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Topological Group Construction in Proximity and Descriptive Proximity Spaces

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Abstract. This paper introduces the topological group structure in proximity and especially descriptive proximity spaces, that is, the concepts of proximal group and descriptive proximal group are introduced. In addition, the concepts of homomorphism and isomorphism, which give important results in group theory, are discussed by interpreting the concepts of continuity in the theory of (descriptive) proximity.

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1. Introduction

Topology concerns about the study of properties preserved under continuous transformations, capturing the concept of nearness between elements of a set. Over the years, various approaches to topological spaces have been explored, each offering unique perspectives on the fundamental notions of continuity and proximity [1, 3, 10, 14, 22–24]. One such approach that has gained significant attention is the nearness theory, which provides an alternative framework for analyzing topological structures through the concept of descriptive proximity [17].

In nearness theory, the traditional notion of open sets is replaced with a more intuitive concept of near sets, characterized by a binary relation that describes the qualitative closeness between elements in a set. This approach introduces the notion of proximity spaces, which generalizes the concept of metric spaces and provides a deeper understanding of the relationships between points based on qualitative descriptions rather than precise distances. Furthermore, in descriptive proximity spaces, proximity relations are tailored to specific features or characteristics, making them particularly suitable for applications in areas such as data analysis and pattern recognition [14, 18, 19].

The aim of this article is to explore the construction of topological groups within the context of nearness theory, with a specific focus on proximity and descriptive proximity spaces. As a slightly different concept, the studies [6, 11, 12] combine the ideas of topological space and near groups in order to define topological (or semitopological) near groups on a nearness approximation space and investigate their features such as group homomorphism of these groups. Topological groups, which combine algebraic and topological structures, offer a natural setting for studying the interplay between group operations and continuous mappings [5]. By leveraging the concepts of proximity and descriptive proximity, we seek to investigate the topological properties of these groups and the implications they hold for the overall structure of the underlying space.

In Section 2, we provide a concise overview of the fundamental concepts and definitions in nearness theory, and establish a framework for the subsequent discussions. This includes introducing the concept of proximity relations and

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their axiomatic properties, as well as delving into the qualitative nature of descriptive proximity relations. The main part of this article presents the construction of topological groups in proximity and descriptive proximity spaces. We explore the compatibility of group operations with nearness structures, investigating the behavior of near sets under group multiplication and inversion. We also give interesting examples in terms of the properties of (descriptive) proximal groups. One of the important results explicitly investigates a homomorphism, or more strongly an isomorphism, between proximal groups. Furthermore, there are important implications about the proximal group setting of isomorphism theorems of groups. Section 4 is dedicated to introducing the concept of descriptive proximal groups. Along with exciting examples in this section, we clearly state that our investigation is not only of theoretical interest but also holds practical implications. Topological groups constructed within descriptive proximity spaces have the potential to find applications in diverse fields, ranging from data analysis and pattern recognition to the study of social networks and cognitive sciences.

In summary, this article endeavors to contribute to the burgeoning field of nearness theory by exploring the topological group construction within proximity and descriptive proximity spaces. By offering a fresh perspective on the interplay between group structures and nearness relations, we aim to enrich the understanding of topological properties in nearness-based settings and open up new avenues for future research.

2. Preliminaries

In this section, we provide a framework for proximity and descriptive proximity spaces. These facts will be frequently used in Section 3 and 4. We first start with presenting the definition of proximity spaces with respect to Lodato [10], Čech [1], and Efremovič [3].

Let Y be a space. Then, 2^Y denotes the *power set* of Y, i.e., 2^Y is the collection of all subsets of Y.

Definition 2.1 ([10]). Given a nonempty space Y, a relation δ on 2^Y is said to be a *Lodato proximity*, provided that the following properties hold for all subsets B_1, B_2, and B_3 in Y.

- **L1.** $B_1\ \delta\ B_2$ implies $B_2\ \delta\ B_1$.
- **L2.** $B_1\ \delta\ B_2$ implies that B_1 and B_2 are nonempty.
- **L3.** $B_1 \cap B_2 \neq \emptyset$ implies $B_1\ \delta\ B_2$.
- **L4.** $B_1\ \delta\ (B_2 \cup B_3)$ if and only if $B_1\ \delta\ B_2$ or $B_1\ \delta\ B_3$.
- **L5.** For each $b_2 \in B_2$, $B_1\ \delta\ B_2$ and $\{b_2\}\ \delta\ B_3$ imply $B_1\ \delta\ B_3$.

Here, $B_1\ \delta\ B_2$ is interpreted as "B_1 is near B_2", whereas $B_1\ \underline{\delta}\ B_2$ is read as "B_1 is far from B_2". Another definition of the proximity is given by E. Čech [1]: The relation δ on 2^Y is said to be a *Čech proximity* if the properties $L1 - L4$ hold. In addition, δ is called an *Efremovič Proximity* [3] provided that the properties of Čech proximity ($L1 - L4$) satisfy the extra condition.

EF $B_1\ \underline{\delta}\ B_2$ implies that there exists a subset K of Y such that $B_1\ \underline{\delta}\ K$ and $(Y - K)\ \underline{\delta}\ B_2$.

In this paper, if we want to emphasize the Lodato proximity relation or Čech proximity relation, we use the expression (L-proximity) or (Č-proximity) for short, respectively. Unless otherwise emphasized, the simple expression δ refers to Efremovič proximity. Therefore, (Y, δ) is called a *proximity space* and simply denoted by (*pspc*). Notice that Efremovič proximity is stronger than (L-proximity) or (Č-proximity). The *discrete proximity* (Y, δ), one of the basic proximity examples, is given by $B_1\ \delta\ B_2$ if and only if $B_1 \cap B_2 \neq \emptyset$ for all $B_1, B_2 \in 2^Y$ [13].

It is possible to get a topology $\tau(\delta)$ induced by a proximity δ on Y. Let K be a subset of (Y, δ). Then it is said to be closed if and only if cl$K$ = K. Thus, the collection of the complements of closed subsets of Y yields a topology $\tau(\delta)$ on Y (see Theorem 2.2 of [13]). The set of all points in Y that are near B_1 which is defined by

$$B_1^{\delta} = \{y \in Y : \{y\}\ \delta\ B_1\},$$

is known as the closure of a subset B_1, indicated by the symbol clB_1 [13]. Mathematically, one has $B_1^{\delta} = \mathrm{cl}B_1$. Then, by considering Kuratowski closure axioms [8], a topology $\tau(\delta)$ can be associated with the (pspc) (Y, δ).

In proximity spaces, *continuity* is defined using the proximity relation instead of open sets. Explicitly, a map $k : (Y_1, \delta_1) \to (Y_2, \delta_2)$ between two (pspc)s is considered continuous (we generally say *proximally continuous* and simply denoted by (pcont)) if it preserves proximity; that is, for any subsets B_1 and B_2 of Y, if B_1 is near B_2 with respect to δ_1, then $k(B_1)$ is near $k(B_2)$ with respect to δ_2 [3, 23]. If k_1 and k_2 are two (pcont) maps, then so is their composition $k_1 \circ k_2$ [13]. A (pcont) map $k : (Y_1, \delta_1) \to (Y_2, \delta_2)$ is said to be a *proximal isomorphism* if its inverse $k^{-1} : (Y_2, \delta_2) \to (Y_1, \delta_1)$ is also (pcont) [13].

