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A Different Perspective for Geometric Series with Binomial Coefficients

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ABSTRACT

The study of mathematical series has long been a fascinating and essential component of mathematics, providing valuable insights into numerous real-world applications and theoretical concepts. Among the various types of series, the "Geometric Series with Binomial Coefficients" stands out as a particularly intriguing and powerful subject of investigation.

A geometric series is a sequence of terms in which each successive term is obtained by multiplying the previous one by a constant factor, known as the common ratio. This classical concept has found extensive applications in fields like finance, physics, engineering, and computer science, making it an indispensable tool for solving a wide array of problems.

However, in the context of the "Geometric Series with Binomial Coefficients," we encounter a fascinating twist that elevates the complexity and versatility of the series. Instead of dealing with constant factors as in the traditional geometric series, the coefficients in this new variant are given by the binomial coefficient formula. Binomial coefficients, also known as "n choose k," are fundamental in combinatorial mathematics and represent the number of ways to choose k elements from a set of n elements.

This work presents a new approach for the computation to geometric series with binomial coefficients. The geometric series with binomial coefficients is derived from the multiple summations of a geometric series. In this article, several theorems and corollaries are established on the innovative geometric series and its binomial coefficients to get practical way for calculations.

MSC Classification codes : 11B65, 05A10, 65B10, 40A05

Keywords: Binomial coefficients; Factorials; Geometric series; Combinatorial functions; Numerical summation of series; Convergence and divergence of series and sequences; Computations**.**

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Binom Katsayılı Geometrik Serilere Farklı Bir Bakış

ÖZ

Matematiksel serilerin incelenmesi, matematiğin uzun süredir büyüleyici ve temel bir bileşeni olmuştur ve birçok gerçek dünya uygulaması ve teorik kavramlar konusunda değerli içgörüler sunmaktadır. Çeşitli seriler arasında, "Binom Katsayılı Geometrik Seri", özellikle ilgi çekici ve güçlü bir araştırma konusu olarak ön plana çıkar.

Geometrik bir seri, her bir ardışık terimin bir öncekinin bir sabit çarpanla çarpılmasıyla elde edildiği bir terim dizisidir; bu sabit çarpana "ortalama oran" denir. Bu klasik kavram, finans, fizik, mühendislik ve bilgisayar bilimleri gibi birçok alanda geniş uygulama alanı bulmuş olup, geniş bir yelpazedeki problemlerin çözümünde vazgeçilmez bir araç haline gelmiştir.

Ancak, "Binom Katsayılı Geometrik Seri" bağlamında, serinin karmaşıklığını ve çok yönlülüğünü artıran büyüleyici bir farkla karşılaşırız. Geleneksel geometrik serilerdeki gibi sabit çarpanlarla uğraşmak yerine, bu yeni türe göre katsayılar, binom katsayısı formülü tarafından belirlenir. Binom katsayıları, kombinatorik matematikte temel bir rol oynar ve n elemandan oluşan bir kümeden k eleman seçmenin kaç farklı yol olduğunu temsil eder.

Bu çalışma, binom katsayılı geometrik serilerin hesaplanması için yeni bir yaklaşım sunmaktadır. Binom katsayılı geometrik seriler, geometrik serilerin çeşitli toplamlarından elde edilir. Bu makalede, hesaplamalarda pratik bir yol elde etmek amacıyla yenilikçi geometrik seriler ve onun binom katsayıları üzerine çeşitli teoremler ve sonuçlar oluşturulmuştur.

Anahtar Sözcükler: Binom katsayıları; Faktöriyeller; Geometrik seriler; Kombinatoryal fonksiyonlar; Serilerin sayısal toplamı; Seri ve dizilerin yakınsaklığı ve ıraksaması; Hesaplamalar.

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1. INTRODUCTION

The study of this subject involves various approaches and methods to understand and analyze its properties, convergence behavior, and applications. There are some common approaches and methods used in exploring this topic as follows:

The fundamental starting point for studying the geometric series with binomial coefficients is the binomial coefficient formula, also known as "n choose k." This formula calculates the number of ways to choose k elements from a set of n elements. Understanding the properties and applications of binomial coefficients lays the groundwork for working with geometric series featuring these coefficients.

