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RESEARCH ARTICLE

Clairaut and Einstein conditions for locally conformal Kaehler submersions

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ABSTRACT

In the present paper, we study Clairaut submersions and Einstein conditions whose total manifolds are locally conformal Kaehler manifolds. We first give a necessary and sufficient condition for a curve to be geodesic on total manifold of a locally conformal Kaehler submersion. Then, we investigate conditions for a locally conformal Kaehler submersion to be a Clairaut submersion.We find the Ricci and scalar curvature formulas between any fiber of the total manifold and the base manifold of a locally conformal Kaehler submersion and give necessary and sufficient conditions for the total manifold of a locally conformal Kaehler submersion to be Einstein. Finally, we obtain some formulas for sectional and holomorphic sectional curvatures for a locally conformal Kaehler submersion.

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1. INTRODUCTION

The notion of Riemannian submersion was introduced by O'Neill (1966) and Gray (1967), independently. Watson (1976) introduced almost Hermitian submersions by adding the condition to be almost complex mappings for Riemannian submersions, and proved that the vertical and the horizontal distributions are invariant with respect to the almost complex structure of the total space of the submersion. Then various kinds of Riemannian and almost Hermitian submersions Falcitelli et al. (2004), Şahin (2017) have been introduced and studied widely such as anti-invariant submersions, Lagrangian submersions, slant submersions, semi-slant submersions, hemi-slant submersions, etc. Moreover, these submersions have been studied for different kinds of manifolds like Kaehler, almost Kaehler, Sasakian and examined under some particular conditions, for example Einstein and Clairaut Lee et al. (2015). Especially, Clairaut submersions were studied in Lorentizian manifolds Allison (1996), Sasakian and Kenmotsu manifolds Taştan and Gerdan (2016), cosymplectic manifolds Taştan and Gerdan Aydın (2019), and locally product Riemannian manifolds Gündüzalp (2020). An important class of these manifolds is locally conformal Kaehler manifold, whose metric is conformal to a Kaehler metric locally. Vaisman studied locally conformal Kaehler manifolds and obtained some curvature properties of these manifolds Vaisman (1980). A comprehensive review for locally conformal Kaehler manifolds was made by Dragomir and Ornea (1998). An almost Hermitian submersion whose total manifold is a locally conformal Kaehler is called a locally conformal Kaehler submersion. Marrero and Rocha (1994) gave some conditions for the fibers of a locally conformal Kaehler submersion to be minimal and studied some relations between the Betti numbers of the total space and the base space of this submersion. In a recent paper Çimen et al. (2023) obtained Gauss and Weingarten equations for a locally conformal Kaehler submersion.

In this paper, we study Clairaut submersions and Einstein conditions whose total manifolds are locally conformal Kaehler manifolds. In section 2, we give basic informations about Riemannian submersions, almost Hermitian submersions and locally conformal Kaehler manifolds. In section 3, we derive conditions for a curve to be geodesic, with respect to two connections which are determined by the Riemannian metric and its conformally related Kaehler metric, on total manifold of a locally conformal Kaehler submersion. After that, we give a necessary and sufficient condition for a locally conformal Kaehler submersion to be Clairaut. In section 4, we derive the formulas for Ricci and scalar curvatures between any fiber of the total manifold and the base manifold of a locally conformal Kaehler submersion. Afterwards, we give necessary and sufficient conditions for the total manifold of a

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locally conformal Kaehler submersion to be Einstein. At the end of this section, we obtain the sectional and holomorphic sectional curvatures for a locally conformal Kaehler submersion.

2. PRELIMINAIRES

In this section, we will give some informations about locally conformal Kaehler manifolds.

Let (M, g) and (N, g') be Riemannian manifolds. A mapping π of (M, g) onto (N, g') is called a *Riemannian submersion* if it satisfies the following conditions:

(i) For every $p \in M$, the derivative map π_* of π is surjective;

hence for each $q \in N$, $\pi^{-1}(q)$ is a submanifold of dimension $dim(M) - dim(N)$. These submanifolds are called *fibers* of the submersion and a vector field on M which is tangent (resp. orthogonal) to fibers is called *vertical* (resp. *horizontal*). Thus, we can write a vector field E on M uniquely as $E = E^v + E^h$, where E^v and E^h are vertical and horizontal parts of E, respectively.

(ii) For every horizontal vector fields X, Y we have $g(X, Y) = g'(\pi_* X, \pi_* Y)$;

that is, π_* is a linear isometry of horizontal distribution.

