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RESEARCH ARTICLE

Notes on multipliers on weighted Orlicz spaces

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ABSTRACT
Let G be a locally compact abelian group with Haar measure μ, Φ be a Young function and ω be a weight function. In this paper, we consider the weighted Orlicz space $L^{\Phi}(G,\omega)$ and we investigate the relationship between the multipliers $L1(G,\omega)$-module and the multipliers on a certain Banach algebra. For this purpose, we firstly define temperate function space with respect to the weighted Orlicz space $L^{\Phi}(G,\omega)$ which we denote by $L^{\Phi}t(G,\omega)$ and give its basic properties. Later, we define a subalgebra of the space of multipliers on $L^{\Phi}(G,\omega)$ and study its basic properties. We also show that this subalgebra is isometrically isomorphic to the space of multipliers of a certain Banach algebra. Moreover, we obtain a characterization for the space of multipliers of $L1(G,\omega)\cap L^{\Phi}(G,\omega)$.

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1. INTRODUCTION

An Orlicz space is a type of function space generalizing the L^p-space. Besides the L^p spaces, a variety of function spaces arises naturally in analysis in this way such as $L\log^+ L$, which is a Banach space related to Hardy-Littlewood maximal functions. Orlicz spaces could also contain certain Sobolev spaces as subspaces. Linear properties of Orlicz spaces have been studied thoroughly (see Başar E., Öztop, S., Uysal, B.H., Yaşar, Ş. (2023); Osançlıol, A., Öztop, S. (2015); Öztop, S., Samei, E. (2017); Öztop, S., Samei, E. (2019); Rao, M. M., Ren, Z. D. (1991) for example). Similar to L^p spaces, one could also consider weighted Orlicz spaces and studied their properties. Very recently the weighted Orlicz space is studied as Banach algebra with respect to convolution for which the corresponding space becomes an algebra and studied their properties such as existence of an approximate identity in compactly supported continuous function spaces of norm one (see Osançlıol, A., Öztop, S. (2015)).

On the other hand, there are a lot of results in abstract harmonic analysis on locally compact groups regarding multipliers for various function spaces. The multipliers of the group algebras of L^p were studied by many authors (see Feichtinger, H. (1976); Fisher, M. J. (1974); Griffin, J., McKennon, K. (1973); McKennon, K. (1972)). In Öztop, S. (2003), Öztop studied the space of multiplier of $L^1(G,A)\cap L^p(G,A)$ where A is a commutative Banach algebra and G is a locally compact abelian group. In Üster, R., Öztop, S. (2020), Üster and Öztop studied compact multiplier problem for $L^{\Phi}(G)$ and in Üster, R. (2021), this concept is extended to $L^{\Phi}(G,\omega)$ spaces by Üster.

Let A be a Banach algebra and E be an A-module. Then, E is essential if the linear span of the elements a, x for $a\in A$ and $x\in E$ is dense in E. A Banach algebra A is called without order, if for all $x\in A$, $xA=Ax=\{0\}$ implies $x=0$. It is known that if A has an approximate identity, then it is without order (see (Larsen, R. 1971, p.13)). A multiplier of A is a mapping $T: A\to A$ such that

$$T(fg)=fT(g)=(Tf)g, \qquad f,g\in A. \tag{1}$$

Let us denote the collection of all multipliers of A by $M(A)$. Then, every multiplier turns out to be a bounded linear operator on A. If A is commutative Banach algebra without order, then $M(A)$ is a commutative operator algebra and $M(A)$ is called the multiplier algebra of A (see (Wang, J. K. 1961, Theorem 2.2)).

Our goal in this paper is to study the relationship between the multipliers $L^1(G,\omega)$-module and the multipliers on a certain Banach algebra. It is well known that $L^{\Phi}(G,\omega)$ is an essential Banach $L^1(G,\omega)$-module with respect to convolution product (see (Öztop, S., Samei, E. 2017, Lemma 3.2)). Moreover, we obtain a characterization for the space of multipliers of $L^1(G,\omega)\cap L^{\Phi}(G,\omega)$.

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This paper is organized as follows. First we present necessary definitions and some basic results that will be used in this paper. In Section 3, we construct the space of Φ-temperate functions for $L^{\Phi}(G,w)$ and study their basic properties. In Section 4, we characterize the space of multipliers of $L^{\Phi}(G,w)$ as a certain Banach algebra and extend the results in Öztop, S. (2003) to weighted Orlicz space. In Section 5, we study the space of multipliers for $L^1(G,\omega)\cap L^{\Phi}(G,\omega)$.

2. PRELIMINARIES

Let us recall some facts concerning Young functions and Orlicz spaces.

An Orlicz space is determined by a Young function. A convex function $\Phi:[0,\infty)\to[0,\infty]$ is called a Young function if $\Phi(0)=0$, $\lim\limits_{x\to 0^+}\Phi(x)=0$ and $\lim\limits_{x\to\infty}\Phi(x)=\infty$.

