,1}(\mathbb{K})$. The set of the left eigenvalues of A is called the left spectrum of A.

Note that we can similarly define the right eigenvalue ($Ax = x\lambda$, $\lambda \in \mathbb{K}$) and the right spectrum of A because of the noncommutativity.

Example 3.3. (i) For the hybrid matrix $A \in M_2(\mathbb{K})$,

$$A = \begin{bmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{bmatrix}$$

$\{0, \varepsilon, -\varepsilon\}$ is the subset of the intersection of the left and the right spectrums of A.

(ii) For the hybrid matrix $B \in M_2(\mathbb{K})$,

$$B = \begin{bmatrix} 0 & h \\ -h & 0 \end{bmatrix}$$

some of the left eigenvalues of B are $\{\mp(i + \varepsilon)\}$ but none of them is the element of the right spectrum of B. Similarly, some of right eigenvalues of B are $\{\mp i\}$ but not the left eigenvalues of B.

The theorems below list several properties of hybrid matrices. The first theorem show the properties which are generally correct, therefore, we construct an example for an explicit proof. On the other hand, the direct proof method can be used for the consecutive theorem.

Theorem 3.4. *If $A \in M_{m,n}(\mathbb{K})$ and $B \in M_{n,s}(\mathbb{K})$, then the properties listed below are true in general.*

i. $\left(\overline{A}\right)^{-1} \neq \overline{(A^{-1})}$,

ii. $\left(A^T\right)^{-1} \neq \left(A^{-1}\right)^T$,

iii. $AB \neq BA$,
iv. $\overline{AB} \neq \overline{B}\,\overline{A}$,
v. $(AB)^T \neq B^T A^T$.

Example 3.5. Let the two hybrid matrices be $A = \begin{bmatrix} i & \varepsilon \\ 0 & h \end{bmatrix}$ and $B = \begin{bmatrix} \varepsilon & 0 \\ 0 & 0 \end{bmatrix}$.
We obtain the following:
i. $\left(\overline{A}\right)^{-1} = \begin{bmatrix} i & 1-h \\ 0 & -h \end{bmatrix} \neq \begin{bmatrix} i & -1-h \\ 0 & -h \end{bmatrix} = \overline{(A^{-1})}$,
ii. $\left(A^T\right)^{-1} = \begin{bmatrix} -i & 0 \\ 1+h & h \end{bmatrix} \neq \begin{bmatrix} -i & 0 \\ -1+h & h \end{bmatrix} = \left(A^{-1}\right)^T$,
iii. $AB = \begin{bmatrix} 1-h & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1+h & 0 \\ 0 & 0 \end{bmatrix} = BA$
iv. $\overline{AB} = \begin{bmatrix} 1+h & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1-h & 0 \\ 0 & 0 \end{bmatrix} = \overline{B}\,\overline{A}$,
v. $(AB)^T = \begin{bmatrix} 1-h & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1+h & 0 \\ 0 & 0 \end{bmatrix} = B^T A^T$.

Remark 3.6. The sufficient condition for the existence of the third case of theorem 4 occur with the invertible matrices. Moreover, if $AB = I$ for any $A = A_1 + A_2 h$, $B = B_1 + B_2 h \in M_n(\mathbb{K})$, this provides $BA = I$. From the hypothesis we get

$$A_1 B_1 + A_2 \overline{B_2} + \left(A_1 B_2 + A_2 \overline{B_1}\right) h = I_n \tag{3.4}$$

and (3.3) is equal to the following matrix product

$$\begin{bmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ \overline{B_2} & \overline{B_1} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \tag{3.5}$$

which yields $BA = I$ since the left hand side of (3.4) is the product of $2n \times 2n$ complex matrices and the hypothesis is true for the complex matrices.

Theorem 3.7. *If $A \in M_{m,n}(\mathbb{K})$, $B \in M_{n,s}(\mathbb{K})$ and $X \in \mathbb{K}$, then the properties listed below are true.*
i. $\left(\overline{A}\right)^T = \overline{(A^T)}$,
ii. $\overline{(\overline{A})} = \left(A^T\right)^T = (A^*)^* = A$,
iii. $(XA)^* = A^* \overline{X}$,
iv. $(A+B)^* = A^* + B^*$,
v. $(AB)^* = B^* A^*$,
vi. $(AB)^{-1} = B^{-1} A^{-1}$ *if A and B are invertible,*
vii. $(A^*)^{-1} = \left(A^{-1}\right)^*$ *if A invertible.*