The problem which is defined as Deriving closed-form expressions for geometric series with binomial coefficients is a crucial aspect of the analysis. Researchers often seek algebraic formulas that succinctly express the sum of such series, allowing for efficient computations and theoretical insights. Various mathematical techniques, such as combinatorial identities, generating functions, and calculus, can aid in finding closed-form expressions.

For convergence of the series, investigating the convergence behavior of geometric series with binomial coefficients is essential to ascertain when such series converge or diverge. Analyzing the relationship between the common ratio (determined by binomial coefficients) and the series sum can help determine under what conditions the series converges to a finite value. Geometric series with binomial coefficients can often be described using recurrence relations, where each term in the series depends on previous terms. Developing and solving these recurrence relations can provide valuable insights into the long-term behavior of the series.

Moreover, geometric series with binomial coefficients find applications in probability and statistics, particularly in problems related to counting and probability distributions. Techniques from probability theory and combinatorial analysis are frequently employed to tackle such problems. As binomial coefficients have a strong connection to combinatorial mathematics, researchers frequently draw on combinatorial techniques to analyze and understand the properties of geometric series with binomial coefficients. Generating functions provide a powerful tool for studying combinatorial structures and series. By defining generating functions for geometric series with binomial coefficients, researchers can efficiently compute coefficients, derive closed-form expressions, and explore various combinatorial properties.

Differential equations can arise when studying certain types of geometric series with binomial coefficients. By formulating and solving relevant differential equations, researchers can gain insights into the behavior of the series. Also, mathematical induction is a common technique used to prove statements about geometric series with binomial coefficients. By establishing base cases and proving the inductive step, researchers can verify properties that hold for all positive integers.

Also it is seen that there are many applıcatıons of the work ın scıence and real life too. For example, Geometric series with binomial coefficients are used in financial mathematics to calculate future values of investments, annuities, and loans. They are also applicable in the field of option pricing and risk assessment. Binomial coefficients are fundamental in combinatorial mathematics, and their use in geometric series extends to various combinatorial problems. These applications can include counting the number of paths in graphs, arrangements of objects, and partitioning sets into subsets.

Geometric series with binomial coefficients are utilized in various biological studies, such as understanding genetic inheritance patterns, population genetics, and analysing DNA sequences. Geometric series with binomial coefficients play a role in feature engineering, probability distributions, and analysing patterns in large datasets. In game theory, these series are used to model decision-making processes and strategies in competitive interactions. In the field of cryptography, binomial coefficients are used in various cryptographic algorithms and protocols to ensure security and confidentiality.

Astawa, Budayasa, and Juniati (2018) delve into the process of student cognition in constructing mathematical conjectures, likely related to binomial coefficients' properties or their applications. Che (2017) establishes a relation between binomial coefficients and Fibonacci numbers raised to higher powers, exploring a specific mathematical connection. Desh Ranjan, John E. Savage, and Mohammad Zubair (2011) discuss strong lower bounds for computation graphs related to binomial coefficients and Fast Fourier Transform (FFT), offering insights into computational complexity.

Echi (2006) explores the historical link between binomial coefficients and Nasir al-Din al-Tusi, providing a historical context for their study. Ferreira (2010) introduces the Integer Binomial Plan, a generalization involving factorials and binomial coefficients, extending their traditional application. Flusser and Francia (2000) focus on deriving and visualizing the binomial theorem, likely emphasizing graphical or visual representations of these coefficients. Gavrikov (2018) discusses properties of binomial coefficients and their application to growth modeling, showing their relevance in modeling real-world phenomena.

Goss (2011) presents an overview of the ongoing developments in the field of binomial coefficients, indicating a continuously evolving area of study. Harne, Badshah, and Verma (2015) explore polynomial identities related to binomial coefficients and Pascal's Triangle, likely providing new insights into their algebraic properties. Hwang (2009) offers a simple proof of the binomial theorem using differential calculus, showcasing alternative approaches to understanding and proving fundamental theorems. Khmelnitskaya, van der Laan, and Talman (2016) discuss generalizations of binomial coefficients to numbers on graphs, possibly extending their application to graph theory.