When one has two (pspc)s (Y_1, δ_1) and (Y_2, δ_2), it is possible to obtain a new (pspc) $(Y_1 \times Y_2, \delta)$ by the cartesian product of them. The *cartesian product proximity relation* δ is given as follows [9]. For any given subsets $(B_1 \times B_2)$, $(C_1 \times C_2) \in 2^{Y_1 \times Y_2}$, $(B_1 \times B_2)\ \delta\ (C_1 \times C_2)$ if and only if $B_1\ \delta_1\ C_1$ and $B_2\ \delta_2\ C_2$. Assume that (Y, δ) is a (pspc) and V is a subset of Y. Another new proximity δ_V, called an *induced or subspace proximity*, is defined by $B_1\ \delta_V\ B_2$ if and only if $B_1\ \delta\ B_2$ for all $B_1, B_2 \in 2^V$ [13].

The isomorphism theorems are fundamental results in group theory that describe the relationship between groups and their subgroups, as well as the structure of factor groups. They are also powerful tools in group theory and help us understand the structural aspects of groups, especially when dealing with homomorphisms and factor groups. They provide valuable insights into the relationship between groups and their quotients, which allows us to analyze the structures of groups more effectively. Recall that, given a group $(G_1, \cdot)$ with a normal subgroup $N_1 \subseteq G_1$, G_1/N_1 is defined as the set $\{aN_1 : a \in G_1\}$.

Theorem 2.2 ([4]). ***i)*** *Assume that $\beta : G_1 \to H_1$ is a homomorphism of groups, the kernel of β, given by*

$$Ker(\beta) = \{g_1 \in G_1 : \beta(g_1) = e_{H_1}\} \subseteq G_1,$$

is a normal subgroup, and $\beta(G_1) \subseteq H_1$ is a subgroup. Then, $G_1/Ker(\beta)$ is isomorphic to $Im(\beta)$.

ii) *Given a group G_1, and subgroups H_1 and N_1 of G with N_1 being a normal subgroup of G_1, we have that $H_1N_1 \subseteq G_1$ is a subgroup, $N_1 \subseteq H_1N_1$ is a normal subgroup, and the intersection $H_1 \cap N_1 \subseteq H_1$ is a normal subgroup. Then H_1N_1/N_1 is isomorphic to $H_1/(H_1 \cap N_1)$.*

iii) *Assume that G_1 is a group, and N_1 and K_1 are normal subgroups of G_1 with $N_1 \subseteq K_1$. Then, $(G_1/N_1)/(K_1/N_1)$ is isomorphic to G_1/K_1.*

In Theorem 2.2, **i)**, **ii)**, and **iii)** are generally known as *First Isomorphism Theorem*, *Second Isomorphism Theorem*, and *Third Isomorphism Theorem*, respectively.

A descriptive proximity relation, denoted by δ_Φ in general, on a nonempty set Y is a binary relation that captures the concept of nearness or closeness between elements in the set by their descriptions [15–17]. It provides a qualitative way to compare how close or similar two elements are to each other, without involving precise distance measurements as in metric spaces.

Consider the nonempty set Y with any element (object) $y \in Y$. Define J as a set of features of the object y. As an illustration of this, we observe the shape feature of a box as an object in Figure 1. For each $j \in J$, ϕ_j is a function from Y to $\mathbb{R}$ and takes any element y to the feature value of it. The set of probe functions is denoted by $\Phi = \{\phi_j\}_{j \in J}$. An object's description can be found in a feature vector Φ. For any subsets $B_1, B_2 \in 2^Y$, $B_1\ \delta_\Phi\ B_2$ if and only if $\Phi(B_1) \cap \Phi(B_2) \neq \emptyset$, where $\Phi(C_1)$ is given by the sets $\{\Phi(c_1) : c_1 \in C_1\}$. Here, $B_1\ \delta_\Phi\ B_2$ means that B_1 is *descriptively near* B_2 (similarly, $B_1\ \underline{\delta_\Phi}\ B_2$ is used to say B_1 is *descriptively far from* B_2) and δ_Φ is called a *descriptive proximity relation* on the subsets of Y. The *descriptive intersection* for the subsets B_1 and B_2 of Y is defined by $\{b \in B_1 \cup B_2 : \Phi(b) \in \Phi(B_1) \cap \Phi(B_2)\}$ and generally denoted by $B_1 \underset{\Phi}{\bigcap} B_2$.

Definition 2.3 ([2]). Let Y be a nonempty space and B_1, B_2, and B_3 in 2^Y. Then, a relation δ_Φ is said to be a *descriptive Lodato proximity* provided that the following properties hold.

DL1. $B_1\ \delta_\Phi\ B_2$ implies $B_2\ \delta_\Phi\ B_1$.
DL2. $B_1\ \underline{\delta_\Phi}\ \emptyset$ for all B_1 in 2^Y.
DL3. The fact that the descriptive intersection of B_1 and B_2 is nonempty implies $B_1\ \delta_\Phi\ B_2$.
DL4. $B_1\ \delta_\Phi\ (B_2 \cup B_3)$ if and only if $B_1\ \delta_\Phi\ B_2$ or $B_1\ \delta_\Phi\ B_3$.
DL5. For each $b_2 \in B_2$, $B_1\ \delta_\Phi\ B_2$ and $\{b_2\}\ \delta_\Phi\ B_3$ imply that $B_1\ \delta_\Phi\ B_3$.

The relation δ_Φ on 2^Y is said to be a *descriptive Efremovič proximity* if the properties **DL1-DL4** hold, and in addition,

DEF $B_1\ \underline{\delta_\Phi}\ B_2$ implies that there exists a subset K of Y such that $B_1\ \underline{\delta_\Phi}\ K$ and $(Y - K)\ \underline{\delta_\Phi}\ B_2$

satisfies.

(Y, δ_Φ) is called a *descriptive proximity space* and simply denoted by (dpspc). A map $k : (Y_1, \delta_{\Phi_1}) \to (Y_2, \delta_{\Phi_2})$ between two (dpspc)s is considered continuous (we generally say *descriptive proximally continuous* and simply denoted by (dpcont)) if it preserves descriptive proximity; that is, for any subsets B_1 and B_2 of Y, if B_1 is descriptively near B_2

with respect to δ_{Φ_1}, then $k(B_1)$ is descriptively near $k(B_2)$ with respect to δ_{Φ_2} [20]. If k_1 and k_2 are two (dpcont) maps, then so is their composition. A (dpcont) map $k : (Y_1, \delta_{\Phi_1}) \to (Y_2, \delta_{\Phi_2})$ is called a *descriptive proximal isomorphism* if its inverse $k^{-1} : (Y_2, \delta_{\Phi_2}) \to (Y_1, \delta_{\Phi_1})$ is also (dpcont) [20].