Proof. The proof of the first four properties and (vi) can be easily shown with using the properties of complex matrix theory and *Theorem 2* in the previous section. Let us prove (v) with $A = A_1 + A_2 h$ and $B = B_1 + B_2 h$, where A_1, A_2, B_1 and B_2 are process-compatible complex matrices, then we have

$$\begin{aligned}
(AB)^* &= \left[A_1 B_1 + A_2 \overline{B_2} + \left(A_1 B_2 + A_2 \overline{B_1}\right) h\right]^* \\
&= \left[A_1 B_1 + A_2 \overline{B_2}\right]^* - \left[A_1 B_2 + A_2 \overline{B_1}\right]^T h \\
&= (A_1 B_1)^* + \left(A_2 \overline{B_2}\right)^* - (A_1 B_2)^T h - \left(A_2 \overline{B_1}\right)^T h \\
&= B_1^* A_1^* + \left(\overline{B_2}\right)^* A_2^* - B_2^T A_1 h - \left(\overline{B_1}\right)^T A_2^T h \\
&= B^* A^*.
\end{aligned}$$

As a consequence of the fifth property, we can obtain the case (vii). $\square$

4. Complex Matrix Equivalence For Hybrid Matrices

The linear map $\psi_X : \mathbb{K} \to \mathbb{K}$ defined in the second section gives us the opportunity to examine the properties of hybrid numbers over 2×2 complex matrices. Since the hybrid multiplication is noncommutative, there are also limitations in questioning the linear algebra over hybrid matrices. In this section, our aim is to turn a hybrid matrix into a complex matrix to use the several properties of linear algebra over the complex field.

In this way, we define a map, Ψ_n, that is between $M_n(\mathbb{K})$ and $M_{2n}(\mathbb{C})$, such as

$$\Psi_n(A) = \begin{bmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{bmatrix}$$

where $A = A_1 + A_2 h \in M_n(\mathbb{K})$ and $A_1, A_2 \in M_n(\mathbb{C})$.

Notice that Ψ_n is a continuous, injective ring homomorphism and described with respect to the linear map ψ_X. For $n = 1$, we can have the corresponding complex matrix of a hybrid number. We call that $\Psi_n(A)$ is the adjoint matrix of $A \in M_n(\mathbb{K})$, and denote by $\tilde{A} \in M_{2n}(\mathbb{C})$.

Example 4.1. Let $A = \begin{bmatrix} 1+\varepsilon & i+\varepsilon+h \\ 1+h & 1 \end{bmatrix}$ be a hybrid matrix. Then we can rewrite it by using the $\mathbb{C}-$types of the components as in following form:

$$A = \begin{bmatrix} 1-i & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} i & 1+i \\ 1 & 0 \end{bmatrix} h. \tag{4.1}$$

Thus the conjugate matrix of A is

$$\tilde{A} = \begin{bmatrix} 1-i & 0 & i & 1+i \\ 1 & 1 & 1 & 0 \\ -i & 1-i & 1+i & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}. \tag{4.2}$$

The following theorem summarizes the properties of adjoint matrices.

Theorem 4.2. *Let $A = A_1 + A_2 h$, $B = B_1 + B_2 h \in M_n(\mathbb{K})$ and their adjoint matrices be $\tilde{A}, \tilde{B} \in M_n(\mathbb{C})$, then the followings are true.*

i. If $A = I_n$, then $\tilde{A} = I_{2n}$,

ii. $\widetilde{A+B} = \tilde{A} + \tilde{B}$,

iii. $\widetilde{A\ B} = \tilde{A}\ \tilde{B}$,

iv. $\widetilde{A^{-1}} = \left(\tilde{A}\right)^{-1}$ if A^{-1} exists,

v. $\left(\tilde{A}\right)^T = \widetilde{(A^T)}$ if $A_2 \in M_n(\mathbb{R})$,

vi. $\widetilde{(\overline{A})} = \overline{\left(\tilde{A}\right)}$ if A_2 is a pure complex matrix,

vii. $\widetilde{(A^*)} = \left(\tilde{A}\right)^*$ if $A \in M_n(\mathbb{C})$.