For a Young function Φ, its complementary function Ψ is given by

$$\Psi(y)=\sup\{xy-\Phi(x):x\geq 0\},\quad y\geq 0$$

and Ψ is also a Young function. Then, (Φ,Ψ) is called a complementary Young pair.

By our definition, a Young function can have the value ∞ at a point, and hence be discontinuous at such a point. However, we always consider a pair of complementary Young functions (Φ,Ψ) with both Φ and Ψ being continuous and strictly increasing. In particular, they attain positive values on $(0,\infty)$. Note that even though Φ is continuous, it may happen that Ψ is not continuous.

A Young function Φ satisfies the Δ_2 condition if there exist a constant $K>0$ and an $x_0\geq 0$ such that $\Phi(2x)\leq K\Phi(x)$ for all $x\geq x_0$. In this case, we write $\Phi\in\Delta_2$.

Let G be a locally compact abelian group with a Haar measure μ. Given a Young function Φ, the Orlicz space $L^{\Phi}(G)$ on G is defined by

$$L^{\Phi}(G)=\Big\{f:G\to\mathbb{C}:\int\limits_G\Phi(\alpha|f(x)|)d\mu(x)<\infty \text{ for some } \alpha>0\Big\}.$$

The Orlicz space is a Banach space under the Orlicz norm $\|\cdot\|_{\Phi}$ defined for $f\in L^{\Phi}(G)$ by

$$\|f\|_{\Phi}=\sup\left\{\int\limits_G|f(x)v(x)|d\mu(x):\int\limits_G\Psi(|v(x)|)d\mu(x)\leq 1\right\}$$

where Ψ is the complementary Young function of Φ.

Let (Φ,Ψ) be a complementary Young pair. If $\Phi\in\Delta_2$, then the dual space $L^{\Phi}(G)^*$ is $L^{\Psi}(G)$ (Rao, M. M., Ren, Z. D. 1991, Corollary 3.4.5). If in addition $\Psi\in\Delta_2$, then the Orlicz space $L^{\Phi}(G)$ is a reflexive Banach space. We have already mentioned that Orlicz spaces are generalizations of Lebesgue spaces. For $1\leq p<\infty$ and $\Phi(x)=\frac{x^p}{p}$, the space $L^{\Phi}(G)$ becomes the Lebesgue space $L^p(G)$ and the norm $\|\cdot\|_{\Phi}$ is equivalent to the classical norm $\|\cdot\|_p$. Particularly, if $p=1$ and $\Phi(x)=x$, then Ψ the complementary Young function of Φ is 0 when $0\leq x\leq 1$, and ∞ when $1<x<\infty$. In this case $\|f\|_{\Phi}=\|f\|_1$ for all $f\in L^1(G)$. If $p=\infty$, then for the defined function Ψ, the space $L^{\Psi}(G)$ is equal to the space $L^{\infty}(G)$ and we have $\|f\|_{\Psi}=\|f\|_{\infty}$ for all $f\in L^{\infty}(G)$.

For further information on Orlicz spaces, the reader is referred to Rao, M. M., Ren, Z. D. (1991).

On the other hand, weights and weighted function spaces play an important role in mathematical analysis and their applications. In addition to this, weights appear naturally in analysis.

Let G be a locally compact group. In this paper, we consider a weight function as a function $\omega:G\to\mathbb{R}^+$ with $\omega(xy)\leq\omega(x)\omega(y)$, $(x,y\in G)$ that $\omega(e)=1$ and $\frac{1}{\omega}\in L^{\infty}_{\text{loc}}(G)$, here $L^{\infty}_{\text{loc}}(G)$ denotes the space of all locally essentially bounded functions on G. There is no loss of generality in assuming that the weight ω is continuous (see (Reiter H., Stegeman J. D. 2000, Section 3.7)).

In Osançlıol, A., Öztop, S. (2015), Osançlıol and Öztop introduced the weighted Orlicz space $L^{\Phi}(G,\omega)$ on a locally compact group G as

$$L^{\Phi}(G,\omega)=\{f:f\omega\in L^{\Phi}(G)\}$$

with the norm

$$\|f\|_{\Phi,\omega}=\|f\omega\|_{\Phi}$$

for $f\in L^{\Phi}(G,\omega)$. Also, they studied them as Banach algebras with respect to the convolution product. One can observe that if $\omega=1$, then the weighted Orlicz spaces $L^{\Phi}_{\omega}(G)$ become the space $L^{\Phi}(G)$.