To find the Gauss and Weingarten formulas of a Riemannian submersion, O'Neill introduced two new tensors of types $(1, 2)$ as follows;

$$\mathcal{T}_E F = (\nabla_{E^v} F^h)^v + (\nabla_{E^v} F^v)^h,$$

$$\mathcal{A}_E F = (\nabla_{E^h} F^h)^v + (\nabla_{E^h} F^v)^h,$$

where E and F are vector fields on M and ∇ is the Levi-Civita connection of g (see for the properties $\mathcal{T}$ and $\mathcal{A}$ in O'Neill (1966)). It is easy to see that,

$$\nabla_U V = (\nabla_U V)^v + \mathcal{T}_U V, \tag{1}$$
$$\nabla_U X = \mathcal{T}_U X + (\nabla_U X)^h, \tag{2}$$
$$\nabla_X U = (\nabla_X U)^v + \mathcal{A}_X U, \tag{3}$$
$$\nabla_X Y = \mathcal{A}_X Y + (\nabla_X Y)^h, \tag{4}$$

where U and V are vertical, and X and Y are horizontal vector fields on M.

Let (M, J, g) and (N, J', g') be almost Hermitian manifolds and $\pi : (M, J, g) \longrightarrow (N, J', g')$ be a Riemannian submersion. π is called an *almost Hermitian submersion* if $\pi_* \circ J = J' \circ \pi_*$, i.e., π is an almost complex mapping. The vertical and horizontal distributions are invariant under the almost complex structure J (see Proposition 2.1 in Watson (1976)).

Let a Hermitian manifold (M, J, g) is called a *locally conformal Kaehler* manifold (briefly l.c.K.), if M has an open cover $\{\mathcal{U}_i\}_{i \in I}$ and for every $i \in I$ with family of positive differentiable functions $\sigma_i : \mathcal{U}_i \longrightarrow \mathbb{R}$ such that

$$g_i = e^{-\sigma_i} g|_{\mathcal{U}_i}$$

are Kaehler metrics on $\mathcal{U}_i$.

Let (M, J, g) be a Hermitian manifold and let Ω be a 2-form defined by $\Omega(E, F) = g(E, JF)$ where E and F are vector fields on M. Dragomir and Ornea (1998) showed that (M, J, g) is a l.c.K. manifold if and only if there exists a globally defined closed 1-form ω such that

$$d\Omega = \omega \wedge \Omega.$$

The 1-form ω is called the *Lee form* and the vector field B defined by

$$\omega(E) = g(B, E), \tag{5}$$

is called *Lee vector field* of M, where E is a vector field of M.

Let ∇^i be the Levi-Civita connection of the locally conformal Kaehler metrics g_i, for every $i \in I$. Then the Levi-Civita connections ∇^i glue up to a globally defined linear connection $\tilde{\nabla}$ on M (see Theorem 2.1 (Dragomir and Ornea (1998))) is given by

$$\tilde{\nabla}_E F = \nabla_E F - \frac{1}{2}\Big\{\omega(E)F + \omega(F)E - g(E, F)B\Big\}, \tag{6}$$

for any vector fields E and F on M. One can see that $\tilde{\nabla}$ is torsion-free and satisfies

$$\tilde{\nabla} g = \omega \otimes g, \tag{7}$$

and

$$\tilde{\nabla} J = 0. \tag{8}$$

$\tilde{\nabla}$ is called *Weyl connection* of the l.c.K manifold M. From (6) and (8), it can be obtained

$$(\nabla_E J)F = \frac{1}{2}\Big\{\omega(JF)E - \omega(F)JE - g(E, JF)B + g(E, F)JB\Big\}. \tag{9}$$

Çimen et al. (2023) showed that the following equations hold for a l.c.K. submersion:

$$\mathcal{T}_U JV = J\mathcal{T}_U V + \frac{1}{2}\Big\{g(U, V)(JB)^h - g(U, JV)B^h\Big\}, \tag{10}$$

$$\mathcal{T}_V JX = J\mathcal{T}_V X + \frac{1}{2}\Big\{\omega(JX)V - \omega(X)JV\Big\}, \tag{11}$$

$$\mathcal{A}_X JV = J\mathcal{A}_X V + \frac{1}{2}\Big\{\omega(JV)X - \omega(V)JX\Big\}, \tag{12}$$

$$\mathcal{A}_X JY = J\mathcal{A}_X Y + \frac{1}{2}\Big\{g(X, Y)(JB)^v - g(X, JY)B^v\Big\}, \tag{13}$$

where U and V are vertical, X and Y are horizontal, and B is the Lee vector field of the total manifold of the submersion.