Leavitt (2011) examines the complexity of Pascal's Triangle, likely shedding light on intricate patterns and structures within this mathematical construct.Lundow and Rosengren (2010) study the p, q-binomial distribution and its connection to the Ising model, bridging binomial coefficients with statistical physics.

Mazur (1997) contributes to conjectural mathematics, potentially involving conjectures related to binomial coefficients or related mathematical structures. Milenkovic, Popovic, Dimitrijevic, and Stojanovic (2019) explore binomial coefficients and their visualization, likely offering graphical representations or visualization techniques.

Norton (2000) discusses student conjectures in geometry, possibly involving hypotheses or conjectures related to geometric applications of binomial coefficients. Nurhasanah, Kusumah, and Sabandar (2017) exemplify mathematical abstraction using the concept of triangles, potentially demonstrating how binomial coefficients can be applied in diverse contexts. Ossanna (2015) investigates fractal dimensions within Pascal's Triangle under square-free moduli, exploring the fractal properties of binomial coefficients. Rosalky (2007) presents a simple probabilistic proof of the binomial theorem, offering an alternative proof method involving probability concepts. Ross (2010) and Rudin (1962) are textbooks likely offering foundational knowledge on probability theory and Fourier analysis, respectively, which may include sections on binomial coefficients within their contexts. All information can be found in the [1-35] references.

When the authors of this article was trying to compute the multiple summations of a geometric series, a new idea stimulated his mind to create a new type of geometric series. As a result, a geometric with binomial coefficients, called 'combinatorial geometric series', was developed with a new approach of binomial coefficient. In this study, we have obtained a useful formulation that enables very practical calculations by expressing a simpler writing on multiple sums of geometric series from a different perspective.

2. MAIN RESULTS

MULTIPLE SUMMATIONS OF GEOMETRIC SERIES

Before theorems, we can briefly remember the topic mentioned with different techniques in [1-35] with numerical examples given as follows:

Example 2.1. (Binomial Coefficients) Consider the combinatorial geometric series:

$$S = \sum_{k=0}^{n} \binom{n}{k} 2^k \tag{1}$$

This series arises when you expand $(1+2)^n$ using the binomial theorem. Let us calculate the series for a specific value of *n*. Then ıt is obtained that S = 81

So, when *n* = 4, the sum of this combinatorial geometric series is 81.

Note: A Fibonacci sequence is a series of numbers where each number is the sum of the two preceding ones. In mathematical terms, it can be defined as:

$$F(0) = 0,\ F(1) = 1 \ \text{(initial conditions) and } F(n) = F(n-1) + F(n-2) \text{ for } n > 1 \tag{2}$$

To express the Fibonacci sequence as a combinatorial geometric series, we can use the following identity:

$$\mathrm{F(n)} = (1/\sqrt{5}).\left([(1 + \sqrt{5})/2]^n - [(1 - \sqrt{5})/2]^n\right) \tag{3}$$

where (1 + √5)/2 is the golden ratio (approximately 1.61803398875). This expression is derived from the Binet formula for the Fibonacci sequence. So, it can be expressed the Fibonacci sequence as a combinatorial geometric series using the Binet formula as shown above.

Example 2.2. (Triple Summation of a Geometric Series) In a similar way, let us consider a triple summation:

$$\mathrm{S} = \sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{k=0}^{r} 2^i \,.\, 3^j \,.\, 4^k \tag{4}$$

For p = 1, q = 1 and r = 1:

$$\mathrm{S} = \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1} 2^i \,.\, 3^j \,.\, 4^k \tag{5}$$

By calculating each term, we obtain S = 135. Thus, when p = 1, q = 1 and r = 1, the triple summation is 135.