Proof. Truth of (i) and (ii) are clear. Let us prove (iii). The adjoint matrices of A and B are

$$\tilde{A} = \begin{bmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{bmatrix} \text{ and } \tilde{B} = \begin{bmatrix} B_1 & B_2 \\ \overline{B_2} & \overline{B_1} \end{bmatrix}. \tag{4.3}$$

Multiplying the adjoints, we obtain the complex matrix

$$\tilde{A}\ \tilde{B} = \begin{bmatrix} A_1B_1 + A_2\overline{B_2} & A_1B_2 + A_2\overline{B_1} \\ \overline{A_1}\ \overline{B_2} + \overline{A_2}B_1 & \overline{A_1}\ \overline{B_1} + \overline{A_2}B_2 \end{bmatrix}. \tag{4.4}$$

Since the hybrid matrix form of (4.4) is

$$\begin{aligned} &A_1B_1 + A_2\overline{B_2} + \left(A_1B_2 + A_2\overline{B_1}\right) h \\ &\textit{or}\ \ (A_1 + A_2h)\,(B_1 + B_2h) \end{aligned} \tag{4.5}$$

then we have $\widetilde{A\,B} = \tilde{A}\ \tilde{B}$.

Applying the third property for the matrices A and A^{-1} then we get (iv), furthermore, the properties given by *Theorem* 5 yields the rest. □

The fourth property of the previous theorem sets out that the image under Ψ_n of an invertible hybrid matrix is an invertible complex matrix. Hence we can talk about the determinant of a hybrid matrix by the combination of det and Ψ_n, then we can conclude the following.

Definition 4.3. Let $A \in M_n(\mathbb{K})$ and $\tilde{A} \in M_{2n}(\mathbb{C})$ be the adjoint matrix of A. The co-determinant of A is the complex determinant of $\tilde{A}$, denoted by $|A|_c$.

Theorem 4.4. *Let $A \in M_n(\mathbb{K})$. The following are equivalent:*
i. A is invertible,
ii. $Ax = 0$ has a unique solution, $x = 0$,
iii. The left (or right) eigenvalues of A do not vanish,
iv. $\tilde{A}$ is invertible.

Proof. (**i** $\Rightarrow$ **ii**) This is a trivial outcome.

(**ii** $\Longleftrightarrow$ **iii**) Assume that A has zero eigenvalue. Then the equation (3.3) satisfies, such as $Ax = 0$, for some non-zero values which is a contradiction.

(**iii** $\Rightarrow$ **iv**) Consider the second case instead of (iii), if $Ax = 0$ for $x = x_1 + x_2h \in M_{n,1}(\mathbb{K})$ then we have

$$A_1x_1 + A_2\overline{x_2} + (A_1x_2 + A_2\overline{x_1})\,h = 0 \text{ or}$$

$$A_1x_1 + A_2\overline{x_2} = 0 \text{ and } \overline{A_2}x_1 + \overline{A_1}\overline{x_2} = 0.$$

This means that the determinant of $\begin{bmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{bmatrix}$ is different from zero due to the unique solution.

(**iv** $\Rightarrow$ **i**) If $\tilde{A}$ is invertible, then there exist a complex matrix such that

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{bmatrix} = I_{2n}.$$

It follows that

$$Z_1A_1 + Z_2\overline{A_2} = I \text{ and } Z_1A_2 + Z_2\overline{A_1} = 0$$

which yields $\left(Z_1A_1 + Z_2\overline{A_2}\right) + \left(Z_1A_2 + Z_2\overline{A_1}\right) h = I$, then the hybrid matrix $Z = Z_1 + Z_2h$ is the inverse of A from *Remark 2*. □

Note that the last case of the previous equivalence theorem implies that a hybrid matrix is invertible if and only if its co-determinant must be different from zero.

Thus we can state the concept of general linear, special linear, and symplectic groups to the hybrid numbers, respectively, as follows:

$$\begin{aligned} GL_n(\mathbb{K}) &= \{A \in M_n(\mathbb{K}) \mid |A|_c \neq 0\}, \\ SL_n(\mathbb{K}) &= \{A \in M_n(\mathbb{K}) \mid |A|_c = 1\}, \\ SP_n(\mathbb{K}) &= \left\{A \in GL_n(\mathbb{K}) \mid A^{-1} = A^*\right\}. \end{aligned} \tag{4.6}$$

Furthermore, since any closed subgroup of $GL_n(\mathbb{C})$ is a Lie group, these groups are Lie groups. Then we can obtain the Lie algebras along with the bracket operation over matrices such as $[A,B] = AB - BA$, $A, B \in M_n(\mathbb{K})$. For example, the Lie algebra of $GL_n(\mathbb{K})$ is the set of all $n \times n$ matrices with entries in $\mathbb{K}$, that is

$$gl_n(\mathbb{K}) = M_n(\mathbb{K}) \tag{4.7}$$

and the Lie algebra of $SP_n(\mathbb{K})$ is the set

$$sp_n(\mathbb{K}) = \{A \in M_n(\mathbb{K}) \mid A + A^* = 0\} \tag{4.8}$$

and finally the Lie algebra of $SL_n(\mathbb{K})$ is the set

$$sl_n(\mathbb{K}) = \left\{A \in M_n(\mathbb{K}) \mid tr\left(\tilde{A}\right) = 0\right\} \tag{4.9}$$

where it can be easily seen that $tr\left(\tilde{A}\right) = 0$ if and only if scalar part of $tr(A)$ is zero.