Now, for each $f\in L^1(G,\omega)$ define the mapping T_f by $T_f(g)=f*g$ whenever $g\in L^{\Phi}(G,\omega)$. T_f is an element of $B(L^{\Phi}(G,\omega))$, which is Banach algebra of all continuous linear operators from $L^{\Phi}(G,\omega)$ to $L^{\Phi}(G,\omega)$, and $\|T_f\|\leq\|f\|_{1,\omega}$. Identifying $f\mapsto T_f$, we obtain an embedding of $L^1(G,\omega)$ in $B(L^{\Phi}(G,\omega))$. We denote the space of all of $L^1(G,\omega)$-module homomorphisms of

$L^{\Phi}(G,\omega)$ by $\mathrm{Hom}_{L^1(G,\omega)}(L^{\Phi}(G,\omega))$, that is, an operator $T\in B(L^{\Phi}(G,\omega))$ satisfies $T(f*g)=f*T(g)$ for each $f\in L^1(G,\omega)$ and $g\in L^{\Phi}(G,\omega)$.

We define

$$(f\circ T)(g)=f*T(g)=T(f*g) \tag{2}$$

for all $g\in L^{\Phi}(G,\omega)$ and $f\in L^1(G,\omega)$. The module homomorphisms space $\mathrm{Hom}_{L^1(G,\omega)}(L^{\Phi}(G,\omega))$ is an essential $L^1(G,\omega)$-module with respect to product defined in (2) and is called the space of multipliers of $L^{\Phi}(G,\omega)$.

Throughout this paper, G is an abelian locally compact group and we are mainly interested in weighted Orlicz spaces $L^{\Phi}(G,\omega)$ with the weight ω and the Δ_2-condition on a Young function Φ.

The definitions, notations and results of this section are adjusted according to the corresponding content of Section 2 of Öztop, S. (2003).

3. THE Φ-TEMPERATE SPACE

In this section, we define the Φ-temperate function space $L_t^{\Phi}(G,\omega)$ and give a closed linear subspace of $B(L^{\Phi}(G,\omega))$ by using the Φ-temperate functions. Moreover, we study some basic properities of these spaces.

Definition 3.1. An element $f\in L^{\Phi}(G,\omega)$ is called Φ-temperate function if

$$\|f\|^t_{\Phi,\omega}=\sup\{\|g*f\|_{\Phi,\omega}:g\in L^{\Phi}(G,\omega),\|g\|_{\Phi,\omega}\le 1\}<\infty$$

or equivalently

$$\|f\|^t_{\Phi,\omega}=\sup\{\|g*f\|_{\Phi,\omega}:g\in C_c(G),\|g\|_{\Phi,\omega}\le 1\}<\infty.$$

The space of all Φ-temperate functions f is denoted by $L_t^{\Phi}(G,w)$. One can observe that

$$(L_t^{\Phi}(G,w),\|\cdot\|^t_{\Phi,\omega})$$

is a normed space. Indeed, let $f\in L_t^{\Phi}(G,w)$. By the definition of the norm $\|\cdot\|^t_{\Phi,\omega}$, we obtain $\|f\|^t_{\Phi,\omega}\ge 0$. On the other hand, if $f=0$, then $\|f\|^t_{\Phi,\omega}=0$ is obvious. Conversely, let $\|f\|^t_{\Phi,\omega}=0$. Then, we have $\sup\{\|g*f\|_{\Phi,\omega}:g\in C_c(G),\|g\|_{\Phi,\omega}\le 1\}=0$ and so $g*f(x)=0$ for all $x\in G$. Since $g\in C_c(G)$ and $C_c(G)$ is dense in $L^1(G,\omega)$, we obtain $g*f\longrightarrow f$ and so $f=0$. For each $f\in L_t^{\Phi}(G,w)$ and $\alpha\in\mathbb{K}$, we have

$$\begin{aligned}\|\alpha f\|^t_{\Phi,\omega}&=\sup\{\|g*(\alpha f)\|_{\Phi,\omega}:g\in C_c(G),\|g\|_{\Phi,\omega}\le 1\}\\&=\sup\{\|\alpha(g*f)\|_{\Phi,\omega}:g\in C_c(G),\|g\|_{\Phi,\omega}\le 1\}\\&=|\alpha|\sup\{\|g*f\|_{\Phi,\omega}:g\in C_c(G),\|g\|_{\Phi,\omega}\le 1\}\\&=|\alpha|\|f\|^t_{\Phi,\omega}.\end{aligned}$$

Finally, for any $f_1,f_2\in L_t^{\Phi}(G,w)$, we have

$$\begin{aligned}\|f_1+f_2\|^t_{\Phi,\omega}&=\sup\{\|g*(f_1+f_2)\|_{\Phi,\omega}:g\in C_c(G),\|g\|_{\Phi,\omega}\le 1\}\\&\le\sup\{\|(g*f_1)\|_{\Phi,\omega}+\|(g*f_2)\|_{\Phi,\omega}:g\in C_c(G),\|g\|_{\Phi,\omega}\le 1\}\\&\le\|f_1\|^t_{\Phi,\omega}+\|f_2\|^t_{\Phi,\omega}.\end{aligned}$$

For each $f\in L_t^{\Phi}(G,w)$, there exists precisely one bounded linear operator on $L^{\Phi}(G,\omega)$, denoted by W_f, such that $W_f: L^{\Phi}(G,\omega)\to L^{\Phi}(G,\omega)$

$$W_f(g)=g*f \text{ and } \|W_f\|=\|f\|^t_{\Phi,\omega}. \tag{3}$$

The linearity of W_f is obvious and since we have

$$\|W_f\|=\|f\|^t_{\Phi,\omega}=\sup\{\|g*f\|_{\Phi,\omega}:g\in L^{\Phi}(G,\omega),\|g\|_{\Phi,\omega}\le 1\}\le\|f\|_{\Phi,\omega},$$

then W_f is bounded.