Let (Y_1, δ_{Φ_1}) and (Y_2, δ_{Φ_2}) be any (dpspc)s. Then, their cartesian product $Y_1 \times Y_2$ admits a *cartesian product descriptive proximity relation* δ_Φ defined as follows [21]. For any $(B_1 \times B_2), (C_1 \times C_2) \in 2^{Y_1 \times Y_2}$, $(B_1 \times B_2)\ \delta_\Phi\ (C_1 \times C_2)$ if and only if $B_1\ \delta_{\Phi_1}\ C_1$ and $B_2\ \delta_{\Phi_2}\ C_2$. Assume that (Y, δ_Φ) is a (dpspc) and V is a subset of Y. A *descriptive induced (or subspace) proximity*, denoted by δ_{Φ_V} is defined by $B_1\ \delta_{\Phi_V}\ B_2$ if and only if $B_1\ \delta_\Phi\ B_2$ for all $B_1, B_2 \in 2^V$.

Given two (dpspc)s, (Y_1, δ_{Φ_1}) and (Y_2, δ_{Φ_2}), the descriptive proximal mapping space $Y_2^{Y_1}$ is defined as the set

$$\{\beta : Y_1 \to Y_2 \mid \beta \text{ is a (dpcont)-map}\}$$

having the following descriptive proximity relation δ_Φ on itself [7]: Let B_1, $B_2 \subseteq Y$ and $\{\gamma_j\}_{j \in J}$ and $\{\gamma_k^{'}\}_{k \in K}$ be any subsets of (dpcont) maps in $Y_2^{Y_1}$. We say that $\{\gamma_j\}_{j \in J}\ \delta_\Phi\ \{\gamma_k^{'}\}_{k \in K}$ provided that $B_1\ \delta_{\Phi_1}\ B_2$ implies that $\gamma_j(B_1)\ \delta_{\Phi_2}\ \gamma_k^{'}(B_2)$ for all j and k.

3. Proximal Groups

Definition 3.1. Let $(G_1, \cdot)$ be a group and δ a proximity relation on G_1. Then, $(G_1, \delta, \cdot)$ is said to be a proximal group when

$$\mu_1 : G_1 \times G_1 \to G_1,$$

defined by $\mu_1(g_1, g_1^{'}) = g_1 \cdot g_1^{'}$ for any $g_1, g_1^{'} \in G_1$, and

$$\mu_2 : G_1 \to G_1,$$

defined by $\mu_2(g_1) = g_1^{-1}$ for any $g_1 \in G_1$, are (pcont) maps.

Recall that, for any subsets B_1, B_2 and B_3 of a topological group G_1, B_1^{-1} and $B_2 \cdot B_3$ are given by $\{b_1^{-1} : b_1 \in B_1\}$ and $\{b_2 \cdot b_3 \mid b_2 \in B_2,\ b_3 \in B_3\}$, respectively.

Example 3.2. Consider $\mathbb{R} - \{0\}$ with a proximity δ, defined by

$$B_1\ \delta\ B_2 \text{ if and only if } D(B_1, B_2) = 0,$$

where $D(B_1, B_2)$ represents the distance between sets B_1 and B_2 in $\mathbb{R} - \{0\}$, and the group operation $\cdot$ for the subsets of $\mathbb{R} - \{0\}$. Then we shall show that $(\mathbb{R} - \{0\}, \delta, \cdot)$ is a proximal group. Define the maps

$$\mu_1 : \mathbb{R} - \{0\} \times \mathbb{R} - \{0\} \to \mathbb{R} - \{0\} \quad \text{and} \quad \mu_2 : \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$$

with $\mu_1(g_1, g_1^{'}) = g_1 \cdot g_1^{'}$ for any $g_1, g_1^{'} \in \mathbb{R} - \{0\}$, and, $\mu_2(g_1) = g_1^{-1}$ for any $g_1 \in \mathbb{R} - \{0\}$, respectively. First, μ_1 is (pcont). Indeed, for any subsets $B_1 \times B_2, C_1 \times C_2 \in \mathbb{R} - \{0\} \times \mathbb{R} - \{0\}$, the fact $B_1 \times B_2$ is near $C_1 \times C_2$ implies that B_1 is near C_1 and B_2 is near C_2. This means that $D(B_1, C_1) = 0$ and $D(B_2, C_2) = 0$, respectively. Therefore, there exist $b_1 \in B_1$, $b_2 \in B_2$, $c_1 \in C_1$, and $c_2 \in C_2$ such that $d(b_1, c_1) = 0$ and $d(b_2, c_2) = 0$. It follows that $b_1 = c_1$ and $b_2 = c_2$, i.e., $b_1 \cdot b_2 = c_1 \cdot c_2$. Thus, $(B_1 \cdot B_2) \cap (C_1 \cdot C_2) \neq \emptyset$, which says that $(B_1 \cdot B_2)\ \delta\ (C_1 \cdot C_2)$. Next, we claim that μ_2 is (pcont). Let B_1 and B_2 be any subsets of $\mathbb{R} - \{0\}$ such that $B_1\ \delta\ B_2$. Then, $D(B_1, B_2) = 0$, that is, there exist $b_1 \in B_1$ and $b_2 \in B_2$ such that $d(b_1, b_2) = 0$. It follows that $b_1 = b_2$, i.e., $b_1^{-1} = b_2^{-1}$. Thus, $B_1^{-1} \cap B_2^{-1} \neq \emptyset$, which says that $B_1^{-1}\ \delta\ B_2^{-1}$. Consequently, $(\mathbb{R} - \{0\}, \delta, \cdot)$ forms a proximal group.

Example 3.3. Let G_2 be an abelian group, and define the proximity relation δ on G_2 as follows: For any sets B_1, $B_2 \in G_2$, we say $B_1\ \delta\ B_2$ if and only if $B_1^{-1} \cdot B_2$ is of finite order in G_2. The map $\mu_1 : G_2 \times G_2 \to G_2$, $\mu_1(g_2, g_2^{'}) = g_2 \cdot g_2^{'}$, is (pcont): Let $B_1 \times B_2$, $C_1 \times C_2 \in G_2 \times G_2$ be subsets such that $B_1 \times B_2$ is near $C_1 \times C_2$. Then, we have that $B_1\ \delta\ C_1$ and $B_2\ \delta\ C_2$, i.e., $B_1^{-1}C_1$ and $B_2^{-1}C_2$ are of finite order in G_2, respectively. Therefore, $(B_2^{-1}C_2) \cdot (B_1^{-1}C_1)$ is of finite order in G_2. Since G_2 is an abelian group, $(B_1B_2)^{-1}C_1C_2$ is of finite order in G_2, which means that $B_1 \cdot B_2$ is near $C_1 \cdot C_2$ in $G_2 \times G_2$. Moreover, the map $\mu_2 : G_2 \times G_2 \to G_2$, $\mu_2(g_2) = g_2^{-1}$, is (pcont): Let B_1, $B_2 \in G$ with $B_1\ \delta\ B_2$. Then $B_1^{-1}B$ is of finite order n in G_2, i.e., the $n-$times product $(B_1^{-1}B_2)(B_1^{-1}B_2) \cdots (B_1^{-1}B_2)$ is the identity e_{G_2} of G_2. Since G_2 is abelian, it follows that

$$\begin{aligned}(B_1^{-1}B_2)(B_1^{-1}B_2)\cdots(B_1^{-1}B_2) = e_{G_2} &\Rightarrow (B_2B_1^{-1})(B_2B_1^{-1})\cdots(B_2B_1^{-1}) = e_{G_2}\\ &\Rightarrow B_2B_1^{-1}B_2B_1^{-1}\cdots B_2B_1^{-1} = e_{G_2}\\ &\Rightarrow e_{G_2} = B_1B_2^{-1}B_1B_2^{-1}\cdots B_1B_2^{-1}\\ &\Rightarrow (B_1B_2^{-1})(B_1B_2^{-1})\cdots(B_1B_2^{-1}) = e_{G_2}.\end{aligned}$$

Hence, $B_1B_2^{-1}$ is of finite order n in G_2, namely, $B_1^{-1}\ \delta\ B_2^{-1}$ in G_2. As a result, $(G_2, \delta, \cdot)$ forms a proximal group.

