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RESEARCH ARTICLE

Unitary weighted composition operators on Bergman-Besov and Hardy Hilbert spaces on the ball

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ABSTRACT
On weighted Bergman and Hardy Hilbert spaces on the unit ball of the complex N-space, we consider weighted compositon operators T_ψ in which the composition is by an automorphism ψ of the unit ball and the weight is a power of the Jacobian of ψ in such a way that the operator is unitary. Assuming that the homogeneous expansion of an f in one of these spaces contains only terms with total degree even (odd, respectively) and the homogeneous expansion of $T_\psi f$ contains only terms with total degree odd (even, respectively), we prove that f is the zero function. We also find related operators on the remaining Bergman-Besov Hilbert spaces including the Drury-Arveson space and the Dirichlet space for which the same result holds. Our results generalize the results obtained in Montes-Rodríguez (2023) on three function spaces on the unit disc to a wider family of function spaces on the unit ball.

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1. INTRODUCTION

In a recent paper Montes-Rodríguez (2023), the author proves that specific unitary weighted composition operators by the automorphisms of the unit disc on three (Bergman, Hardy, and Dirichlet) Hilbert spaces of holomorphic functions have the property that if a function in one of these spaces and its image under the corresponding operator have different parity, then it is the zero function.

Our objective in this paper is to extend this result to a wider classes of Hilbert spaces of holomorphic functions and to the case of the unit ball of $\mathbb{C}^N$. Moving up to arbitrary-dimensional balls where mappings of several complex variables are used complicate matters considerably. The geometry of Möbius transformations in the ball is more complicated and simple derivatives in the disc need to be replaced by complex Jacobians whose fractional powers are used in the generalizations of the operators of interest to the weighted spaces. Further, Besov Hilbert spaces such as the Dirichlet space have to be handled differently, because the derivatives used in the integral norms of such spaces simply are not compatible with the natural unitary weighted composition operators on them.

To present our result, we now introduce the necessary definitions and notation. Let $\mathbb{B}$ be the open unit ball in $\mathbb{C}^N$ with respect to the usual Hermitian inner product $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_N\overline{w}_N$, and the associated norm $|z| = \sqrt{\langle z, z\rangle}$. When $N = 1$, the unit ball is the unit disc $\mathbb{D}$ in the complex plane.

Definition 1.1. For $q \in \mathbb{R}$ and $z, w \in \mathbb{B}$, the *Bergman-Besov kernels* are

$$K_q(z,w) := \begin{cases} \dfrac{1}{(1-\langle z,w\rangle)^{1+N+q}} = \displaystyle\sum_{k=0}^{\infty} \frac{(1+N+q)_k}{k!}\,\langle z,w\rangle^k, & q > -(1+N), \\ {}_2F_1(1,1;1-(N+q);\langle z,w\rangle) = \displaystyle\sum_{k=0}^{\infty} \frac{k!\,\langle z,w\rangle^k}{(1-(N+q))_k}, & q \le -(1+N), \end{cases}$$

where ${}_2F_1 \in H(\mathbb{D})$ is the Gauss hypergeometric function and $(a)_b$ is the Pochhammer symbol.

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Definition 1.2. For $q \in \mathbb{R}$, the *Bergman-Besov Hilbert space* $\mathcal{D}_q$ is the reproducing kernel Hilbert space on $\mathbb{B}$ generated by the kernel K_q endowed with the inner product and norm induced by K_q.

The kernels K_q are sesquiholomorphic on $\mathbb{B}^2$ and hence the functions in the $\mathcal{D}_q$ are holomorphic on $\mathbb{B}$. In particular, $\mathcal{D}_q$ is the standard weighted Bergman space A_q^2 for $q > -1$, the Hardy space H^2 for $q = -1$, the Drury-Arveson space $\mathcal{A}$ for $q = -N$, and the Dirichlet space $\mathcal{D}$ for $q = -(1+N)$, that is, $\mathcal{D}_{-(1+N)} = \mathcal{D}$.

Let $H(\mathbb{B})$ be the space of all holomorphic functions on $\mathbb{B}$. Every $f \in H(\mathbb{B})$ and thus every $f \in \mathcal{D}_q$ has homogeneous and Taylor expansions

$$f(z) = \sum_{k=0}^{\infty} f_k(z) = \sum_{|\alpha|=0}^{\infty} f_\alpha z^\alpha \qquad (z \in \mathbb{B}) \tag{1}$$

converging absolutely and uniformly on compact subsets of $\mathbb{B}$, where f_k is a homogeneous polynomial of degree k in $z_1, \dots, z_N$, α is a multi-index, and $k = |\alpha|$. We use the expression $f \in H(\mathbb{B})$ has *even parity* (respectively, *odd parity*) to mean that the homogeneous expansion of f as in (1) has f_k with only even k (respectively, only odd k).

Denote by $\mathcal{M}$ the group of all one-to-one onto holomorphic maps (*automorphisms*) of $\mathbb{B}$. Let $J\psi$ be the complex Jacobian of $\psi \in \mathcal{M}$. For $\psi \in \mathcal{M}$, also $\psi^{-1} \in \mathcal{M}$ and $J\psi \neq 0$ on $\mathbb{B}$.