Also, we observe that $W_f(h*g)=(h*g)*f=h*(g*f)=h*W_f(g)$ for each $f\in L_t^{\Phi}(G,w)$, $g\in L^{\Phi}(G,\omega)$. Hence, we obtain $W_f\in\mathrm{Hom}_{L^1(G,\omega)}(L^{\Phi}(G,\omega))$.

Proposition 3.2. *Let Φ be a Young function. Then $L_t^{\Phi}(G,w)$ is a dense subspace of $L^{\Phi}(G,\omega)$.*

Proof. Since each $f\in C_c(G)$ belongs to $L_t^{\Phi}(G,w)$ and $C_c(G)$ is dense in $L^{\Phi}(G,\omega)$, we have the required result. □

Lemma 3.3. *The space $L_t^{\Phi}(G,w)$ is a normed algebra with the convolution product.*

Proof. By (3), we have

$$\begin{aligned}\|f*g\|^t_{\Phi,\omega} &= \sup\{\|h*(f*g)\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \leq 1\}\\ &= \sup\{\|g*(h*f)\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \leq 1\}\\ &= \sup\{\|W_g(h*f)\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \leq 1\}\\ &\leq \|W_g\| \sup\{\|h*f\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \leq 1\}\\ &= \|g\|^t_{\Phi,\omega}\|f\|^t_{\Phi,\omega}\end{aligned}$$

for all $f, g \in L^\Phi_t(G,w)$. Hence, $(L^\Phi_t(G,w), \|\cdot\|^t_{\Phi,\omega})$ is a normed algebra. □

Note that

$$W_{f*g} = W_f \circ W_g = W_g \circ W_f \tag{4}$$

for all $f, g \in L^\Phi_t(G,w)$. Moreover, the closed linear subspace of $B(L^\Phi(G,\omega))$ spanned by $\{W_{f*g} : f \in L^\Phi_t(G,w), g \in C_c(G)\}$ is denoted by $\Lambda_{L^\Phi(G,\omega)}$.

Theorem 3.4. *The space $\Lambda_{L^\Phi(G,\omega)}$ is a complete subalgebra of $\mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega))$ and it has a minimal approximate identity, that is, a net $\{T_\alpha\}_\alpha$ such that $\overline{\lim}_\alpha\|T_\alpha\| \leq 1$ and $\lim_\alpha \|T_\alpha \circ T - T\| = 0$ for all $T \in \Lambda_{L^\Phi(G,\omega)}$.*

Proof. If $f \in L^\Phi_t(G,w)$, then $W_f \in B(L^\Phi(G,\omega))$. Since $L^\Phi(G,\omega)$ is an $L^1(G,\omega)$-module, we have

$$W_f(g*h) = g*h*f = g*W_f(h)$$

for all $g \in L^1(G,\omega)$ and $h \in L^\Phi(G,\omega)$.

Hence W_f belongs to $\mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega))$. Since $\mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega))$ is a Banach algebra under the usual operator norm, $\Lambda_{L^\Phi(G,\omega)}$ is a complete subalgebra of $\mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega))$.

Now, we show the existence of minimal approximate identity of $\Lambda_{L^\Phi(G,\omega)}$. Let $\{e_{U_\alpha}\}$ be a minimal approximate identity for $L^1(G,\omega)$ Dinculeanu, N. (1974). If $\{e_\alpha\}$ denotes the product net of $\{e_{U_\alpha}\}$ with itself, then $\{e_\alpha\}$ is also minimal approximate identity for $L^1(G,\omega)$. It can be observed that the net $W_{e_\alpha} \in \Lambda_{L^\Phi(G,\omega)}$ and $\overline{\lim}_\alpha\|W_{e_\alpha}\| \leq 1$.