Theorem 3.4. *Let $(G_1, \delta, \cdot)$ be a proximal group and $x_1 \in G_1$. Then,*

$$L_{x_1} : G_1 \to G_1,$$

defined by $L_{x_1}(g_1) = x_1 \cdot g_1$, and

$$R_{x_1} : G_1 \to G_1,$$

defined by $R_{x_1}(g_1) = g_1 \cdot x_1$, are proximal isomorphisms.

Proof. First, we shall show that L_{x_1} is a proximal isomorphism. Define a map

$$\nu_{x_1} : G_1 \to G_1 \times G_1$$

by $\nu_{x_1}(y_1) = (x_1, y_1)$. $B_1\ \delta\ B_2$ implies that $(\{x_1\}, B_1)\ \delta'\ (\{x_1\}, B_2)$ for any subsets B_1, $B_2 \in G_1$, where δ' is a proximity on $G_1 \times G_1$. It follows that ν_{x_1} is (pcont). Since G_1 is a proximal group, μ_1 is (pcont). Therefore, L_{x_1} is (pcont) because $L_{x_1} = \mu_1 \circ \nu_{x_1}$. With the same method, the proximal continuity of $L_{x_1}^{-1}$ can be easily shown by considering the fact

$$L_{x_1}^{-1} = L_{x_1^{-1}}.$$

Hence, L_{x_1} is a proximal isomorphism. The argument is similar for R_{x_1}. □

We say that a group G_1 has the invertible subset property with respect to a subset $B_1 \subseteq G_1$ provided that $B_1 \cdot B_1^{-1}$ is $\{e_{G_1}\}$ and $B_1^{-1} \cdot B_1$ is $\{e_{G_1}\}$. Note that a group always has an invertible subset property with respect to its one-point subsets.

Lemma 3.5. *Let δ be a proximity and $\cdot$ a group operation on a set G_1, respectively. Assume that $\mu_1 : G_1 \times G_1 \to G_1$, $\mu_1(g_1 \cdot g_1') = g_1 \cdot g_1'$, is (pcont) and G_1 has the invertible subset property with respect to any subset of it. Then $(G_1, \delta, \cdot)$ is a proximal group.*

Proof. We shall show that $\mu_2 : G_1 \to G_1$, $\mu_2(x_1) = x_1^{-1}$ is a (pcont) map. Let $B_1, B_2 \subseteq G_1$ with $B_1\ \delta\ B_2$. Then, we have $(B_1^{-1} \cdot B_1)\ \delta\ (B_1^{-1} \cdot B_2)$. Since B_1 is invertible, $\{e_{G_1}\}\ \delta\ (B_1^{-1} \cdot B_2)$. It follows that $B_2^{-1}\ \delta\ (B_1^{-1} \cdot (B_2 \cdot B_2^{-1}))$. Therefore, we get $B_2^{-1}\ \delta\ B_1^{-1}$ because B_2 is invertible. This proves that μ_2 is (pcont). □

Theorem 3.6. *Let δ be a proximity and $\cdot$ a group operation on a set G_1, respectively. Assume that G_1 has the invertible subset property with respect to any subset of it. If δ admits the transitivity property, i.e., $B_1\ \delta\ B_2$ and $B_2\ \delta\ B_3$ imply that $B_1\ \delta\ B_3$ for any subsets B_1, B_2, $B_3 \subseteq G_1$, then $(G_1, \delta, \cdot)$ is a proximal group.*

Proof. It is enough to show that μ_1 in Definition 3.1 is (pcont) by Lemma 3.5. Let $B_1 \times B_2\ \delta\ C_1 \times C_2$ in $G_1 \times G_1$. Then, $B_1\ \delta\ C_1$ and $B_2\ \delta\ C_2$. $B_1\ \delta\ C_1$ and the proximal continuity of R_{x_1} imply that $B_1 \cdot B_2$ is near $C_1 \cdot B_2$ in $G_1 \times G_1$. Similarly, $B_2\ \delta\ C_2$ and the proximal continuity of L_{x_1} imply that $C_1 \cdot B_2$ is near $C_1 \cdot C_2$ in $G_1 \times G_1$. The transitivity property of δ says that $B_1 \cdot B_2$ is near $C_1 \cdot C_2$ in $G_1 \times G_1$. It follows that $\mu_1(B_1 \times B_2)\ \delta\ \mu_1(C_1 \times C_2)$, which means that μ_1 is (pcont). □

In Theorem 3.6, if we specifically choose the Lodato proximity δ' on G_1, we need a slightly weaker condition instead of the transitivity property as follows:

Corollary 3.7. *Let δ' be a Lodato proximity and $\cdot$ a group operation on a set G_1, respectively. Assume that G_1 has the invertible subset property with respect to any subset of it. If δ' admits that $B_1\ \delta\ B_2$ imply $\{x_1\}\ \delta\ B_2$ for all $x_1 \in B_1$, then $(G_1, \delta', \cdot)$ is a proximal group.*

Proof. Assume that $B_1 \cdot B_2$ is near $C_1 \cdot B_2$ and $C_1 \cdot B_2$ is near $C_1 \cdot C_2$ for any $B_1 \times B_2$ and $C_1 \times C_2$ in $G_1 \times G_1$. Since $C_1 \cdot B_2$ is near $C_1 \cdot C_2$, it follows that $\{c_1b_2\}$ is near $C_1 \cdot C_2$ for all $c_1b_2 \in C_1 \cdot B_2$. Therefore, we get $B_1 \cdot B_2$ is near $C_1 \cdot C_2$, which proves that μ_1 is (pcont). □

Proposition 3.8. *Let $(G_1, \delta, \cdot)$ be a proximal group and H_1 a subgroup of G_1. Then, $(H_1, \delta_{H_1}, \cdot)$ is a proximal group.*

Proof. Since $(G_1, \delta, \cdot)$ is a proximal group,

$$\mu_1 : G_1 \times G_1 \to G_1, \quad \mu_1(g_1, g_1^{'}) = g_1 \cdot g_1^{'}$$

and

$$\mu_2 : G_1 \to G_1, \quad \mu_2(g_1) = g_1^{-1}$$

are (pcont). Then, the restrictions

$$\mu_1|_{H_1 \times H_1} : H_1 \times H_1 \to H_1$$

defined by $\mu_1|_{H_1 \times H_1}(h_1, h_1^{'}) = h_1 \cdot h_1^{'}$ and

$$\mu_2|_{H_1} : H_1 \to H_1$$

defined by $\mu_2|_{H_1}(h_1) = h_1^{-1}$ are (pcont), respectively. This shows that $(H_1, \delta_{H_1}, \cdot)$ is a proximal group. □

Note that, in Proposition 3.8, $(H_1, \delta_{H_1}, \cdot)$ is said to be a proximal subgroup of $(G_1, \delta, \cdot)$. As an example, $\mathbb{R}^+$ is a proximal subgroup of $\mathbb{R} - \{0\}$ in Example 3.2.