Definition 1.3. For $q \geq -(1+N)$, $\psi \in \mathcal{M}$ and $f \in \mathcal{D}_q$, define the operator $T_\psi^q : \mathcal{D}_q \to \mathcal{D}_q$ by

$$T_\psi^q f(z) := f(\psi(z))(J\psi(z))^{1+\frac{q}{1+N}}$$

using an appropriate, say the principal, branch of the logarithm for the fractional power of $J\psi(z)$.

So T_ψ^q is the product

$$T_\psi^q = M_{\theta_\psi^q} C_\psi,$$

where C_ψ is the *composition operator* given by $C_\psi f = f \circ \psi$ and $M_{\theta_\psi^q}$ is the *multiplication operator* by

$$\theta_\psi^q(z) = (J\psi(z))^{1+\frac{q}{1+N}}. \tag{2}$$

When $q = -(1+N)$, $T_\psi^{-(1+N)}$ reduces simply to C_ψ on the Dirichlet space. When $q = 0$, $\theta_\psi^0 = J\psi$ for the unweighted Bergman space. When $q = -1$, $\theta_\psi^{-1} = (J\psi)^{\frac{N}{1+N}}$ for the Hardy space. When $q = -N$, $\theta_\psi^{-N} = (J\psi)^{\frac{1}{1+N}}$ for the Drury-Arveson space.

In (Beatrous and Burbea 1989, Theorem 1.10), it is proved that T_ψ^q is a unitary operator for $q > -(1+N)$ with respect to the standard inner product that the kernel K_q induces on $\mathcal{D}_q$ given in (5) below. By (Zhu 2005, Section 6.4), $T_\psi^{-(1+N)}$ is unitary on the space $\mathcal{D}_0 = \mathcal{D}/\mathbb{C}$ with respect to the slightly different inner product (6).

Our main result is the following.

Theorem 1.4. *Let $q \geq -1$ and $\psi \in \mathcal{M}$. Suppose the homogeneous expansions (1) of an $f \in \mathcal{D}_q$ and of $T_\psi^q f$ are of different parity, that is, one contains terms only with k even and the other only with k odd. Then $f = 0$. There are also operators on $\mathcal{D}_q$ for $q < -1$ for which the same conclusion is true.*

We prove Theorem 1.4 in Section 3. In the next Section 2, we provide further details and properties on notation, the spaces, and the automorphisms.

2. PRELIMINARIES

In multi-index notation, $\alpha = (\alpha_1, \dots, \alpha_N)$ is an N-tuple of nonnegative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha! = \alpha_1! \cdots \alpha_N!$, $0^0 = 1$, and $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$. An overbar $\overline{(\ )}$ indicates complex conjugate for numbers and functions and closure for sets. The boundary of $\mathbb{B}$ is the unit sphere $\mathbb{S}$.

The *Pochhammer symbol* $(a)_b$ is defined by

$$(a)_b := \frac{\Gamma(a+b)}{\Gamma(a)}$$

when a and $a+b$ are off the pole set $-\mathbb{N}$ of the gamma function Γ. This is a shifted rising factorial since $(a)_k = a(a+1)\cdots(a+k-1)$ for positive integer k. In particular, $(1)_k = k!$ and $(a)_0 = 1$. Stirling formula gives

$$\frac{\Gamma(c+a)}{\Gamma(c+b)} \sim c^{a-b}, \qquad \frac{(a)_c}{(b)_c} \sim c^{a-b}, \qquad \frac{(c)_a}{(c)_b q} \sim c^{a-b} \qquad (\operatorname{Re} c \to \infty), \tag{3}$$

where $A \sim B$ means that $|A/B|$ is bounded above and below by two strictly positive constants, that is, $A = O(B)$ and $B = O(A)$ for all A, B of interest.

The *Gauss hypergeometric function* ${}_2F_1 \in H(\mathbb{D})$ is defined by

$$ {}_2F_1(a,b;c;z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k. $$

2.1. Spaces

A function $K(z,w)$ is called the *reproducing kernel* of a Hilbert space H of functions defined on $\mathbb{B}$ and with inner product $\langle\cdot,\cdot\rangle_H$ if $K(\cdot,w) \in H$ for each $w \in \mathbb{B}$ and

$$ u(z) = \langle u(\cdot), K(z,\cdot)\rangle_H \qquad (u \in H,\ z \in \mathbb{B}). $$

There is a one-to-one correspondence between reproducing kernel Hilbert spaces and positive definite kernels.

Let $c_k(q)$ be the coefficient of $\langle z,w\rangle^k$ in the series for $K_q(z,w)$. Then $c_0(q) = 1$, $c_k(q) > 0$ for al k, and by (3),

$$ c_k(q) \sim k^{N+q} \qquad (k \to \infty), \tag{4} $$

for every q. This explains the choice of the parameters of the hypergeometric function in K_q for $q < -(1+N)$. The positive definiteness of $\langle z,w\rangle$ and the positivity of the $c_k(q)$ yield that the K_q are positive definite and thus reproducing kernels. The kernels K_q for $q < -(1+N)$ appear in the literature first in (Beatrous and Burbea 1989, p. 13). The kernels K_q for $q > -(1+N)$ can also be written as ${}_2F_1(1, 1+(N+q); 1; \langle z,w\rangle)$. For $q < -(1+N)$, the functions in $\mathcal{D}_q$ are bounded on $\mathbb{B}$ while the other $\mathcal{D}_q$ contain unbounded functions.