Let $f \in L^\Phi_t(G,w)$ and $g \in C_c(G)$. Since $\{e_\alpha\}$ is a minimal approximate identity for $L^1(G,\omega)$, using (4) we obtain

$$\begin{aligned}\overline{\lim}_\alpha\|W_{e_\alpha} \circ W_{f*g} - W_{f*g}\| &= \overline{\lim}_\alpha\|(W_{e_\alpha} \circ W_g - W_g) \circ W_f\|\\ &\leq \overline{\lim}_\alpha\|W_{g*e_\alpha - g}\|\|W_f\|\\ &\leq \overline{\lim}_\alpha\|g*e_\alpha - g\|_{1,\omega}\|W_f\| = 0.\end{aligned}$$

Thus we have $\overline{\lim}_\alpha\|W_{e_\alpha} \circ T - T\| = 0$ for all $T \in \Lambda_{L^\Phi(G,\omega)}$. □

Let $g \in L^1(G,\omega)$, $f \in L^\Phi_t(G,\omega)$ and $W_f \in \Lambda_{L^\Phi(G,\omega)}$. We define the module action $g \circ W_f$ of $L^1(G,\omega)$ from $L^\Phi(G,\omega)$ to $L^\Phi(G,\omega)$ by

$$(g \circ W_f)(h) = W_f(h*g) = W_f(g*h)$$

for each $h \in L^\Phi(G,\omega)$.

Proposition 3.5. *The space $\Lambda_{L^\Phi(G,\omega)}$ is an essential $L^1(G,\omega)$-module.*

Proof. Let $g \in L^1(G,\omega)$, $f \in L^\Phi_t(G,\omega)$ and $W_f \in \Lambda_{L^\Phi(G,\omega)}$. We have

$$\|g \circ W_f\| = \sup\{\|W_f(g*h)\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \leq 1\} \leq \|f\|^t_{\Phi,\omega}\|g\|_{1,\omega}.$$

Hence, $\Lambda_{L^\Phi(G,\omega)}$ is an $L^1(G,\omega)$-module. On the other hand, since $L^1(G,\omega)$ has a minimal approximate identity $\{e_\alpha\}$ with a compact support, it is also an approximate identity in $L^\Phi(G,\omega)$.

For any $W_f \in \Lambda_{L^\Phi(G,\omega)}$, we have

$$\begin{aligned}\|e_\alpha \circ W_f - W_f\| &= \sup\{\|(e_\alpha \circ W_f - W_f)(h)\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \leq 1\}\\ &= \sup\{\|W_f(e_\alpha * h - h)\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \leq 1\}\\ &\leq \|f\|^t_{\Phi,\omega}\|e_\alpha * h - h\|_{\Phi,\omega} = 0\end{aligned}$$

for all $h \in L^\Phi(G,\omega)$. Thus, $\Lambda_{L^\Phi(G,\omega)}$ is an essential $L^1(G,\omega)$-module. Moreover, $\Lambda_{L^\Phi(G,\omega)}$ contains $L^1(G,\omega)$. □

4. A CHARACTERIZATION FOR THE SPACE OF MULTIPLIERS OF $\Lambda_{L^{\Phi}(G,\omega)}$

In this section, we give an identification for the space of $L^1(G,\omega)$-module multiplier with the space of multipliers of certain normed algebra.

The definitions, notations and proofs of this section are adjusted according to the corresponding content of Section 3 of Öztop, S. (2003).

Proposition 4.1. *Let $T \in \mathrm{Hom}_{L^1(G,\omega)}(L^{\Phi}(G,\omega))$.*

(i) If $f \in L_t^{\Phi}(G,\omega)$, then $T(f) \in L_t^{\Phi}(G,\omega)$.
(ii) If $g \in L_t^{\Phi}(G,\omega)$, then $T(f * g) = f * T(g)$

for all $f, g \in L^{\Phi}(G,\omega)$.

Proof. (i) Let $f \in L_t^{\Phi}(G,\omega)$. Since $T \in \mathrm{Hom}_{L^1(G,\omega)}(L^{\Phi}(G,\omega))$ we have

$$\begin{aligned}
\|T(f)\|_{\Phi,\omega}^t &= \sup\{\|h * T(f)\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \le 1\} \\
&= \sup\{\|T(h * f)\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \le 1\} \\
&\le \|T\| \sup\{\|h * f\|_{\Phi,\omega} : h \in C_c(G), \|h\|_{\Phi,\omega} \le 1\} \\
&= \|T\|\|f\|_{\Phi,\omega}^t < \infty.
\end{aligned}$$

(ii) Let $g \in L_t^{\Phi}(G,\omega)$. Since $\overline{C_c(G)} = L^{\Phi}(G,\omega)$, for each $f \in L^{\Phi}(G,\omega)$ there exists $(f_n) \subseteq C_c(G)$ such that $\lim_{n\to\infty} \|f_n - f\|_{\Phi,\omega} = 0$. Using (3), we obtain $\lim_{n\to\infty} \|f_n * g - f * g\|_{\Phi,\omega} = 0$. By (i), we have

$$\lim_{n\to\infty} \|f_n * T(g) - f * T(g)\|_{\Phi,\omega} = 0$$

and $f * T(g) = \lim_{n\to\infty} f_n * T(g) = \lim_{n\to\infty} T(f_n * g) = T(f * g)$. □