3. CLAIRAUT LOCALLY CONFORMAL KAEHLER SUBMERSIONS

In this section we shall give a necessary and sufficient condition for a locally conformal Kaehler submersion to be Clairaut. First, we recall the definition of a Clairaut submersion.

Let $\rho(p)$ be the distance from a point p on a surface of revolution in $\mathbb{R}^3$ to the rotation axis of this surface and α be a geodesic in this surface. Clairaut's theorem says that for the angle $\theta(s)$ between the velocity vector $\dot{\alpha}(s)$ and the meridian through $\alpha(s)$, $(\rho \sin\theta)(s)$ is constant. Motivated by this idea, Bishop (1972) introduced the notion of Clairaut submersion in the following way:

Definition 3.1. A Riemannian submersion $\pi : (M, g) \to (N, g')$ is called a *Clairaut submersion* if there exists a positive function ρ on M such that for any geodesic α on M, the function $\rho sin\theta$ is constant, where θ is the angle between $\dot{\alpha}$ and the horizontal distribution at every point of M.

Bishop (1972) gave the following characterization for Clairaut submersions.

Theorem 3.2. *Let $\pi : (M, g) \to (N, g')$ be a Riemannian submersion with connected fibers. Then π is a Clairaut submersion with $\rho = e^f$ if and only if each fiber is totally umbilical and has the mean curvature vector field $H = -\mathrm{grad} f$.*

We shall obtain a necessary and sufficient condition for a curve on the total space of a l.c.K submersion to be geodesic.

Lemma 3.3. *Let π be a l.c.K. submersion from (M, J, g) onto (N, J', g'), and let α be a curve on M whose tangent vector field has horizontal and vertical components X and V, respectively. Then α is a geodesic with respect to the Weyl connection $\tilde{\nabla}$ if and only if*

$$(\nabla_{\dot{\alpha}} JX)^h + \mathcal{T}_V JV + \mathcal{A}_X JV - \frac{1}{2}\Big\{\omega(\dot{\alpha})JX + \omega(J\dot{\alpha})X\Big\} = 0, \tag{14}$$

$$(\nabla_{\dot{\alpha}} JV)^v + \mathcal{T}_V JX + \mathcal{A}_X JX - \frac{1}{2}\Big\{\omega(\dot{\alpha})JV + \omega(J\dot{\alpha})V\Big\} = 0. \tag{15}$$

Proof. From (6) and (8), we have

$$\begin{aligned}
\tilde{\nabla}_{\dot{\alpha}}\dot{\alpha} &= -J\tilde{\nabla}_{\dot{\alpha}} J\dot{\alpha} \\
&= -J\Big(\nabla_{\dot{\alpha}} J\dot{\alpha} - \frac{1}{2}\Big\{\omega(\dot{\alpha})J\dot{\alpha} + \omega(J\dot{\alpha})\dot{\alpha} - g(\dot{\alpha}, J\dot{\alpha})B\Big\}\Big) \\
&= -J\Big(\nabla_V JV + \nabla_V JX + \nabla_X JV + \nabla_V JV \\
&\qquad - \frac{1}{2}\Big\{\omega(\dot{\alpha})JX + \omega(\dot{\alpha})JV + \omega(J\dot{\alpha})X + \omega(J\dot{\alpha})V\Big\}\Big).
\end{aligned}$$

Then nonsingular J implies that α is geodesic if and only if

$$\nabla_V JV + \nabla_V JX + \nabla_X JV + \nabla_V JV - \frac{1}{2}\Big\{\omega(\dot{\alpha})JX + \omega(\dot{\alpha})JV + \omega(J\dot{\alpha})X + \omega(J\dot{\alpha})V\Big\} = 0.$$

Taking the horizontal and vertical parts of this equation, we get (14) and (15), respectively. □

Lemma 3.4. *Let $\pi : (M, J, g) \to (N, J', g')$ be a l.c.K. submersion with connected fibers. If α is a geodesic on M with respect to both ∇ and $\tilde{\nabla}$, then we have*

$$\omega(\dot{\alpha})\dot{\alpha} = \frac{1}{2}B. \tag{16}$$

Proof. *Suppose that α is a geodesic curve with respect to both ∇ and $\tilde{\nabla}$, that is $\nabla_{\dot{\alpha}}\dot{\alpha} = 0$ and $\tilde{\nabla}_{\dot{\alpha}}\dot{\alpha} = 0$. Then we get (16) immediately from (6).* □

A geodesic curve whose vertical component of its velocity vector is zero is called a *horizontal geodesic* by O'Neill (1967). For a horizontal geodesic of a l.c.K. submersion, we have the following result.