All Bergman-Besov kernels can be written of the form

$$ K_q(z,w) = \sum_{k=0}^{\infty} c_k(q) \langle z,w\rangle^k = \sum_{k=0}^{\infty} c_k(q) \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!} z^\alpha \overline{w}^\alpha. $$

Then by the theory of reproducing kernel Hilbert spaces and (4), the space $\mathcal{D}_q$ consists of all $f \in H(\mathbb{B})$ with Taylor expansions as in (1) for which

$$ \|f\|^2_{\mathcal{D}_q} := \sum_{|\alpha|=0}^{\infty} |f_\alpha|^2 \|z^\alpha\|^2_{\mathcal{D}_q} := \sum_{|\alpha|=0}^{\infty} |f_\alpha|^2 \frac{1}{c_{|\alpha|}(q)} \frac{\alpha!}{|\alpha|!} \sim \sum_{|\alpha|=1}^{\infty} |f_\alpha|^2 \frac{1}{|\alpha|^{N+q}} \frac{\alpha!}{|\alpha|!} < \infty $$

equipped with the inner product

$$ \langle f,g\rangle_{\mathcal{D}_q} := \sum_{|\alpha|=0}^{\infty} \frac{1}{c_{|\alpha|}(q)} \frac{\alpha!}{|\alpha|!} f_\alpha \overline{g}_\alpha. \tag{5} $$

The case of the Drury-Arveson space is especially simple, because then $q = -N$ and $c_k(-N) = 1$ for all $k = 1, 2, \ldots$. For $q > -(1+N)$, it is with respect to the inner product in (5) that the operators T^q_ψ are unitary.

Notice that the reproducing kernel of the Dirichlet space is

$$ K_{-(1+N)}(z,w) = \frac{1}{\langle z,w\rangle} \log \frac{1}{1 - \langle z,w\rangle} = \sum_{k=0}^{\infty} \frac{1}{1+k} \langle z,w\rangle^k $$

and this gives

$$ \langle f,g\rangle_{\mathcal{D}} = \sum_{|\alpha|=0}^{\infty} (1+|\alpha|) \frac{\alpha!}{|\alpha|!} f_\alpha \overline{g}_\alpha $$

with which $\|1\|_{\mathcal{D}} = 1$. The inner product with respect to which the operator $T_\psi^{-(1+N)} = C_\psi$ is unitary on $\mathcal{D}_0 = \mathcal{D}/\mathbb{C}$ is

$$ \langle f,g\rangle_{\mathcal{D}_0} = \sum_{|\alpha|=1}^{\infty} |\alpha| \frac{\alpha!}{|\alpha|!} f_\alpha \overline{g}_\alpha \tag{6} $$

with which $\|1\|_{\mathcal{D}_0} = 0$.

For $s, t \in \mathbb{R}$, we define the *radial fractional differential operator* D^t_s on $H(\mathbb{B})$ by

$$ D^t_s f := \sum_{k=0}^{\infty} d_k(s,t) f_k := \sum_{k=0}^{\infty} \frac{c_k(s+t)}{c_k(s)} f_k. $$

We have $d_0(s,t) = 1$ so that $D_s^t(1) = 1$, $d_k(s,t) > 0$ for any k, and by (4),

$$d_k(s,t) \sim k^t \qquad (k \to \infty),$$

for any s,t. So D_s^t is a continuous operator on $H(\mathbb{B})$ and is of order t. In particular, $D_s^t z^\alpha = d_{|\alpha|}(s,t) z^\alpha$ for any multi-index α. More importantly,

$$D_s^0 = I, \qquad D_{s+t}^u D_s^t = D_s^{t+u}, \qquad \text{and} \qquad (D_s^t)^{-1} = D_{s+t}^{-t} \tag{7}$$

for $s,t,u \in \mathbb{R}$, where the inverse is two-sided. Here and in any other context, I is the identity operator, Any D_s^t maps $H(\mathbb{B})$ onto itself continuously.

The $d_k(s,t)$ are chosen the way they are in order to have

$$D_q^t K_q(z,w) = K_{q+t}(z,w) \qquad (q,t \in \mathbb{R}),$$

where differentiation is performed on the holomorphic variable z. More interestingly, by (Alpay and Kaptanoğlu 2007, Proposition 3.2),

$$D_s^t(\mathcal{D}_q) = \mathcal{D}_{q+2t} \tag{8}$$

is an isomorphism of Hilbert spaces for any s,t and an isometry when the norms are chosen suitably.

The spaces $\mathcal{D}_q$ have also equivalent inner products and norms that are integrals of functions or their sufficiently high-order derivatives. For fixed $q \in \mathbb{R}$, let $s,t \in \mathbb{R}$ be such that $q + 2t > -1$. By (Alpay and Kaptanoğlu 2007, Definition 3.1c), a family of norms each of which is equivalent to $\|\cdot\|_{\mathcal{D}_q}$ is

$$|||f|||^2_{\mathcal{D}_q} := \int_{\mathbb{B}} |D_s^t f(z)|^2 (1-|z|^2)^{q+2t}\, d\nu(z), \tag{9}$$

where ν is the normalized volume measure on $\mathbb{B}$. Setting $d\nu_q(z) := (1-|z|^2)^q\, d\nu(z)$, equivalently $f \in \mathcal{D}_q$ if and only if $f \in H(\mathbb{B})$ and $D_s^t f \in L^2(\nu_{q+2t})$ for some s,t with $q+2t > -1$, where L^p denotes the Lebesgue classes. For Bergman Hilbert spaces, $q > -1$, we take $t = 0$ and obtain the usual integral norms of these spaces as

$$|||f|||^2_{A^2_q} := \int_{\mathbb{B}} |f(z)|^2\, d\nu_q(z) \qquad (q > -1).$$

The Hardy space also has an equivalent norm which is the well-known

$$|||f|||^2_{H^2} := \int_{\mathbb{S}} |f(z)|^2\, d\sigma(z),$$

where σ is the normalized surface measure on $\mathbb{S}$. Each integral norm on every $\mathcal{D}_q$ also has an accompanying integral inner product.