Definition 4.2. For the space $\Lambda_{L^{\Phi}(G,\omega)}$, we define $\Lambda^{\circ}_{L^{\Phi}(G,\omega)}$ by

$$\Lambda^{\circ}_{L^{\Phi}(G,\omega)} = \{T \in \mathrm{Hom}_{L^1(G,\omega)}(L^{\Phi}(G,\omega)) : T \circ W \in \Lambda_{L^{\Phi}(G,\omega)} \text{ for all } W \in \Lambda_{L^{\Phi}(G,\omega)}\}.$$

Lemma 4.3. *The space $\Lambda^{\circ}_{L^{\Phi}(G,\omega)}$ is equal to $\mathrm{Hom}_{L^1(G,\omega)}(L^{\Phi}(G,\omega))$.*

Proof. It is obvious that

$$\Lambda^{\circ}_{L^{\Phi}(G,\omega)} \subseteq \mathrm{Hom}_{L^1(G,\omega)}(L^{\Phi}(G,\omega)). \tag{5}$$

Conversely, let $T \in \mathrm{Hom}_{L^1(G,\omega)}(L^{\Phi}(G,\omega))$. For any $S \in \Lambda_{L^{\Phi}(G,\omega)}$ which is $S = W_{f*g}$ for some $f \in L_t^{\Phi}(G,\omega)$ and $g \in C_c(G)$, we have

$$\begin{aligned}
(T \circ W_{f*g})(h) &= T(h * f * g) \\
&= h * T(f * g) \\
&= W_{T(f*g)}(h) \\
&= W_{f*T(g)}(h),
\end{aligned}$$

for all $h \in L^{\Phi}(G,\omega)$. So, $T \circ S \in \Lambda_{L^{\Phi}(G,\omega)}$ implies that $T \in \Lambda^{\circ}_{L^{\Phi}(G,\omega)}$. Hence, we have $\mathrm{Hom}_{L^1(G,\omega)}(L^{\Phi}(G,\omega)) \subseteq \Lambda^{\circ}_{L^{\Phi}(G,\omega)}$ by the continuity of T. □

Let us note that we have the inclusion $M(\Lambda_{L^{\Phi}(G,\omega)}) \subset \mathrm{Hom}_{L^1(G,\omega)}(\Lambda_{L^{\Phi}(G,\omega)})$.

Theorem 4.4. *The space of multipliers $M(\Lambda_{L^{\Phi}(G,\omega)})$ is isometrically isomorphic to the space $\Lambda^{\circ}_{L^{\Phi}(G,\omega)}$.*

Proof. Define the mapping $F : \Lambda^{\circ}_{L^{\Phi}(G,\omega)} \to M(\Lambda_{L^{\Phi}(G,\omega)})$ by letting $F(T) = \rho_T$ for each $T \in \Lambda^{\circ}_{L^{\Phi}(G,\omega)}$, where $\rho_T(S) = T \circ S$ for all $S \in \Lambda_{L^{\Phi}(G,\omega)}$. Thus F is well-defined and moreover $\rho_T(S \circ K) = T \circ S \circ K = \rho_T(S) \circ K$ for all $S, K \in \Lambda_{L^{\Phi}(G,\omega)}$, so $\rho_T \in M(\Lambda_{L^{\Phi}(G,\omega)})$, since G is an abelian group and so the convolution multiplication is commutative.

It is clear that the mapping $T \mapsto \rho_T$ is linear and that $\|\rho_T\| \le \|T\|$. Since W_{e_α} is minimal approximate identity for $\Lambda_{L^{\Phi}(G,\omega)}$ by Theorem 3.4, we have

$$\begin{aligned}
\|\rho_T\| = \sup_{S \in \Lambda_{L^{\Phi}(G,\omega)}} \frac{\|\rho_T(S)\|}{\|S\|} &= \sup_{S \in \Lambda_{L^{\Phi}(G,\omega)}} \frac{\|T \circ S\|}{\|S\|} \\
&\ge \sup_{S \in \Lambda_{L^{\Phi}(G,\omega)}} \frac{\|T \circ W_{e_\alpha}\|}{\|W_{e_\alpha}\|} \ge \|T\|.
\end{aligned}$$

Thus $\|\rho_T\| = \|T\|$.