Proposition 3.9. *Given any proximal groups $(G_1, \delta_1, \circ)$ and $(G_2, \delta_2, *)$, their cartesian product $G_1 \times G_2$ is also a proximal group.*

Proof. Since G_1 is a proximal group with a proximity δ_1 and a group operation $\circ$, we have that

$$\mu_1 : G_1 \times G_1 \to G_1, \ \mu_1(g_1, g_1^{'}) = g_1 \circ g_1^{'} \quad \text{and} \quad \mu_2 : G_1 \to G_1, \ \mu_2(g_1) = g_1^{-1}$$

are (pcont). Similarly, from the proximal group construction of G_2, we have that

$$\mu_1^{'} : G_2 \times G_2 \to G_2, \ \mu_1^{'}(g_2, g_2^{'}) = g_2 * g_2^{'} \quad \text{and} \quad \mu_2^{'} : G_2 \to G_2, \ \mu_2^{'}(g_2) = g_2^{-1}$$

are (pcont). Define two maps

$$\mu_3 : (G_1 \times G_2) \times (G_1 \times G_2) \to G_1 \times G_2$$

and

$$\mu_4 : G_1 \times G_2 \to G_1 \times G_2$$

by $\mu_3((g_1, g_2), (g_1^{'}, g_2^{'})) = (\mu_1(g_1, g_1^{'}), \mu_2(g_2, g_2^{'}))$ and $\mu_4(g_1, g_2) = (\mu_1^{'}(g_1), \mu_2^{'}(g_2))$, respectively. Then, μ_3 and μ_4 are (pcont) from the definition of cartesian product proximity. Thus, $G_1 \times G_2$ is a proximal group having the product proximity $\delta_1 \times \delta_2$ on itself. □

Definition 3.10. Let $(G_1, \delta_1, \cdot_1)$ and $(G_2, \delta_2, \cdot_2)$ be proximal groups. Then, $\eta : G_1 \to G_2$ is called a homomorphism of proximal groups, provided that it is (pcont) group homomorphism. Furthermore, η is called an isomorphism of proximal groups, if it is a group isomorphism and also a proximal isomorphism.

Example 3.11. Consider the antipodal map $\eta : (\mathbb{R}, \delta, +) \to (\mathbb{R}, \delta, +)$, $\eta(x) = -x$, where δ is given in Example 3.2, and $+$ is the usual additive group operation. For $B_1, B_2 \in \mathbb{R}$, $B_1 \ \delta \ B_2$ means that $B_1 \cap B_2 \neq \emptyset$. Then, there exists $r \in \mathbb{R}$ such that r belongs to both B_1 and B_2. Since $(\mathbb{R}, +)$ is a group, r has an inverse $-r$ in $\mathbb{R}$. It follows that $-r$ belongs to both $-B_1$ and $-B_2$, i.e., $(-B_1) \cap (-B_2) \neq \emptyset$. Therefore,

$$\eta(B_1) = (-B_1) \ \delta \ (-B_2) = \eta(B_2).$$

Thus, η is (pcont). Similarly, it can be easily shown that η^{-1} is a (pcont) map. On the other hand, we observe

$$\eta(B_1 + B_2) = -(B_1 + B_2) = -B_1 + (-B_2) = \eta(B_1) + \eta(B_2),$$

which shows that η is a group homomorphism. As a consequence, η is a proximal group isomorphism.

Theorem 3.12. *Let $\eta : (G_1, \delta_1, \cdot_1) \to (G_2, \delta_2, \cdot_2)$ be a group homomorphism between two proximal groups G_1 and G_2 such that they have the invertible subset property with respect to any subset of them. Then, η is a proximal homomorphism provided that $B_1 \ \delta_1 \ \{e_{G_1}\}$ implies that $\eta(B_1) \ \delta_2 \ \{e_{G_2}\}$.*

Proof. Let $B_1\ \delta_1\ B_2$ for any B_1, $B_2 \in G_1$. Then, $(B_1B_2^{-1})\ \delta_1\ (B_2B_2^{-1}) = \{e_{G_1}\}$ because G_1 is a proximal group. It follows that $\eta(B_1B_2^{-1})\ \delta_2\ \eta(\{e_{G_2}\})$. Since η is a group homomorphism, $\eta(B_1B_2^{-1}) = \eta(B_1)\eta(B_2)^{-1}$ and $\eta(\{e_{G_1}\}) = \{e_{G_2}\}$. Therefore, we get $(\eta(B_1)\eta(B_2)^{-1})\ \delta_2\ \{e_{G_2}\}$. G_2 is a proximal group, so we find $\eta(B_1)\ \delta_2\ \eta(B_2)$, which shows that η is (pcont). □

In the topological setting, the isomorphism theorems from ordinary group theory are not necessarily valid because an isomorphism of topological groups need not be a bijective homomorphism. For topological groups, for instance, a native version of the first isomorphism theorem is false: if $\eta : G_1 \to G_2$ is a morphism of topological groups (i.e., a continuous homomorphism), it is not always true that the induced homomorphism $\widetilde{\eta} : G_1/Ker(\eta) \to Im(\eta)$ is an isomorphism of topological groups; it will be a bijective, continuous homomorphism, but it need not be a homeomorphism. There is a version of the first isomorphism theorem for topological groups, which may be stated as follows: if $\eta : G_1 \to G_2$ is a continuous homomorphism, then the induced homomorphism $\widetilde{\eta} : G_1/Ker(\eta) \to Im(\eta)$ is an isomorphism if and only if η is open onto its image. Similarly, the second isomorphism theorem does not hold for topological groups. However, it is easy to verify that the third isomorphism theorem holds true for topological groups.