2.2. Möbius Transformations

Following (Rudin 1980, Chapter 2), the *Möbius transformation* that exchanges 0 and $0 \neq a \in \mathbb{B}$ is the map

$$\varphi_a(z) := \frac{a - P_a(z) - \sqrt{1-|a|^2}(I - P_a)(z)}{1 - \langle z, a\rangle} \qquad (z \in \overline{\mathbb{B}}),$$

where $P_a(z) := \langle z, a\rangle a/|a|^2$ is the projection on the complex line passing through 0 and a. It reduces to $\varphi_a(z) = (a-z)/(1-\overline{a}z)$ for $a, z \in \mathbb{D}$ when $N = 1$. Each φ_a is an involution, that is, $\varphi_a^{-1} = \varphi_a$. An extremely useful identity for φ_a is

$$1 - \langle \varphi_a(z), \varphi_a(w)\rangle = \frac{(1-|a|^2)\,(1-\langle z,w\rangle)}{(1-\langle z,a\rangle)\,(1-\langle a,w\rangle)} \qquad (z,w \in \overline{\mathbb{B}}). \tag{10}$$

The complex Jacobian of $\varphi_a(z)$ is $\det \varphi_a'(z)$ and equals

$$J\varphi_a(z) = \gamma(z)\left(\frac{1-|\varphi_a(z)|^2}{1-|z|^2}\right)^{(1+N)/2} = \gamma(z)\left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{(1+N)/2} \qquad (z \in \overline{\mathbb{B}})$$

for some $\gamma(z) \in \mathbb{C}$ with $|\gamma(z)| = 1$, where in obtaining the second form, (10) is used. Its real Jacobian is

$$J_{\mathbb{R}}\varphi_a(z) = |J\varphi_a(z)|^2 > 0.$$

We need two changes of variables formulas involving $\psi \in \mathcal{M}$. Let $G \subset \mathbb{B}$ and $Q \subset \mathbb{S}$ be Borel sets, $f \in L^1(\nu_q)$, and $F \in L^1(\sigma)$. The first is the usual

$$\int_G f\, d\nu = \int_{\psi^{-1}(G)} f(\psi(w)) J_{\mathbb{R}}\psi(w)\, d\nu(w). \tag{11}$$

The less common second one is obtained by explicitly writing (Rudin 1980, p. 45, (5)) using (10) and is

$$\int_Q F\, d\sigma = \int_{\psi^{-1}(Q)} F(\psi(\eta))(J_{\mathbb{R}}\psi(\eta))^{\frac{N}{1+N}}\, d\sigma(\eta). \tag{12}$$

Let also $\mathcal{U}$ denote the group of all unitary transformations of $\mathbb{C}^N$. All $U \in \mathcal{U}$ are characterized by $\langle Uz, Uw\rangle = \langle z, w\rangle$. If $\psi \in \mathcal{M}$ and $a = \psi^{-1}(0)$, then there is a unique $U \in \mathcal{U}$ such that

$$\psi(z) = U(\varphi_a(z)) \qquad (z \in \mathbb{B}). \tag{13}$$

Since $JU \in \mathbb{C}$ with $|JU| = 1$, we see that $J\psi$ has the same form as $J\varphi_a$ with a (possibly) different $\tilde{\gamma}(z)$ in place of $\gamma(z)$.

If $U \in \mathcal{U}$, then

$$\varphi_a = U^{-1}\varphi_{Ua}U; \tag{14}$$

this is (Cowen and MacCluer 1991, Lemma 2.71). This is useful, because $\mathcal{U}$ acts on $\mathbb{S}$ transitively and we can choose U in such way that Ua has only the first component nonzero and real. The automorphism that maps such a Ua to 0 is especially simple. For example, we use $\varphi_r(z) = -\varphi_{-b}(z)$ with $b = (r, 0, \dots, 0)$ and $0 < r < 1$ that has the explicit form

$$\varphi_r(z) := \left(\frac{r + z_1}{1 + rz_1}, \frac{\sqrt{1 - r^2}}{1 + rz_1} z'\right) \qquad (z \in \mathbb{B}), \tag{15}$$

where $z = (z_1, z')$ and z' denotes the remaining $N - 1$ components; see (Cowen and MacCluer 1991, p. 98). This φ_r has exactly 2 fixed points, $e_1 = (1, 0, \dots, 0)$ and $-e_1$, both on $\mathbb{S}$ and none in $\mathbb{B}$.