Theorem 3.5. *Let $\pi : (M, J, g) \to (N, J', g')$ be a l.c.K. submersion with connected fibers. If a curve α is a horizontal geodesic on M with respect to both ∇ and $\tilde{\nabla}$, then the dimension of horizontal distribution is equal to 2 or the submersion π is a Kaehler submersion, i.e., its total manifold is Kaehler.*

Proof. *Let $\{X_1, ..., X_m\}$ be an orthonormal basis of the horizontal distribution of the submersion π at $p \in \pi^{-1}(q)$, where $q \in N$. Then there exist horizontal geodesic curves $\alpha_1, ..., \alpha_m$ such that $\dot{\alpha}_i = X_i$, $i = 1, ..., m$. Thus, for every $i = 1, ..., m$, we have*

$$g(B, X_i)X_i = \frac{1}{2}B^h \tag{17}$$

from (5) and (16). Taking summation of the equation (17) over i, we obtain

$$\Big(1 - \frac{m}{2}\Big)B^h = 0.$$

Hence, it follows that $m = 2$ or $B^h = 0$. In the case of $B^h = 0$, B is a zero vector field since B cannot be vertical by Theorem 2 of Çimen et al. (2023). It means that M is Kaehler. □

Now, we shall give the condition for a l.c.K. submersion to be Clairaut.

Theorem 3.6. *Let π be a l.c.K. submersion from (M, J, g) onto (N, J', g'). Then π is a Clairaut submersion with $\rho = e^f$ if and only if*

$$g(\dot{\alpha}, \mathrm{grad} f)g(V, V) + \frac{1}{2}\omega(\dot{\alpha})g(V, V) + \frac{1}{2}||V||^2\omega(X) - \frac{1}{2}||X||^2\omega(V) - g(\mathcal{T}_V X, V) = 0, \tag{18}$$

where X and V denote the horizontal and vertical components of $\dot{\alpha}$ of the geodesic α on M with respect to $\tilde{\nabla}$, respectively.

Proof. *Let α be a geodesic on M. Then we have*

$$g(X, X) = cos^2\theta \quad and \quad g(V, V) = sin^2\theta.$$

From (6) and (7) , we have

$$\begin{aligned}\omega(\dot{\alpha})g(V, V) &= (\tilde{\nabla}_{\dot{\alpha}} g)(V, V)\\ &= \tilde{\nabla}_{\dot{\alpha}} g(V, V) - 2g(\tilde{\nabla}_{\dot{\alpha}} JV, JV)\\ &= \tilde{\nabla}_{\dot{\alpha}} g(V, V) - 2g(\nabla_{\dot{\alpha}} JV, JV) + g(\omega(\dot{\alpha})JV + \omega(JV)\dot{\alpha} - g(JV, \dot{\alpha})B, JV).\end{aligned}$$

Then we obtain,

$$\tilde{\nabla}_{\dot{\alpha}} g(V, V) = 2g(\nabla_{\dot{\alpha}} JV, JV). \tag{19}$$

On the other hand, we have

$$\tilde{\nabla}_{\dot{\alpha}} g(V, V) = \nabla_{\dot{\alpha}} g(V, V) = sin\theta cos\theta\frac{d\theta}{dt}. \tag{20}$$

Then, π is Clairaut if and only if $\frac{d}{ds}(e^f sin\theta) = 0$. Hence from (20), we get

$$\tilde{\nabla}_{\dot{\alpha}} g(V, V) = -2\frac{df}{ds}sin^2\theta. \tag{21}$$

By using (15) and (21), in (19) we obtain

$$g(\dot{\alpha}, grad f)g(V,V) + \frac{1}{2}\omega(\dot{\alpha})||V||^2 - g(\mathcal{T}_V JX, JV) - g(\mathcal{A}_X JX, JV) = 0. \tag{22}$$

With the help of (11) and (13), the equation (18) follows from (22). □

Let $\pi : (M, J, g) \longrightarrow (N, J', g')$ be a l.c.K. submersion with totally umbilical fibers. Then, for any vertical vector fields U and V, and horizontal vector field X, we have

$$g(\mathcal{T}_U V, X) = -g(U,V)g(H,X),$$

from Theorem (3.2). Hence, using (6) and (1), we obtain

$$g(\tilde{\nabla}_U V, X) = -g(U,V)g(H,X) + \frac{1}{2}g(U,V)g(B^h, X).$$

Here, we know that $H = -\frac{1}{2}B^h$ from Proposition 3.34 of Falcitelli et al. (2004). Thus, we get

$$g(\tilde{\nabla}_U V, X) = 0,$$

and so we have

$$g(\tilde{\nabla}_U X, V) = 0.$$

Hence, we obtain the following result.