Now, with Proposition 3.13, we check whether the isomorphism theorems on proximal groups are satisfied. Before that, we shall show that $(\mathbb{R}, \delta, +)$ is a proximal group whenever δ is a discrete proximity or a proximity defined by

$$B_1\ \delta\ B_2 \ \text{ if and only if } \ D(B_1, B_2) = 0,$$

where $D(B_1, B_2)$ represents the distance between sets B_1 and B_2 in $\mathbb{R}$, and $B_1 + B_2$ is given by the set

$$\{b_1 + b_2 \mid b_1 \in B_1, b_2 \in B_2\}.$$

First, assume that δ is a discrete proximity. Define the maps

$$\mu_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad \text{ and } \quad \mu_2 : \mathbb{R} \to \mathbb{R}$$

with $\mu_1(g_1, g_1^{'}) = g_1 + g_1^{'}$ for any g_1, $g_1^{'} \in \mathbb{R}$, and, $\mu_2(g_1) = -g_1$ for any $g_1 \in \mathbb{R}$, respectively. For any subsets $B_1 \times B_2$, $C_1 \times C_2 \in \mathbb{R} \times \mathbb{R}$, the fact $B_1 \times B_2$ is near $C_1 \times C_2$ implies that $B_1\ \delta\ C_1$ and $B_2\ \delta\ C_2$. This means that $B_1 \cap C_1 \neq \emptyset$ and $B_2 \cap C_2 \neq \emptyset$, respectively. Therefore, there exist two elements x_1, x_2 in $\mathbb{R}$ such that $x_1 \in B_1$, $x_1 \in C_1$, $x_2 \in B_2$, and $x_2 \in C_2$. It follows that $x_1 + x_2 \in B_1 + B_2$ and $x_1 + x_2 \in C_1 + C_2$, i.e., $(B_1 + B_2) \cap (C_1 + C_2) \neq \emptyset$. Thus, $(B_1 + B_2)\ \delta\ (C_1 + C_2)$, which says that μ_1 is (pcont). Next, we claim that μ_2 is (pcont). Let B_1 and B_2 be any subsets of $\mathbb{R}$ such that $B_1\ \delta\ B_2$. Then, $B_1 \cap B_2 \neq \emptyset$, that is, there exist an element x in $\mathbb{R}$ such that x belongs to both B_1 and B_2. It follows that $-x$ belongs to both $-B_1$ and $-B_2$, i.e., $(-B_1) \cap (-B_2) \neq \emptyset$. Thus, $(-B_1)\ \delta\ (-B_2)$, which says that μ_2 is (pcont). Consequently, $(\mathbb{R}, \delta, +)$ forms a proximal group.

Now, assume that δ is a proximity defined by

$$B_1\ \delta\ B_2 \ \text{ if and only if } \ D(B_1, B_2) = 0.$$

We quickly have that $(\mathbb{R}, \delta, +)$ is also a proximal group when we consider the usual addition $+$ on $\mathbb{R}$ as the group operation in Example 3.2 by using the facts that $b_1 = c_1$ and $b_2 = c_2$ imply that $b_1 + b_2 = c_1 + c_2 \in (B_1 + B_2) \cap (C_1 + C_2)$, and $b_1 = b_2$ implies that $-b_1 = -b_2 \in (-B_1) \cap (-B_2)$.

Note also that, by Proposition 3.8, we observe that $(\mathbb{R} - \{0\}, \delta, +)$ is also a proximal group, where δ is a discrete proximity or a proximity defined by

$$B_1\ \delta\ B_2 \ \text{ if and only if } \ D(B_1, B_2) = 0.$$

Proposition 3.13. ***i)*** *The First Isomorphism Theorem does not hold for proximal groups.*
ii) *The Second Isomorphism Theorem does not hold for proximal groups.*
iii) *The Third Isomorphism Theorem holds for proximal groups.*

Proof. **i)** Let $(\mathbb{R}, \delta_1, +)$ and $(\mathbb{R}, \delta_2, +)$ be two proximal groups, where δ_1 is the discrete proximity and δ_2 is given by $B_1\ \delta_2\ B_2 \Leftrightarrow D(B_1, B_2) = 0$. The identity map

$$id : (\mathbb{R}, \delta_1, +) \to (\mathbb{R}, \delta_2, +)$$

is both a (pcont) map and a group homomorphism. Also, the identity map is surjective and $Ker(id)$ only consists of the identity element of $(\mathbb{R}, +)$. However, $\mathbb{R}/Ker(id)$, that is proximally isomorphic to $\mathbb{R}$ as proximal groups, with the discrete proximity is not proximally isomorphic to $\mathbb{R}$ with the proximity δ_2: Since $D(B_1, B_2) = 0$ does not always imply $B_1 \cap B_2 \neq \emptyset$ for any B_1, $B_2 \subseteq \mathbb{R}$, the inverse of the identity map id is not (pcont).

ii) Let $G_1 = (\mathbb{R}, \delta, +)$ be a proximal group with its subgroup $H_1 = \{2s \mid s \in \mathbb{Q}^c\}$ and its normal subgroup $N_1 = \mathbb{Z}$, where δ is the proximity δ_2 given in i). Then, the intersection of H_1 and N_1 is empty, which follows that, as proximal groups, H_1 is proximally isomorphic to $H_1/(H_1 \cap N_1)$ as proximal groups. Since $D(B_1, B_2) > 0$ for all $B_1, B_2 \subseteq H_1$, H_1 must have the discrete proximity. On the other hand, we have that $(H_1 + N_1)^\delta = \mathbb{R}$. Therefore, $[(H_1 + N_1)/N_1]^\delta = \mathbb{R}/\mathbb{Z}$, which means that $(H_1 + N_1)/N_1$ cannot have the discrete proximity. Consequently, the map

$$(H_1 + N_1)/N_1 \to H_1/(H_1 \cap N_1)$$

is not a group isomorphism of proximal groups.

iii) The proof is similar to the case of topological groups. □

Note that, a continuous map need not be (pcont). However, a (pcont) map is always continuous with respect to corresponding topologies. Hence, given a proximal group $(Y, \delta, \cdot)$, we have that $(Y, \tau(\delta), \cdot)$ is a topological group since (pcont) maps μ_1 and μ_2 in Definition 3.1 are also continuous maps with respect to the corresponding topologies. We say that $(Y, \tau(\delta), \cdot)$ is a topological group induced by δ. For instance, when we consider δ as the discrete proximity, $(Y, \tau(\delta), \cdot)$ forms a discrete topological group because $\tau(\delta)$ is a discrete topology induced by the discrete proximity δ.

Theorem 3.14. *Let $(G_1, \delta, \cdot)$ be a proximal group. Then, G_1 admits an Hausdorff topological group if and only if $\{e_{G_1}\}\ \delta\ B_1$ implies $B_1 = \{e_{G_1}\}$ for $B_1 \subseteq G_1$.*

Proof. The assertion is clear from the fact that, for a topological group G_1, it is Hausdorff if and only if $\{e_{G_1}\}$ is closed. □

In a topological group, the axioms T_0, T_1, and T_2 (Hausdorffness) coincide. This means that one can consider the topological group $(G_1, \delta(\tau), \cdot)$ as T_0 or T_1 instead of T_2 in Theorem 3.14.

4. Descriptive Proximal Groups

Definition 4.1. Let $(G_1, \cdot)$ be a group and δ_Φ a descriptive proximity relation on G_1. Then, $(G_1, \delta_\Phi, \cdot)$ is said to be a descriptive proximal group when

$$\mu_1 : G_1 \times G_1 \to G_1,$$

defined by $\mu_1(g_1, g_1^{'}) = g_1 \cdot g_1^{'}$ for any $g_1, g_1^{'} \in G$, and

$$\mu_2 : G_1 \to G_1,$$

defined by $\mu_2(g_1) = g_1^{-1}$ for any $g_1 \in G_1$, are (dpcont) maps.

Recall that a descriptive proximal path from any point g to any point $g^{'}$ in (G, δ_Φ) is a (dpcont) map

$$\mu : I = [0, 1] \to G$$

with $\mu(0) = g$ and $\mu(1) = g^{'}$.