Möbius transformations map balls onto ellipsoids. We need the ellipsoids described in (Cowen and MacCluer 1991, p. 103) given by

$$E(e_1, u) := \{\, z \in \mathbb{B} : |1 - \langle z, e_1\rangle|^2 \le u(1 - |z|^2) \,\}$$

with $u > 0$. Equivalently, $z \in E(e_1, u)$ if and only if

$$\left|z_1 - \frac{1}{1+u}\right|^2 + \frac{u}{1+u}\,|z'|^2 < \left(\frac{u}{1+u}\right)^2.$$

The ellipsoid $E(e_1, u)$ lies in $\mathbb{B}$, has center $e_1/(1+u)$, and is tangent to $\mathbb{S}$ at e_1.

3. PROOF OF MAIN RESULT

We prove Theorem 1.4 and a corollary to it, and make some further comments.

Proof of Theorem 1.4. We follow the proof of (Montes-Rodríguez 2023, Theorem 1) with many detailed modifications to adapt it to several complex variables. For $\psi \in \mathcal{M}$, note that $C_{\psi^{-1}}(C_\psi f(z)) = C_{\psi^{-1}} f(\psi(z)) = f(\psi(\psi^{-1}(z))) = f(z)$ and hence $C_\psi^{-1} = C_{\psi^{-1}}$.

First we look at the case of weighted Bergman spaces. But the initial stages of the proof work for $q > -(1+N)$ and that is what we assume for now. Let $f \in \mathcal{D}_q$. By (13) and (2), $T_\psi^q = M_{\theta_\psi^q} C_{\varphi_a} C_U$ for some $a \in \mathbb{B}$ and $U \in \mathcal{U}$. By the remarks following (13), also

$$\beta T_\psi^q = M_{\theta_{\varphi_a}^q} C_{\varphi_a} C_U$$

for some $\beta \in \mathbb{C}$ with $|\beta| = 1$. By a simple computation with matrices, each $U \in \mathcal{U}$ carries a monomial z^α to a homogeneous polynomial of the same degree $|\alpha|$. Consequently C_U preserves parity. Thus f and $f_1 = C_U f$ have the same parity, and also $g = T_\psi^q f$ and $g_1 = \beta T_\psi^q f$ have the same parity that is opposite to that of f_1 by hypothesis. So without loss of generality we can replace T_ψ^q by $T_{\varphi_a}^q$ and it suffices to consider

$$T_{\varphi_a}^q = M_{\theta_{\varphi_a}^q} C_{\varphi_a}.$$

The fact that $\varphi_a^{-1} = \varphi_a$ implies $C_{\varphi_a}^{-1} = C_{\varphi_a}$ and $C_{\varphi_a}^2 = I$. We have

$$\begin{aligned}(T_{\varphi_a}^q)^2 f(z) &= T_{\varphi_a}^q\big((J\varphi_a(z))^{1+\frac{q}{1+N}} f(\varphi_a(z))\big)\\ &= (J\varphi_a(z))^{1+\frac{q}{1+N}} (J\varphi_a(\varphi_a(z)))^{1+\frac{q}{1+N}} f(\varphi_a(\varphi_a(z))).\end{aligned}$$

Since $\varphi_a(\varphi_a(z)) = z$, by the chain rule, $\varphi_a'(\varphi_a(z))\varphi_a'(z) = I$. Taking determinants give $J\varphi_a(\varphi_a(z))J\varphi_a(z) = 1$. This shows that $(T_{\varphi_a}^q)^2 f(z) = f(z)$ and $(T_{\varphi_a}^q)^2 = I$. Setting $g = T_{\varphi_a}^q f$ gives $f = T_{\varphi_a}^q g$. Thus the case $f \in \mathcal{D}_q$ having even parity and $T_{\varphi_a}^q g \in \mathcal{D}_q$ having odd parity coexists with the case $g \in \mathcal{D}_q$ having even parity and $T_{\varphi_a}^q f \in \mathcal{D}_q$ having odd parity. So it does not matter which case is investigated; let's assume the former and keep the notation $g = T_{\varphi_a}^q f$, Let $V(z) = -z$, which is unitary. Then $C_V f = f$, $C_V g = -g$, and $C_V^{-1} = C_V$. Then also $g = -C_V T^q_{\varphi_a} f$ and $f = C_V T^q_{\varphi_a} g$. Therefore

$$f = -(C_V T^q_{\varphi_a})^2 f; \tag{16}$$

that is, -1 is an eigenvalue of $(C_V T^q_{\varphi_a})^2$ with f as the eigenvector. By the spectral mapping theorem, $+i$ or $-i$ is an eigenvalue of $C_V T^q_{\varphi_a}$ with f as the eigenvector. But

$$C_V T^q_{\varphi_a} f(z) = (T^q_{\varphi_a} f)(-z) = (J\varphi_a(-z))^{1+\frac{q}{1+N}} f(\varphi_a(-z)). \tag{17}$$