Theorem 4.5. *Let $A, B \in M_n(\mathbb{K})$, the co-determinants satisfy the following properties,*
i. $\left|A\ B\right|_c = \left|A\right|_c \left|B\right|_c$,
ii. $\left|A^{-1}\right|_c = \left|A\right|_c^{-1}$, *if* $A \in GL_n(\mathbb{K})$,
iii. $\left|P\ A\ Q\right|_c = \left|A\right|_c$, *for* $P, Q \in SL_n(\mathbb{K})$,
iv. Cayley-Hamilton Theorem for hybrid matrices: Let A be a square hybrid matrix and the characteristic polynomial of A be $P_A(\lambda) = \left|\lambda I_{2n} - \tilde{A}\right|$, $\lambda \in \mathbb{C}$, then $P_A(A) = 0$.

Proof. (i). From the third property of theorem 6, we get $\left|A\ B\right|_c = \left|\widetilde{A\ B}\right| = |\tilde{A}\ \tilde{B}| = \left|A\right|_c \left|B\right|_c$,
(ii). From the fourth property of theorem 6, if $A \in GL_n(\mathbb{K})$ then we have $\left|A^{-1}\right|_c = \left|\widetilde{A^{-1}}\right| = \left|\left(\tilde{A}\right)^{-1}\right| = \left|A\right|_c^{-1}$,
(iii). If the hybrid $P, Q \in SL_n(\mathbb{K})$ which means they are elementary matrices and $\left|P\right|_c = \left|Q\right|_c = 1$, this completes the proof.
(iv). The coefficients of the polynomial $P_A(\lambda)$ are real [26]. Then we have $\widetilde{p(A)} = P_A(\tilde{A})$ for any real coefficient polynomial p. On the other hand, Cayley-Hamilton theorem for the complex matrices proves that $P_A(\tilde{A}) = 0$ for $\tilde{A} \in M_{2n}(\mathbb{C})$. This implies that $\widetilde{p(A)} = 0$, namely $P_A(A) = 0$. □

5. Conclusion

In number theory, different studies are available in which the complex, dual and hyperbolic numbers systems are expressed in the one sentence [3,13]. One of them has arisen recently and called the hybrid number. When the interdisciplinary applications of the constituent number systems are observed, we obviously see that the most effective results are obtained by their matrices. From a technique point of view, matrices can be taken into account as functional tools to organize the accumulated knowledge, accelerate the calculations and finally formulate the conclusions in various developed mathematical frameworks. The result of these motivations, the satisfactory concept of this study rises as the necessity of hybrid matrices.

The present paper is concerned with the linear algebra over hybrid matrices. However, we have to face the limitations on algebraic properties of the hybrid matrices by the fact that the hybrid multiplication is noncommutative. In this way, we firstly use an alternative form for hybrid numbers called the $\mathbb{C}-$type and obtain the subset of 2×2 complex matrices, $\mathcal{K}$ which represents the matrices corresponding to hybrid numbers by the transformation ψ_X. After describing and investigating the basic properties of hybrid matrices, we are aware of the need to rearrange them. Therefore, we define a continuous, injective ring homomorphism Ψ_n between $M_n(\mathbb{K})$ and $M_{2n}(\mathbb{C})$ by taking advantage of the effect of the transformation ψ_X on the hybrid numbers. Thus the adjoint matrices of hybrid matrices are obtained over complex numbers, this gives us the right to implement the properties of linear algebra over the complex field for hybrid matrices. Since Ψ_n turns an invertible hybrid matrix to an invertible complex matrix, we could have the one of the important result that is the calculation of determinant of hybrid matrices. This leads to describe general linear, special linear, and symplectic groups of the hybrid numbers and their corresponding Lie algebras, respectively. Finally, we state Cayley-Hamilton theorem for hybrid matrices.

6. Acknowledgments

The authors would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding

The authors declared that has not received any financial support for the research, authorship or publication of this study.