Example 4.2. Let X be a set illustrated in Figure 1a), which consists of three boxes A, B, and C. Consider G as the set of all descriptive proximal paths on X. For any descriptive proximal paths γ_1, γ_2 in G with $\gamma_1(1) = \gamma_2(0)$, a group operation $*$ on G is defined by

$$\gamma_1 * \gamma_2(s) = \begin{cases} \gamma_1(2s), & 0 \leq s \leq 1/2 \\ \gamma_2(2s-1), & 1/2 \leq s \leq 1. \end{cases}$$

Note that $\gamma_i(s) \in \{A, B, C\}$ for each $i = 1, 2$. Consider Φ as a probe function that determines any descriptive proximal path by the order of the box names of that path. It effectively disregards the specific geometric details of the path and focuses solely on the order in which the boxes are visited. For instance, in Figure 1b), the blue path and the green path are both represented by ABC. Then, δ_Φ is a descriptive proximity on G. Indeed, two paths are considered descriptively near if their associated sequences of box names, as determined by Φ, are similar. In other words, paths are close in terms of δ_Φ if they traverse the same sequence of boxes, even if the specific geometric details of their paths differ. As an example, the blue path is descriptively near the green path. Consider the map $\mu_1 : G \times G \to G$ given by $\mu_1(\gamma_1, \gamma_2) = \gamma_1 * \gamma_2$. For any (γ_1, γ_2), $(\gamma_3, \gamma_4) \in G \times G$, the fact (γ_1, γ_2) is descriptively near (γ_3, γ_4) implies that $\gamma_1\ \delta_\Phi\ \gamma_3$ and $\gamma_2\ \delta_\Phi\ \gamma_4$. Then, for all s_1, $s_2 \in [0, 1]$, we have that $\gamma_1(s_1) = \gamma_3(s_1)$ and $\gamma_2(s_2) = \gamma_4(s_2)$. It follows that $\gamma_1 * \gamma_2 = \gamma_3 * \gamma_4$, which means that $\gamma_1 * \gamma_2$ is descriptively near $\gamma_3 * \gamma_4$. Hence, μ_1 is (dpcont). Moreover, the map

$\mu_2 : G \to G$ defined by $\mu_2(\gamma) = \gamma^{-1}$ is (dpcont). Indeed, for any $\gamma_1, \gamma_2 \in G$, $\gamma_1\ \delta_\Phi\ \gamma_2$ implies that $\gamma_1(s) = \gamma_2(s)$ for all $s \in [0,1]$. Therefore, $\gamma_1^{-1}(s) = \gamma_2^{-1}(s)$ for all $s \in [0,1]$. This shows that $\gamma_1^{-1}\ \delta_\Phi\ \gamma_2^{-1}$, and finally, $(G, \delta_\Phi, \circ)$ is a descriptive proximal group.

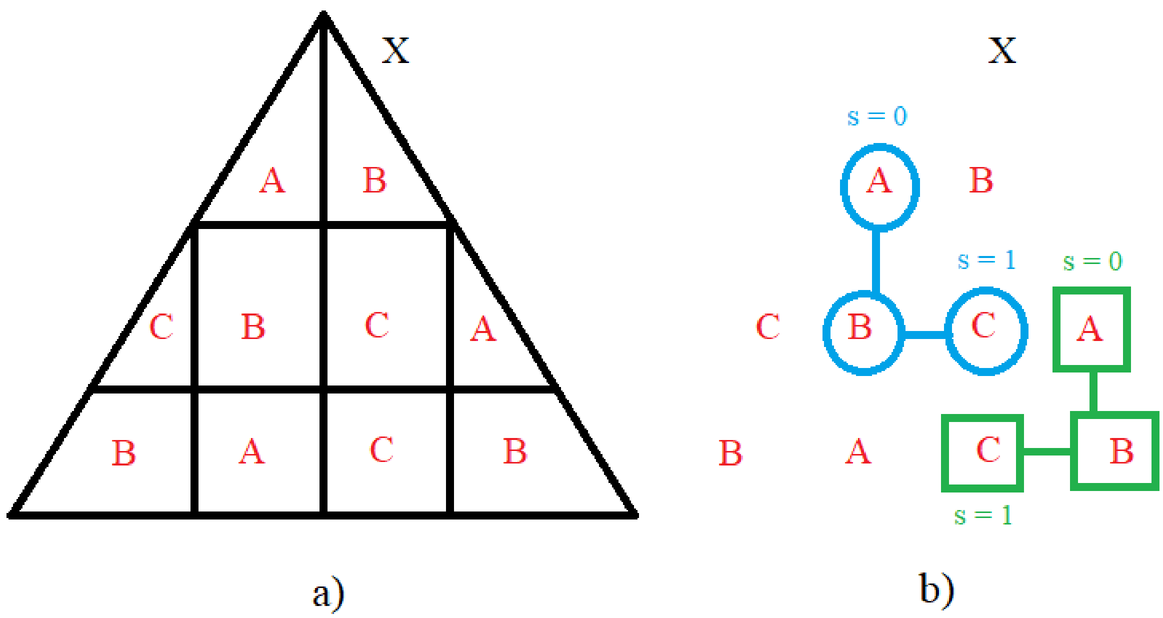

Figure 1. **a)** The picture X consists of boxes labelled A, B, and C.
b) The blue path ABC is descriptively near the green path. Orders of the paths are equal to each other.

Theorem 4.3. *Let $(G_1, \delta_\Phi, \cdot)$ be a descriptive proximal group and $x_1 \in G_1$. Then,*

$$L_{x_1} : G_1 \to G_1,$$

defined by $L_{x_1}(g_1) = x_1 \cdot g_1$, and

$$R_{x_1} : G_1 \to G_1,$$

defined by $R_{x_1}(g_1) = g_1 \cdot x_1$, are descriptive proximal isomorphisms.

Proof. The proof is parallel with Theorem 3.4 since the composition of (dpcont) maps is again (dpcont). □

Remark 4.4. For a descriptive proximal group $(G_1, \delta_\Phi, \cdot)$ and a subgroup H_1 of G_1, we say that $(H_1, \delta_{\Phi_{H_1}}, \cdot)$ is a descriptive proximal subgroup of $(G_1, \delta_\Phi, \cdot)$.

Example 4.5. Consider the additive group $(\mathbb{R}, +)$ and assume that $\mathbb{Z}^c$ denotes the set of all non-integer elements in $\mathbb{R}$. Let $\Phi = \{\phi_1, \phi_2\}$ be the probe function such that $\phi_1 : \mathbb{R} \to \mathbb{R}$ and $\phi_2 : \mathbb{R} \to \mathbb{R}$ are defined by $\phi_1(y) = trunc(y)$ and $\phi_2(y) = \begin{cases} y, & y \in \mathbb{Z}^c \\ y + 0.3, & y \in \mathbb{Z} \end{cases}$ for any $y \in \mathbb{R}$, respectively. Here, the function $trunc(y)$ truncates $y \in \mathbb{R}$ to an integer by removing the fractional part of the number. For example, $trunc(-3.4) = -3$ and $trunc(5.6) = 5$. Note that $\phi_1(y) \in \mathbb{Z}$ and $\phi_2(y) \in \mathbb{Z}^c$. To show that

$$\mu_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ \mu_1(y_1, y_2) = y_1 + y_2,$$

is (dpcont), we shall show that $B_1 \times B_2$ is descriptively near $C_1 \times C_2$ implies that $(B_1 + B_2)\ \delta_\Phi\ (C_1 + C_2)$ for any $B_1 \times B_2$, $C_1 \times C_2 \in \mathbb{R} \times \mathbb{R}$. Since $B_1 \times B_2$ is descriptively near $C_1 \times C_2$, we have that B_1 is descriptively near C_1 and B_2 is descriptively near C_2. When we consider $B_1\ \delta_\Phi\ C_1$ (i.e., $\Phi(B_1) \cap \Phi(C_1) \neq \emptyset$), there are some cases as follows.