Set $\mu_a(z) := \varphi_a(-z) = -\varphi_{-a}(z)$. Then $\mu_a \in \mathcal{M}$, $\mu_a'(z) = \varphi_a'(-z)(-I)$, and $J\mu_a(z) = (-1)^N J\varphi_a(-z)$. Hence

$$\begin{aligned} T^q_{\mu_a} f(z) = (J\mu_a(z))^{1+\frac{q}{1+N}} f(\mu_a(z)) &= (-1)^{N\left(1+\frac{q}{1+N}\right)} (J\varphi_a(-z))^{1+\frac{q}{1+N}} f(\varphi_a(-z)) \\ &= (-1)^{N\left(1+\frac{q}{1+N}\right)} C_V T^q_{\varphi_a} f(z) \end{aligned}$$

using (17), and

$$(T^q_{\mu_a})^2 f(z) = (-1)^{2N\left(1+\frac{q}{1+N}\right)} (C_V T^q_{\varphi_a})^2 f(z) = (-1)^{1+2N\left(1+\frac{q}{1+N}\right)} f(z) = \kappa f(z)$$

using (16), where

$$\kappa = (-1)^{1+2N\left(1+\frac{q}{1+N}\right)}$$

and $|\kappa| = 1$. Thus κ is an eigenvalue of $(T^q_{\mu_a})^2$, and $+\sqrt{\kappa}$ or $-\sqrt{\kappa}$ is an eigenvalue of $T^q_{\mu_a}$, both with eigenvector f. Clearly also $|\pm\sqrt{\kappa}| = 1$.

By (14), $\mu_a = -\varphi_{-a} = -U^{-1}\varphi_{U(-a)}U = U^{-1}(-\varphi_{-U(a)})U$. Choosing $U \in \mathcal{U}$ such that $b := U(a) = (r, 0, \dots, 0)$ with $0 < r < 1$, we obtain $\mu_a = U^{-1}\varphi_r U$, where φ_r is as in (15). Let $\eta = (\det U)^{1+\frac{q}{1+N}}$. For $f \in \mathcal{D}_q$, we have $T^q_{U^{-1}} f(z) = \overline{\eta} f(U^{-1}z)$ and $T^q_U T^q_{U^{-1}} f(z) = \eta\overline{\eta} f(UU^{-1}z) = f(z)$; hence $(T^q_U)^{-1} = T^q_{U^{-1}}$. Further, we compute that $T^q_{\varphi_r} T^q_{U^{-1}} f(z) = \overline{\eta}(J\varphi_r(z))^{1+\frac{q}{1+N}} f(U^{-1}\varphi_r(z))$ and

$$\begin{aligned} T^q_U T^q_{\varphi_r} T^q_{U^{-1}} f(z) &= \eta\overline{\eta}(J\varphi_r(Uz))^{1+\frac{q}{1+N}} f(U^{-1}\varphi_r(Uz)) \\ &= (J\varphi_r(Uz))^{1+\frac{q}{1+N}} f(\mu_a(z)) \\ &= (J\mu_a(z))^{1+\frac{q}{1+N}} f(\mu_a(z)) = T^q_{\mu_a} f(z), \end{aligned}$$

where the equality before the last one can be seen by evaluating $J\mu_a(z)$ using the chain rule. In other words, $T^q_{\varphi_r} = (T^q_U)^{-1} T^q_{\mu_a} T^q_U$. Since a similarity transformation preserves eigenvalues and eigenvectors, we conclude that $+\sqrt{\kappa}$ or $-\sqrt{\kappa}$ is an eigenvalue of $T^q_{\varphi_r}$ with eigenvector $f \in \mathcal{D}_q$.

We have $\lim_{z\to e_1} \varphi_r(z) = e_1$ and let

$$\delta := \lim_{z\to e_1} \frac{1-|\varphi_r(z)|}{1-|z|} = \lim_{z\to e_1} \frac{1-|\varphi_r(z)|^2}{1-|z|^2},$$

where the limits are unrestricted from within $\mathbb{B}$. A quick computation shows that $\delta = 1/(1+r) < 1$. By (Cowen and MacCluer 1991, Lemma 2.77) due to Julia, $\varphi_r(E(e_1, u)) \subset E(e_1, \delta u)$, and by (Cowen and MacCluer 1991, Proposition 2.85), $E(e_1, \delta u) \subset E(e_1, u)$. Together we have the inclusion $\varphi_r(E(e_1, u)) \subset E(e_1, u)$.

For $n = 1, 2, \dots$, denote the *forward iterates* of φ_r by $\varphi_r^n = \varphi_r \circ \varphi_r^{n-1}$, where φ_r^0 is the identity, and its *backward iterates* by $\varphi_r^{-n} = (\varphi_r^n)^{-1}$. By the properties on the ellipsoids $E(e_1, u)$ and of φ_r and the Denjoy-Wolff theorem, as $n \to \infty$, φ_r^n converges uniformly on compact subsets of $\mathbb{B}$ to e_1; see (Cowen and MacCluer 1991, Theorem 2.83 and Proposition 2.88). In other words, e_1 is the attracting fixed point of φ_r and its *Denjoy-Wolff point*. Now fix $u = 1$, call the corresponding $E(e_1, 1) =: E$, and let $G = E \setminus \varphi_r(E)$, which is nonempty by above. As a consequence of all the discussion about the ellipsoids, for any $0 < r < 1$ we have

$$\mathbb{B} = \bigcup_{n\in\mathbb{Z}} \varphi_r^n(G) \tag{18}$$

and the sets $\varphi_r^n(G)$ for different n's are disjoint.