The Declaration of Conflict of Interest/ Common Interest

The authors declared no conflict of interest or common interest

The Declaration of Ethics Committee Approval

This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics

The authors declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the authors declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

References

[1] Y. Alagöz, K. H. Oral, S. Yüce, Split quaternion matrices, Miskolc Mathematical Notes, Vol.13, No.2, pp.223-232 (2012).

[2] S. A. Billings, Nonlinear system identification: NARMAX methods in the time, frequency, and spatio-temporal domains, John Wiley & Sons, (2013).

[3] F. Catoni, R. Cannata, V. Catoni, P. Zampetti, Two dimensional hypercomplex number and related trigonometries, Adv. Appl. Clifford Algebras, Vol.14, No.1, pp.47-68 (2004).

[4] F. Catoni, D. Boccaletti, R. Cannata, V. Catoni, E. Nichelatti, P. Zampetti, The mathematics of Minkowski space-time: with an introduction to commutative hypercomplex numbers, Birkhäuser, Basel, (2008).

[5] Y. Choo, On the generalized bi-periodic Fibonaci and Lucas quaternions, Miskolc Mathematical Notes, Vol.20, No.2, pp.807-821 (2019).

[6] W.K. Clifford, Preliminary sketch of biquaternions, Proc. Lond. Math. Soc., Vol.1, No.1, pp.381-395 (1871).

[7] J. Cockle, III. On a new imaginary in algebra, Lond. Edinb. Dublin Philos. Mag., Vol.34, No.226, pp.37-47 (1849).

[8] N. Cohen, S. De Leo, The quaternionic determinant, The Electronic Journal of Linear Algebra, Vol.7, pp.100-111 (2000).

[9] G. Dattoli, et al., Hybrid complex numbers: the matrix version, Adv. Appl. Clifford Algebras, Vol.28, pp.1-17 (2018).

[10] P.A.M. Dirac, The principles of quantum mechanics, 4th edition, Oxford University Press, Oxford, (1958).

[11] M. Erdoğdu, M. Özdemir, On eigenvalues of split quaternion matrices, Adv. Appl. Clifford Algebras, Vol.23, No.3, pp.615-623 (2013).

[12] W. R. Hamilton, Elements of quaternions, Longmans, Green, & Company, (1866).

[13] A.A. Harkin, J.B. Harkin, Geometry of generalized complex numbers, Math. Mag., Vol.77, No.2, pp.118-29 (2004).

[14] L. Huang, W. So, On left eigenvalues of a quaternionic matrix, Linear algebra and its applications, Vol.323, No.1-3, pp.105-116 (2001).

[15] T. Jiang, Z. Zhang, Z. Jiang, Algebraic techniques for eigenvalues and eigenvectors of a split quaternion matrix in split quaternionic mechanics, Computer Physics Communications,Vol.229, pp.1-7 (2018).

[16] V.V. Kisil, Erlangen program at large-1: geometry of invariants, SIGMA Symmetry Integr. Geom. Methods Appl., Vol.6, No.076, pp.45 (2010).

[17] M. Ozdemir, Introduction to hybrid numbers. Adv. Appl. Clifford Algebras, Vol.28, pp.1-32 (2018).

[18] M. Ozdemir, Finding n-th roots of a 2×2 real matrix using De Moivre's formula, Adv. Appl. Clifford Algebras, Vol.29, No.1, pp. 2 (2019).

[19] Ç. Ramis, Y. Yaylı, Dual split quaternions and Chasles' theorem in 3 dimensional Minkowski space $\mathbb{E}_1^3$, Adv. Appl. Clifford Algebras, Vol.23, pp.951-964 (2013).

[20] B. A. Rosenfeld, A history of Non-Euclidean geometry, Studies in the History of Mathematics and Physical Sciences, Springer, (1988).

[21] G. Sobczyk, New foundations in mathematics: the geometric concept of number, Birkhäuser, Boston, (2013).

[22] A. Szynal-Liana, The Horadam hybrid numbers, Discuss. Math. Gen. Algebra Appl., Vol.38, pp.91-98 (2018).

[23] I.M. Yaglom, Complex Numbers in Geometry, Academic Press, New York, (1968).

[24] I.M. Yaglom, A Simple Non-Euclidean Geometry and its Physical Basis, Heidelberg Science Library, Springer, New York, (1979).

[25] Y. Yazlik, S. Kome, C. Kome, Bicomplex generalized $k-$Horadam quaternions, Miskolc Mathematical Notes, Vol.20, No.2, pp.1315-1330 (2019).

[26] F. Zhang, Quaternions and matrices of quaternions. Linear Algebra Appl., Vol.251, pp.21-57 (1997).

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