Theorem 3.7. *If $\pi : (M, J, g) \to (N, J', g')$ is a Clairaut l.c.K. submersion, then we have*

$$(\tilde{\nabla}_U V)^h = 0 \quad \text{and} \quad (\tilde{\nabla}_U X)^v = 0,$$

where U and V are vertical, and X is horizontal vector fields on M.

4. EINSTEIN LOCALLY CONFORMAL KAEHLER SUBMERSIONS

In this section, we shall give the conditions for the fibers and the base manifold of a l.c.K. submersion to be Einstein.

Definition 4.1. A Riemannian manifold (M, g) with $\dim(M) = m > 2$ is said to be an *Einstein manifold* if its Ricci tensor $S = \frac{r}{m}g$, where r denotes the scalar curvature of M.

Lemma 4.2. *Let $\pi : (M, J, g) \to (N, J', g')$ be a l.c.K. submersion. If the Lee vector field B is horizontal, then the Ricci tensor S is given by*

$$\begin{aligned} S(U,V) = \hat{S}(U,V) &- \sum_{i=1}^{k} g(\mathcal{T}_{U_i}U_i, \mathcal{T}_U V) + \sum_{i=1}^{k} g(\mathcal{T}_U U_i, \mathcal{T}_V U_i) \\ &+ \sum_{j=1}^{l} g((\nabla_{E_j}\mathcal{T})(U,V), E_j) - \sum_{j=1}^{l} g(\mathcal{T}_U E_j, \mathcal{T}_V E_j), \end{aligned} \tag{23}$$

$$S(X,Y) = S^*(X,Y) + \sum_{i=1}^{k} g((\nabla_X T)(U_i, U_i), Y) - \sum_{i=1}^{k} g(T_{U_i}X, T_{U_i}Y), \tag{24}$$

$$S(U,X) = \sum_{i=1}^{k} g((\nabla_U \mathcal{T})(U_i, U_i), X) - \sum_{i=1}^{k} g((\nabla_{U_i}\mathcal{T})(U, U_i), X), \tag{25}$$

and the scalar curvature r is given by

$$\begin{aligned} r = \hat{r} + r^* &- \sum_{i,j=1}^{k} g(\mathcal{T}_{U_i}U_i, \mathcal{T}_{U_j}U_j) - 2\sum_{j=1}^{l}\sum_{i=1}^{k} g(\mathcal{T}_{U_i}E_j, \mathcal{T}_{U_i}E_j) \\ &+ \sum_{i,j=1}^{k} g(\mathcal{T}_{U_i}U_j, \mathcal{T}_{U_i}U_j) + \sum_{j=1}^{l}\sum_{i=1}^{k} g((\nabla_{E_j}\mathcal{T})(U_i, U_i), E_j), \end{aligned} \tag{26}$$

where $\{U_1, ..., U_k\}$ and $\{E_1, ..., E_l\}$ are orthonormal frames of vertical and horizontal distributions, respectively, S^* is the horizontal lift of Ricci tensor of N, $\hat{S}$ is Ricci tensor of any fiber, r^* is the lift of scalar curvature of N and $\hat{r}$ is scalar curvature of any fiber.

Proof. From Proposition 2 of Lee et al. (2015) we have,

$$\begin{aligned} S(U,V) = \hat{S}(U,V) &- \sum_{i=1}^{k} g(\mathcal{T}_{U_i}U_i, \mathcal{T}_U V) + \sum_{i=1}^{k} g(\mathcal{T}_U U_i, \mathcal{T}_V U_i) \\ &+ \sum_{j=1}^{l} g((\nabla_{E_j}\mathcal{T})(U,V), E_j) - \sum_{j=1}^{l} g(\mathcal{T}_U E_j, \mathcal{T}_V E_j) \\ &+ \sum_{j=1}^{l} g(\mathcal{A}_{E_j}U, \mathcal{A}_{E_j}V), \end{aligned}$$