In the remaining part of the proof we first restrict to $q > -1$ for which $\mathcal{D}_q = A^2_q$, weighted Bergman spaces. Now applying the change of variables $z = \varphi_r(w)$, using (11) and that f is an eigenvector yield

$$\begin{aligned}\int_{\varphi_r^{n+1}(G)} |f(z)|^2\,d\nu_q(z) &= \int_{\varphi_r^n(G)} |f(\varphi_r(w))|^2(1-|\varphi_r(w)|^2)^q J_{\mathbb{R}}\varphi_r(w)\,d\nu(w)\\ &= \int_{\varphi_r^n(G)} |f(\varphi_r(w))|^2\frac{(1-|\varphi_r(w)|^2)^{1+N+q}}{(1-|w|^2)^{1+N}}\,d\nu(w)\\ &= \int_{\varphi_r^n(G)} |f(\varphi_r(w))|^2\big|(J\varphi_r(w))^{1+\frac{q}{1+N}}\big|^2(1-|w|^2)^q\,d\nu(w)\\ &= \int_{\varphi_r^n(G)} |T^q_{\varphi_r}f(w)|^2\,d\nu_q(w) = \int_{\varphi_r^n(G)} |\pm\sqrt{\kappa}f(w)|^2\,d\nu_q(w)\\ &= \int_{\varphi_r^n(G)} |f(z)|^2\,d\nu_q(z).\end{aligned}$$

Thus the above integrals have the same value on all the sets $\varphi_r^n(G)$ for $n \in \mathbb{Z}$ which is equal to the value of the integral on $\varphi_r^0(G) = G$. But f is an eigenvector and hence is not the zero function, and since $f \in H(\mathbb{B})$, none of the integrals on the $\varphi_r^n(G)$ is 0. On the other hand, $f \in \mathcal{D}_q$ and hence $|||f|||_{\mathcal{D}_q} < \infty$. But by (18) we also have

$$|||f|||^2_{\mathcal{D}_q} = \int_{\mathbb{B}} |f(z)|^2\,d\nu_q(z) = \sum_{n\in\mathbb{Z}}\int_{\varphi_r^n(G)} |f(z)|^2\,d\nu_q(z) = \sum_{n\in\mathbb{Z}}\int_G |f(z)|^2\,d\nu_q(z) = \infty.$$

This contradiction shows that a nonzero f having the parity properties in the statement of the theorem cannot exist for $q > -1$.

Next we take care of the case $q = -1$, the Hardy space. Let D be the intersection of the ellipsoid $E = E(e_1, 1)$ with the complex line $[e_1]$ through 0 and e_1, which is given by $|z_1 - 1/2| < 1/4$. The set $\widetilde{G} = D \setminus \varphi_r(D)$ is nonempty just like $G \neq \emptyset$. Let also $Q = \{(z_1, z') : z_1 \in \widetilde{G},\ |z_1|^2 + |z'|^2 = 1\}$; this is that part of $\mathbb{S}$ that lies "above" $\widetilde{G}$. We have $\mathbb{B} \cap [e_1] = \bigcup_{n\in\mathbb{Z}} \varphi_r^n(\widetilde{G})$ and $\mathbb{S} = \{(z_1, z') : z_1 \in \mathbb{D},\ |z_1|^2 + |z'|^2 = 1\}$, the second modulo a set of σ-measure 0. Then just like (18), we have $\mathbb{S} = \bigcup_{n\in\mathbb{Z}} \varphi_r^n(Q)$ modulo a set of σ-measure 0, which is a disjoint union.

Now we apply the change of variables $\zeta = \varphi_r(\eta)$, use (12) and that f is an eigenvector to obtain

$$\begin{aligned}\int_{\varphi_r^{n+1}(Q)} |f(\zeta)|^2\,d\sigma(\zeta) &= \int_{\varphi_r^n(Q)} |f(\varphi_r(\eta))|^2\big|(J\varphi_r(\eta))^{1-\frac{1}{1+N}}\big|^2\,d\sigma(\eta)\\ &= \int_{\varphi_r^n(Q)} |T^{-1}_{\varphi_r}f(\eta)|^2\,d\sigma(\eta) = \int_{\varphi_r^n(Q)} |\pm\sqrt{\kappa}f(\eta)|^2\,d\sigma(\eta)\\ &= \int_{\varphi_r^n(Q)} |f(\zeta)|^2\,d\sigma(\zeta).\end{aligned}$$

As in the case $q > -1$, each integral on $\varphi_r^n(Q)$ can be replaced by one on Q. But $f \in H^2$ is an eigenvector, so is not the zero function, and by (Rudin 1980, Theorem 5.6.4 (b)), its boundary values on $\mathbb{S}$ are nonzero σ-a.e.. Then none of the integrals on the $\varphi_r^n(Q)$ is 0. On the other hand, $f \in H^2$ and hence $|||f|||_{H^2} < \infty$. Similar to the case $q > -1$, we have

$$|||f|||^2_{H^2} = \int_{\mathbb{S}} |f(\zeta)|^2\,d\sigma(\zeta) = \sum_{n\in\mathbb{Z}}\int_{\varphi_r^n(Q)} |f(\zeta)|^2\,d\sigma(\zeta) = \sum_{n\in\mathbb{Z}}\int_Q |f(\zeta)|^2\,d\sigma(\zeta) = \infty.$$

By this contradiction, the theorem is proved for the case $q = -1$ too.