In general **[Binomial Coefficient Geometric Series (n Choose r)],** if we consider a series where binomial coefficients $\binom{n}{r}$ are multiplied by a geometric progression:

$$\mathrm{S} = \sum_{r=0}^{n} \binom{n}{r} . \left(\frac{1}{2}\right)^r \tag{6}$$

and calculate it for n = 4, then we get

$$\mathrm{S} = \binom{4}{0}.\left(\frac{1}{2}\right)^0 + \binom{4}{1}.\left(\frac{1}{2}\right)^1 + \binom{4}{2}.\left(\frac{1}{2}\right)^2 + \binom{4}{3}.\left(\frac{1}{2}\right)^3 + \binom{4}{4}.\left(\frac{1}{2}\right)^4 = \frac{31}{16} \tag{7}$$

These are examples of multi-summations of geometric series, where the terms involve nested geometric series with different bases and different summation limits.

A geometric series with binomial coefficients [1-4, 18-29] is derived from the multiple summations of a geometric series. The binomial coefficient V_n^r is defined as a coefficient of x^k in the geometric series $\sum_{i=0}^{n} x^i$ ([5-17]).

Let us compute the multiple summations of a geometric series and its binomial coefficient based-geometric series.

$$\sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\sum_{i_3=i_2}^{n}\cdots\sum_{i_r=i_{r-1}}^{n} x^{i_r} = \sum_{i=0}^{n} V_i^r x^i \text{, where } r \text{ refers to the total number of summations.}$$

Here, $\sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\sum_{i_3=i_2}^{n}\cdots\sum_{i_r=i_{r-1}}^{n} x^{i_r}$ denotes the $(r+1)$summations of a geometric series and

$\sum_{i=0}^{n} V_i^r x^i$ is the geometric series with binomial coefficient V_n^r, whose expansion is given below:

$$V_n^r = \frac{(n+1)(n+2)(n+3)\cdots(n+r)}{r!} \tag{8}$$

where r is a positive integer and n is a non-negative integer ([30-35]).

Remark: Note that $V_n^0 = V_0^0 = 0! = 1$ and $V_0^r = \frac{(0+1)(0+2)(0+3)\cdots(0+r)}{r!} = \frac{r!}{r!} = 1.$

If $r = 0$ in $\sum_{i=0}^{n} V_i^r x^i$, then $\sum_{i=0}^{n} V_i^0 x^i = \sum_{i=0}^{n} x^i$ is the single summation of a geometric series.

If $r = 1$, then $\sum_{i=0}^{n} V_i^1 x^i$ denotes the double summations of a geometric series, that is,

$$\sum_{i=0}^{n}\sum_{j=i}^{n} x^j = \sum_{j=0}^{n} x^j + \sum_{j=1}^{n} x^j + \cdots + \sum_{j=n}^{n} x^j = 1 + 2x + 3x^2 + \cdots + (n+1)x^n = \sum_{i=0}^{n} V_i^1 x^i. \tag{9}$$

If $r = 2$, then $\sum_{i=0}^{n} V_i^1 x^i$ denotes the triple summations of a geometric series, that is,

$$\sum_{i=0}^{n}\sum_{j=i}^{n}\sum_{k=i}^{n} x^k = \sum_{j=0}^{n}\sum_{k=i}^{n} x^k + \sum_{j=1}^{n}\sum_{k=i}^{n} x^k + \sum_{j=2}^{n}\sum_{k=i}^{n} x^k + \cdots + \sum_{j=n}^{n}\sum_{k=i}^{n} x^k$$

$$= \sum_{i=0}^{n} V_i^2 x^i. \tag{10}$$

Similarly, the $(r+1)$ summations of a geometric series (for $r = 0, 1, 2, 3, \cdots$) is as follows [21-28]:

$$\sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\sum_{i_3=i_2}^{n}\cdots\sum_{i_r=i_{r-1}}^{n} x^{i_r} = \sum_{i=0}^{n} V_i^r x^i. \tag{11}$$

As a different way and practically, it is satisfied that multi summation of geometric series can be also determined as follows:

Theorem 2.1: The following equation is provided for $n \in \mathbb{Z}_+$ values.

$$\sum_{i=0}^{n} V_i^{r+1} x^i = \sum_{i=0}^{n} V_i^r x^i + \sum_{i=1}^{n} V_{i-1}^r x^i + \sum_{i=2}^{n} V_{i-2}^r x^i + \cdots + \sum_{i=n}^{n} V_{i-n}^r x^i. \tag{12}$$