Finally, we show that F is onto. Let $\rho \in M(\Lambda_{L^\Phi(G,\omega)})$ and $\{e_\alpha\} \subseteq L^1(G,\omega)$ be a minimal approximate identity of $L^1(G,\omega)$. The limit of ρW_{e_α} exists for strong operator topology. Let $T = \lim_\alpha \rho W_{e_\alpha}$. We prove $\rho_T = \rho$. By (1), we have $(\rho W_{e_\alpha})(f * g) = (\rho W_{e_\alpha})(W_f g) = (\rho W_{e_\alpha * f})(g)$ for every $f \in L^1(G,\omega)$, $g \in L^\Phi(G,\omega)$. So we have

$$T(f * g) = \lim_\alpha (\rho W_{e_\alpha})(f * g) = (\rho W_f)g. \tag{6}$$

Since $L^\Phi(G,\omega)$ is an essential $L^1(G,\omega)$-module, the limit of $(\rho W_{e_\alpha})(f * g)$ exists in $L^\Phi(G,\omega)$. Let this limit be denoted by $T_g \in \mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega))$. From (6) we obtain for all $f \in L^1(G,\omega)$,

$$f \circ T = \rho f. \tag{7}$$

Thus we have

$$T \circ W_{e_\alpha} \circ W = (\rho W_{e_\alpha}) \circ W = \rho(W_{e_\alpha} \circ W) \tag{8}$$

for all $W \in \Lambda_{L^\Phi(G,\omega)}$. Since $L^\Phi(G,\omega)$ is essential $L^1(G,\omega)$-module, we have $T \circ W = \rho(W)$ and so $\rho_T(W) = \rho(W)$ for all $W \in \Lambda_{L^\Phi(G,\omega)}$, which gives $\rho_T = \rho$. □

Corollary 4.5. $M(\Lambda_{L^\Phi(G,\omega)}) \cong \mathrm{Hom}_{L^1(G,\omega)}(L^\Phi(G,\omega))$.

Proof. From Lemma 4.3 and Theorem 4.4, the result is obtained. □

5. THE IDENTIFICATION FOR THE SPACE $L^1(G,\omega) \cap L^\Phi(G,\omega)$

In this section, adapted from Chapter of Öztop, S. (2003), we study some basic properties of the space $L^1(G,\omega) \cap L^\Phi(G,\omega)$ and we characterize the space of multipliers $\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$.

Given a Young function Φ, the space $L^1(G,\omega) \cap L^\Phi(G,\omega)$ is a Banach space with the norm

$$|||f||| = \|f\|_{1,\omega} + \|f\|_{\Phi,\omega} \tag{9}$$

for $f \in L^1(G,\omega) \cap L^\Phi(G,\omega)$.

Lemma 5.1. *For $L^1(G,\omega) \cap L^\Phi(G,\omega)$ the following is true.*

(i) $L^1(G,\omega) \cap L^\Phi(G,\omega)$ is dense in $L^1(G,\omega)$ with respect to the norm $\|\cdot\|_{1,\omega}$.
(ii) For every $f \in L^1(G,\omega) \cap L^\Phi(G,\omega)$ and $x \in G$ the mapping $x \mapsto L_x f$ is continuous where $L_x f(y) = f(x^{-1}y)$ for all $y \in G$.

Proof. (i) Since $C_c(G)$ is dense in $L^1(G,\omega)$ with respect to the norm $\|\cdot\|_{1,\omega}$ and $C_c(G) \subseteq L^1(G,\omega) \cap L^\Phi(G,\omega) \subseteq L^1(G,\omega)$ we have the required result.

(ii) Let $f \in L^1(G,\omega) \cap L^\Phi(G,\omega)$. Observe that by (Osançlıol, A., Öztop, S. 2015, Lemma 2.3) $|||L_x f||| \le w(x)|||f|||$ for all $x \in G$ and the function $x \mapsto L_x f$ is continuous from G into $L^\Phi(G,\omega)$ and $L^1(G,\omega)$. Thus for any $x_0 \in G$ and $\varepsilon > 0$, there exists $U_1 \in V_{(x_0)}$ and $U_2 \in V_{(x_0)}$ such that for every $x \in U_1$

$$\|L_x f - L_{x_0} f\|_{\Phi,\omega} < \frac{\varepsilon}{2}$$

and for every $x \in U_2$

$$\|L_x f - L_{x_0} f\|_{1,\omega} < \frac{\varepsilon}{2},$$

where $V_{(x_0)}$ denotes the neighborhood of x_0. Set $V = U_1 \cap U_2$, then for all $x \in V$ we have $|||L_x f - L_{x_0} f||| < \varepsilon$. □

Since $L^1(G,\omega)$ has a minimal approximate identity and $L^\Phi(G,\omega)$ is an essential $L^1(G,\omega)$-module, the following proposition and lemma are hold trivially.

Proposition 5.2. *The space $L^1(G,\omega) \cap L^\Phi(G,\omega)$ has a minimal approximate identity in $L^1(G,\omega)$.*

Lemma 5.3. *The space $L^1(G,\omega) \cap L^\Phi(G,\omega)$ is an essential $L^1(G,\omega)$-module.*

Corollary 5.4. $L^1(G,\omega) \cap L^\Phi(G,\omega)$ is a Banach ideal in $L^1(G,\omega)$.