Lastly, we consider the case $q < -1$. Pick $s, t \in \mathbb{R}$ such that $p = q + 2t > -1$. Here we prove the result not for T^q_ψ but for $Y^q_\psi = D^{-t}_{s+t}T^p_\psi D^t_s$ where $T^p_\psi : A^2_p \to A^2_p$. By (8) and (7), we have $Y^q_\psi : \mathcal{D}_q \to \mathcal{D}_q$. We have also $T^p_\psi = D^t_s Y^q_\psi D^{-t}_{s+t}$ and that Y^q_ψ and T^p_ψ are similar operators. Now suppose, without loss of generality, that $f \in \mathcal{D}_q$ has even parity and $Y^q_\psi f$ has odd parity. By their very definitions, the D^t_s preserve parity and $g = D^t_s f \in A^2_p$ also has even parity. On the other hand, since $f = D^{-t}_{s+t}g$, we see that $T^p_\psi g = D^t_s Y^q_\psi f$ has odd parity. By the already proved case for the Bergman space A^2_p, we conclude that $g = 0$. Then also $f = 0$. In fact, $Y^q_\psi f = \lambda f$ if and only if $D^{-t}_{s+t}T^p_\psi g = \lambda f$ if and only if $T^p_\psi g = \lambda g$ for some $\lambda \in \mathbb{C}$.

The proof of Theorem 1.4 is now complete.

Remark 3.1. The change of variables that is used in transformung the integrals in the case $q > -1$ can be expressed in the form that the measures ν_q are invariant under the transformations $Z^q_\psi f(z) := f(\psi(z))(J\psi(z))^{2\left(1+\frac{q}{1+N}\right)}$ in the sense that

$$\int_{\mathbb{B}} Z^q_\psi f\,d\nu_q = \int_{\mathbb{B}} f\,d\nu_q \qquad (f \in L^1(\nu_q),\ q \in \mathbb{R}).$$

This is already noted in (Kaptanoğlu 2005, (21)).

Remark 3.2. The last part of the proof involving integrals does not work for $q < -1$, because a positive-order derivative on f is required in the integral norms of $\mathcal{D}_q$ on $\mathbb{B}$ for all $q < -1$; see (Kaptanoğlu and Üreyen 2018, Corollary 7.2). So, for example, if we pick t so that $q + 2t = 0$ for simplicity in (9), we end up with an integral of $|T^0_{\varphi_r} D^t_s f|^2$ on $\varphi^n_r(G)$. For the proof to go through, we need $D^t_s f \in A^2_0$ to be an eigenvector of $T^0_{\varphi_r}$. This would be implied by $T^0_{\varphi_r} D^t_s f$ having odd parity when f and hence $D^t_s f$ have even parity. But what we know is that $T^q_{\varphi_r} f$ has odd parity and this need not imply that $T^0_{\varphi_r} D^t_s f$ has odd parity because of the differences between T^q_ψ and T^0_ψ.

Such differences do not prevent Montes-Rodríguez (2023) from obtaining the theorem for the Dirichlet space, because when $N = 1$, the first-order ordinary derivative and the chain rule are enough to move between that space and the unweighted Bergman space. Neither of these tools is available for $N \geq 2$. These are exactly the reasons why we resort to the other operators Y^q_ψ when $q < -1$.

Remark 3.3. The value κ depends in general on both N and q. For the unweighted Bergman space, $q = 0$, $\kappa = (-1)^{1+2N} = -1$, and the eigenvalues that are shown not to exist in the proof of Theorem 1.4 are $+\sqrt{\kappa} = +i$ and $-\sqrt{\kappa} = -i$ independently of dimension N. For the Hardy space, $q = -1$ and $\kappa = (-1)^{\frac{1+N+2N^2}{1+N}}$. If also $N = 1$, $\kappa = (-1)^2 = +1$ and the eigenvalues that are shown not to exist in the proof of Theorem 1.4 are $+\sqrt{\kappa} = +1$ and $-\sqrt{\kappa} = -1$, contrary to what is claimed in the proof of (Montes-Rodríguez 2023, Theorem 1). But as already noted in (Montes-Rodríguez 2023, Remark 1), the proof of (Montes-Rodríguez 2023, Theorem 1) as well as of Theorem 1.4 here depend only on $|\pm\sqrt{\kappa}| = 1$ and are unaffected.

Corollary 3.4. Let $q \geq -1$ and $\psi \in \mathcal{M}$. There is a nonzero function $f \in \mathcal{D}_q$ such that f and $T^q_\psi f$ have the same parity if and only if $\psi = U \in \mathcal{U}$. For $q < -1$, the same result holds for the operators Y^q_ψ.

Proof. Let $q \geq -1$ first. If $\psi = U \in \mathcal{U}$, since compositions with U and multiplication with complex numbers that are Jacobians of such composition operators preserve parity, there are f as claimed.

Conversely, let $\psi = \varphi_a$ in which $a \neq 0$ and suppose an f as claimed exists. If we repeat the proof of Theorem 1.4 carefully considering the case f even and $T^q_{\varphi_a} f$ even and the case f odd and $T^q_{\varphi_a} f$ odd, we obtain $f = (C_V T^q_{\varphi_a})^2 f$ instead of (16). This in turn yields two complex numbers of modulus 1 one of which is an eigenvalue of $T^q_{\mu_a}$ and of $T^q_{\varphi_r}$ with f as an eigenvector. In the rest of the proof, the only property of the eigenvalues used is that they are of modulus 1. Again we conclude that $f = 0$.

The case $q \leq -1$ is automatic since it depends on the conclusion of the case $q > -1$.

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