Proof: Let's show that the expression on the right-hand side of the theorem is equal to the expression on the left- hand side of the theorem.

$$\sum_{i=0}^{n} V_i^r x^i + \sum_{i=1}^{n} V_{i-1}^r x^i + \sum_{i=2}^{n} V_{i-2}^r x^i + \cdots + \sum_{i=n-1}^{n} V_{i-(n-1)}^r x^i + \sum_{i=n}^{n} V_{i-n}^r x^i$$

$$= (V_0^r + V_1^r x + V_2^r x^2 + V_3^r x^3 + \cdots + V_n^r x^n) + \ (V_0^r x + V_1^r x^2 + V_2^r x^3 + V_3^r x^4 + \cdots + V_{n-1}^r x^n)$$

$$+(V_0^r x^2 + V_1^r x^3 + V_2^r x^4 + V_3^r x^5 + \cdots + V_{n-2}^r x^n) + \cdots + (V_0^r x^{n-1} + V_1^r x^n) + V_0^r x^n$$

$$= V_0^r + (V_0^r + V_1^r)x + (V_0^r + V_1^r + V_2^r)x^2 + \cdots + (V_0^r + V_1^r + V_2^r + V_3^r + \cdots + V_n^r)x^n$$

$$(\text{Here, } V_0^r + V_1^r + V_2^r + \cdots + V_n^r = V_n^{r+1} \text{ for } r = 0, 1, 2, 3, \cdots)$$

$$= V_0^{r+1} + V_1^{r+1}x + V_2^{r+1}x^2 + V_3^{r+1}x^3 + V_4^{r+1}x^4 + \cdots + V_{n-1}^{r+1}x^{n_1} + V_n^{r+1}x^n = \textstyle\sum_{i=0}^n V_i^{r+1}x^i.$$

Hence, theorem is proved.

As conclusions of the Theorem 2.1. following results can be given:

Corollary 2.1: The existence of the following equation is possible

$$\textstyle\sum_{i=0}^n V_i^{r+1} = \sum_{i=0}^n V_i^r + \sum_{i=1}^n V_{i-1}^r + \sum_{i=2}^n V_{i-2}^r + \cdots + \sum_{i=n}^n V_{i-n}^r \tag{13}$$

for $n \in \mathbb{Z}_+$ values.

***Proof*:** Considering $x = 1$ and put into the Theorem 2.2, then Corollary 2.1 is obtained trivially.

Corollary 2.2: The equation given as below is provided for $n \in \mathbb{Z}_+ \cup \{0\}$.

$$V_n^{r+1} = V_0^r + V_1^r + V_2^r + \cdots + V_{n-1}^r + V_n^r. \tag{14}$$

where $r = 0,1,2,3,\cdots$.

Proof. Necessary proving is shown below for the Corollary 2.2.

$$\sum_{i=n}^{n} V_i^{r+1}x^i = \sum_{i=n}^{n} V_i^r x^i + \sum_{i=n}^{n} V_{i-1}^r x^i + \sum_{i=n}^{n} V_{i-2}^r x^i + \cdots + \sum_{i=n}^{n} V_{i-(n-1)}^r x^i + \sum_{i=n}^{n} V_{i-n}^r x^i$$

$$\Rightarrow V_n^{r+1}x^n = V_n^r x^n + V_{n-1}^r x^n + V_{n-2}^r x^n + \cdots + V_3^r x^n + V_2^r x^n + V_0^r x^n +$$

$$\Rightarrow V_n^{r+1}x^n = (V_n^r + V_{n-1}^r + V_{n-2}^r + \cdots + V_3^r + V_2^r + V_0^r)x^n$$

$$\Rightarrow V_n^{r+1} = V_n^r + V_{n-1}^r + V_{n-2}^r + \cdots + V_3^r + V_2^r + V_0^r \tag{15}$$

$$\therefore V_n^{r+1} = V_0^r + V_1^r + V_2^r + \cdots + V_n^r$$

where $r = 0, 1, 2, 3, \cdots$

Theorem 2.2: The equality expressed below is satisfied for nonnegative n and r integer values;

$$\textstyle\sum_{i=0}^n V_i^{r+1}x^i = V_0^r \sum_{i=0}^n x^i + V_1^r \sum_{i=1}^n x^i + V_2^r \sum_{i=2}^n x^i + V_3^r \sum_{i=3}^n x^i + \cdots + V_n^r \sum_{i=n}^n x^i. \tag{16}$$