Proposition 5.5. *$L^1(G,\omega) \cap L^\Phi(G,\omega)$ is a Banach algebra with the norm $|||\cdot|||$.*

Proof. For any $f, g \in L^1(G,\omega) \cap L^\Phi(G,\omega)$ we have

$$\begin{aligned} |||f * g||| &= \|f * g\|_{1,\omega} + \|f * g\|_{\Phi,\omega} \\ &\le \|f\|_{1,\omega}\|g\|_{\Phi,\omega} + \|f\|_{1,\omega}\|g\|_{\Phi,\omega} \le |||f|||\ |||g|||. \end{aligned}$$

□

Note that let G be a locally compact abelian group. A subalgebra $S^1(G)$ of $L^1(G)$ is called a Segal algebra if it satisfies the following conditions (see Reiter H., Stegeman J. D. (2000)).

(i) The space $S^1(G)$ is dense in $L^1(G)$.

(ii) The subalgebra $S^1(G)$ is a Banach algebra which is invariant under translations and for each $f \in L^1(G)$ there is a neighborhood $U = U_\varepsilon$ of the identity element e such that

$$\|L_y f - f\| < \varepsilon, \quad y \in U.$$

Corollary 5.6. The space $L^1(G,\omega) \cap L^\Phi(G,\omega)$ is a Segal algebra.

Proof. By Lemma 5.1 and Proposition 5.5 we obtain that $L^1(G,\omega) \cap L^\Phi(G,\omega)$ is a Segal algebra. □

Remark 5.7. Since $L^1(G,\omega) \cap L^\Phi(G,\omega)$ is an $L^1(G,\omega)$-module and a Banach algebra, using the similar methods in Section 3 we obtain $M(L^1(G,\omega) \cap L^\Phi(G,\omega)) \cong \mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$.

Proposition 5.8. $\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$ *is an essential Banach module over* $L^1(G,\omega)$.

Proof. Let $f \in L^1(G,\omega)$ and $T \in \mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$. Define the operator fT on $L^1(G,\omega) \cap L^\Phi(G,\omega)$ by

$$(fT)(g) = T(f * g) \tag{10}$$

for all $g \in L^1(G,\omega) \cap L^\Phi(G,\omega)$ and $f \in L^1(G,\omega) \cap L^\Phi(G,\omega)$. Since $L^1(G,\omega) \cap L^\Phi(G,\omega)$ is a Banach algebra, the mapping (10) is well defined. On the other hand, we have

$$\begin{aligned}\|fT\| &= \sup_{|||g|||\le 1} |||(fT)(g)|||\\ &\le \sup_{|||g|||\le 1} |||T(f*g)|||\\ &\le \sup_{|||g|||\le 1} \|T\|\, |||f*g|||\\ &\le \sup_{|||g|||\le 1} \|T\|\, \|f\|_{1,\omega}|||g|||\\ &\le \|T\|\, \|f\|_{1,\omega}.\end{aligned}$$

Hence, $\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$ is an $L^1(G,\omega)$ module.

Let $\{e_\alpha\}$ be a minimal approximate identity for $L^1(G,\omega)$ and T be in $\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$. We have

$$\lim_\alpha \|e_\alpha \circ T - T\| = 0.$$

Then, we obtain that $\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$ is an essential Banach module over $L^1(G,\omega)$. □

Define $\mathcal{P}$ to be the closure of $L^1(G,\omega)$ in $\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$ for the operator norm. Clearly we have

$$\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega)) = (\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega)))_e = \mathcal{P} = (\mathcal{P})_e, \tag{11}$$

where $(.)_e$ denotes the essential part and we have

$$\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega)) = (\mathcal{P}).$$

Here $(\mathcal{P})$ is defined as the space of the elements $T \in \mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$ such that $T \circ \mathcal{P} \subset \mathcal{P}$.

Using the same method as in Theorem 4.4, we obtain the following lemma.

Lemma 5.9. *The space of multipliers of Banach algebra* $\mathcal{P}$ *is isometrically isometric to the space* $(\mathcal{P})$.

Corollary 5.10. $\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega)) \cong M(\mathcal{P})$.

Proof. The proof is obtained by using Lemma 4.3 and Theorem 4.4. □

Remark 5.11. It is evident that every measure $\mu \in M(G)$ defines multiplier for $L^1(G,\omega) \cap L^\Phi(G,\omega)$. This is obvious from the fact that $\|\mu * f\| \le \|\mu\|\|f\|$, $f \in L^1(G,\omega) \cap L^\Phi(G,\omega)$. On the other hand, for $\mu \in M(G)$, we have $\mu \circ L^1(G,\omega) \subset L^1(G,\omega)$, the inclusion in the space $\mathrm{Hom}_{L^1(G,\omega)}(L^1(G,\omega) \cap L^\Phi(G,\omega))$. Thus, $\mu \circ \mathcal{P} \subset \mathcal{P}$ and $M(G)$ can be embeded into $(\mathcal{P})$.

Moreover, if G is noncompact locally compact abelian, we have the more general results than Corollary in (Larsen, R. 1971, Corollary 3.5.1).

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