- There exists $b_1 \in \mathbb{R}$ such that $b_1 \in B_1 \cap C_1$.
- There exist $b_1 \in \mathbb{Z}$ and $c_1 \in \mathbb{Z}^c$ such that $b_1 \in B_1$ and $c_1 \in C_1$ with $trunc(c_1) = b_1$.
- There exist $b_1 \in \mathbb{Z}$ and $c_1 \in \mathbb{Z}^c$ such that $b_1 \in B_1$ and $c_1 \in C_1$ with $c_1 = b_1 + 0.3$.

The cases are hold when we focus on $B_2 \ \delta_\Phi \ D_2$ (i.e., $\Phi(B_2) \cap \Phi(D_2) \neq \emptyset$):

- There exists $b_2 \in \mathbb{R}$ such that $b_2 \in B_2 \cap C_2$.
- There exist $b_2 \in \mathbb{Z}$ and $c_2 \in \mathbb{Z}^c$ such that $b_2 \in B_2$ and $c_2 \in C_2$ with $trunc(c_2) = b_2$.
- There exist $b_2 \in \mathbb{Z}$ and $c_2 \in \mathbb{Z}^c$ such that $b_2 \in B_2$ and $c_2 \in C_2$ with $c_2 = b_2 + 0.3$.

For all cases, we have that $\Phi(B_1 + B_2) \cap \Phi(C_1 + C_2) \neq \emptyset$. This means that $B_1 + B_2$ is descriptively near $C_1 + C_2$. Now, define

$$\mu_2 : \mathbb{R} \to \mathbb{R}, \ \mu_2(y) = -y.$$

Let $B_1, B_2 \in 2^{\mathbb{R}}$ with $B_1 \ \delta_\Phi \ B_2$. Then, there are three cases again.

- There exists $b_1 \in \mathbb{R}$ such that $b_1 \in B_1 \cap B_2$.
- There exist $b_1 \in \mathbb{Z}$ and $b_2 \in \mathbb{Z}^c$ such that $b_1 \in B_1$ and $b_2 \in B_2$ with $trunc(b_2) = b_1$.
- There exist $b_1 \in \mathbb{Z}$ and $b_2 \in \mathbb{Z}^c$ such that $b_1 \in B_1$ and $b_2 \in B_2$ with $b_2 = b_1 + 0.3$.

In each case we find that $-B_1$ is descriptively near $-B_2$:

- If there is a real number $b_1 \in B_1 \cap B_2$, then the real number $-b_1$ belongs to both $-B_1$ and $-B_2$.
- If $trunc(b_2) = b_1$, then $trunc(-b_2) = -b_1$.
- If $b_2 = b_1 + 0.3$, then $-b_1 = -b_2 + 0.3$.

Therefore, we observe that $-b_1 \in \Phi(-B_1) \cap \Phi(-B_2)$ for all cases, namely that, $(-B_1) \ \delta_\Phi \ (-B_2)$. This shows that μ_2 is (dpcont). Hence, $(\mathbb{R}, \delta_\Phi, +)$ is a descriptive proximal group. Moreover, the fact $(\mathbb{Q}, +)$ is a subgroup of $(\mathbb{R}, +)$ shows that $(\mathbb{Q}, \delta_{\Phi_{\mathbb{Q}}}, +)$ is a descriptive proximal subgroup.

Similar to Proposition 3.9, the cartesian product of two descriptive proximal groups is a descriptive proximal group. Consider the descriptive proximal group $(\mathbb{R}, \delta_\Phi, +)$ given in Example 4.5. Then, we have that $(\mathbb{R}^2, \delta_{\Phi'}, +)$ is also a descriptive proximal group, where $\delta_{\Phi'}$ is the cartesian product descriptive proximity $\delta_\Phi \times \delta_\Phi$.

Definition 4.6. Let $(G_1, \delta_{\Phi_1}, \circ)$ and $(G_2, \delta_{\Phi_2}, *)$ be any descriptive proximal groups. Then, $\eta : G_1 \to G_2$ is called a homomorphism of descriptive proximal groups provided that it is (dpcont) group homomorphism. Furthermore, η is called an isomorphism of descriptive proximal groups if it is a group isomorphism and also a descriptive proximal isomorphism.

Example 4.7. Let $(Y_1, \delta_{\Phi_1}, +)$ and $(Y_2, \delta_{\Phi_2}, +)$ be any proximal groups. Then, define a (dpcont) map

$$\nu : (Y_1, \delta_{\Phi_1}, +) \times (Y_2, \delta_{\Phi_2}, +) \to (Y_1, \delta_{\Phi_1}, +)$$

by $\nu(y_1, y_2) = y_1$. ν is a homomorphism of descriptive proximal groups. Indeed, for $B_1 \times B_2, C_1 \times C_2 \in 2^{Y_1 \times Y_2}$,

$$\begin{aligned} \nu[(B_1 \times B_2) + (C_1 \times C_2)] &= \nu[(B_1 + C_1), (B_2 + C_2)] \\ &= B_1 + C_1 \\ &= \nu(B_1 \times B_2) + \eta(C_1 \times C_2). \end{aligned}$$

However, ν is not an isomorphism of descriptive proximal groups: For any $B_1, C_1 \in 2^{Y_1}$, if $B_1 \ \delta_{\Phi_1} \ C_1$ and $B_2 \ \underline{\delta_{\Phi_2}} \ C_2$, then $B_1 \times B_2$ cannot be descriptively near $C_1 \times C_2$.

5. Conclusion

The study of topological groups in proximity and descriptive proximity spaces marks an important step in the process of nearness theory, providing a fresh perspective on the interplay between algebraic and topological structures. As we continue to explore the potential of nearness-based settings, this research opens up new avenues for future investigations, promising further advancements and exciting discoveries in the fascinating realm of topological groups.

It is necessary to mention an open problem on isomorphism theorems for groups. Intuitively, it can be thought that the first and second isomorphism theorem are not satisfied, but the third isomorphism theorem is satisfied, just as in proximity spaces. Making this clear with examples or proofs is to take the matter one step further. For another open problem, Lie groups setting in the theory of proximity (or descriptive proximity) can be considered. However, for this problem, first of all, the concept of a manifold and its related invariants should be studied extensively in the theory of nearness. As a result, it is very possible to obtain interesting results using the (descriptive) proximal group results on descriptive proximity theory.

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed the published version of the manuscript.

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