Proof. Let us prove this theorem using the following binomial expansion.

$$\textstyle\sum_{i=0}^n V_i^{r+1}x^i = \sum_{i=0}^n V_i^r x^i + \sum_{i=1}^n V_{i-1}^r x^i + \sum_{i=2}^n V_{i-2}^r x^i + \cdots + \sum_{i=n-1}^n V_{i-(n-1)}^r x^i + \sum_{i=n}^n V_{i-n}^r x^i.$$

The binomial expansion on the right-hand side of the equation is further expressed as follows:

$$\sum_{i=0}^{n} V_i^r x^i + \sum_{i=1}^{n} V_{i-1}^r x^i + \sum_{i=2}^{n} V_{i-2}^r x^i + \cdots + \sum_{i=n}^{n} V_{i-n}^r x^i$$
$$= (V_0^r x^0 + V_0^r x^1 + V_0^r x^2 + \cdots + V_0^r x^n) + (V_1^r x^1 + V_1^r x^2 + V_1^r x^2 + \cdots + V_1^r x^n)$$
$$+ (V_2^r x^2 + V_2^r x^2 + V_2^r x^2 + \cdots + V_2^r x^n) + \cdots + V_n^r x^n.$$

By simplifying this expression, we get

$$\sum_{i=0}^{n} V_i^{r+1} x^i = V_0^r \sum_{i=0}^{n} x^i + V_1^r \sum_{i=1}^{n} x^i + V_2^r \sum_{i=2}^{n} x^i + V_3^r \sum_{i=3}^{n} x^i + \cdots + V_{n-1}^r \sum_{i=n-1}^{n} x^i + V_n^r \sum_{i=n}^{n} x^i.$$

Proof is completed.

Following conclusion is extreated from the Theorem 2.2.

Corollary 2.3:

$$\sum_{i=0}^{n} V_i^{r+1} x^i = \sum_{i=0}^{n} (n+1-i) V_i^n. \tag{17}$$

Proof. Trivial… if x = 1 put into the Theorem 2.2, then result is obtained.

3. CONCLUSION

In conclusion, the study on combinatorial geometric series has yielded a valuable formulation that provides a fresh perspective on simplifying complex expressions involving multiple sums of geometric series. The research has introduced a practical approach for conducting calculations with greater ease and efficiency.

By reimagining the representation of multiple sums of geometric series, the study has contributed to the field's mathematical tools and problem-solving techniques. This new formulation can be particularly advantageous in various mathematical and scientific disciplines where these types of series commonly arise. The practicality of this formulation lies in its ability to streamline calculations that would otherwise be laborious and time-consuming. It offers researchers and practitioners a more accessible way to tackle problems that involve nested geometric series with combinatorial coefficients, saving both time and computational resources.

In summary, the study's achievement in providing a simplified and practical approach to handling combinatorial geometric series signifies a significant contribution to the field of mathematics and related disciplines. This novel perspective opens up new avenues for solving complex problems efficiently and enhances the toolbox of mathematicians, scientists, and researchers.

In this article, a technique has been introduced for computing the multiple summation of a geometric series into a single geometric series with binomial coefficients. Also, some theorems were provided using the geometric series.

DECLARATION

This study does not require ethics committee approval.

CONFLICT OF INTEREST

There is no conflict of authors in this work.

CONTRIBUTIONS OF AUTHORS

C.A.: Methodology, formal analysis, investigation, resources, original draft preparation.

Ö.Ö.: Conceptualization, methodology, validation, formal analysis, investigation, resources, writing-review, extending and editing.

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