$$\begin{aligned} S(X,Y) = S^*(X,Y) &+ \sum_{i=1}^{k} g((\nabla_X T)(U_i,U_i), Y) - \sum_{i=1}^{k} g(T_{U_i}X, T_{U_i}Y) \\ &+ \sum_{i=1}^{k} g((\nabla_{U_i}\mathcal{A})(X,Y), U_i) + \sum_{i=1}^{k} g(\mathcal{A}_X U_i, \mathcal{A}_Y U_i) \\ &-3\sum_{j=1}^{l} g(\mathcal{A}_{E_j}X, \mathcal{A}_{E_j}Y), \end{aligned}$$

$$\begin{aligned} S(U,X) = &\sum_{i=1}^{k} g((\nabla_U\mathcal{T})(U_i,U_i), X) - \sum_{i=1}^{k} g((\nabla_{U_i}\mathcal{T})(U,U_i), X) \\ &+ \sum_{j=1}^{l} g((\nabla_{E_j}\mathcal{A})(X,E_j), U) - 2\sum_{j=1}^{l} g(\mathcal{A}_X E_j, \mathcal{T}_U E_j), \end{aligned}$$

and

$$\begin{aligned} r = \hat{r} + r^* &- \sum_{i,j=1}^{k} g(\mathcal{T}_{U_i}U_i, \mathcal{T}_{U_j}U_j) - 2\sum_{j=1}^{l}\sum_{i=1}^{k} g(\mathcal{T}_{U_i}E_j, \mathcal{T}_{U_i}E_j) \\ &+ \sum_{i,j=1}^{k} g(\mathcal{T}_{U_i}U_j, \mathcal{T}_{U_i}U_j) + \sum_{j=1}^{l}\sum_{i=1}^{k} g((\nabla_{E_j}\mathcal{T})(U_i,U_i), E_j) \\ &+2\sum_{j=1}^{l}\sum_{i=1}^{k} g(\mathcal{A}_{E_j}U_i, \mathcal{A}_{E_j}U_i) - 3\sum_{i,j=1}^{l} g(\mathcal{A}_{E_i}E_j, \mathcal{A}_{E_i}E_j). \end{aligned}$$

Since the Lee vector field B is horizontal, then we have $\mathcal{A} \equiv 0$, see Proposition 4.3 of Marrero and Rocha (1994). Thus, (23) ~ (26) can be obtained from the above equations, respectively. □

Theorem 4.3. *$\pi : (M,J,g) \to (N,J',g')$ be a l.c.K. submersion with horizontal Lee vector field B. Then (M,J,g) is an Einstein manifold if and only if the following relations hold:*

$$\begin{aligned} \hat{S}(U,V) = &\frac{r}{m}g(U,V) + \sum_{i=1}^{k} g(\mathcal{T}_{U_i}U_i, \mathcal{T}_U V) - \sum_{i=1}^{k} g(\mathcal{T}_U U_i, \mathcal{T}_V U_i) \\ &- \sum_{j=1}^{l} g((\nabla_{E_j}\mathcal{T})(U,V), E_j) + \sum_{j=1}^{l} g(\mathcal{T}_U E_j, \mathcal{T}_V E_j), \end{aligned}$$

$$S^*(X,Y) = \frac{r}{m}g(X,Y) - \sum_{i=1}^{k} g((\nabla_X T)(U_i,U_i), Y) + \sum_{i=1}^{k} g(T_{U_i}X, T_{U_i}Y),$$

and

$$\sum_{i=1}^{k} g((\nabla_U\mathcal{T})(U_i,U_i), X) - \sum_{i=1}^{k} g((\nabla_{U_i}\mathcal{T})(U,U_i), X) = 0.$$

Proof. If π is a l.c.K. submersion with horizontal Lee vector field B, then $\mathcal{A}$ vanishes. So, from (23), (24) and (25), we have the result. □

For a l.c.K. manifold (M, J, g), the relation between the curvature tensors R and $\tilde{R}$ of ∇ and $\tilde{\nabla}$ respectively, is given by Vaisman (1980)

$$\begin{aligned}\tilde{R}(X,Y)Z = R(X,Y)Z & \\ -\frac{1}{2}\Big\{ L(X,Z)Y - L(Y,Z)X - g(Y,Z)\Big[\nabla_X B + \frac{1}{2}\omega(X)B\Big] & \\ + g(X,Z)\Big[\nabla_Y B + \frac{1}{2}\omega(Y)B\Big]\Big\} & \\ -\frac{||\omega||^2}{4}\Big\{ g(Y,Z)X - g(X,Z)Y\Big\}, &\end{aligned} \tag{27}$$

where

$$L(X,Y) = (\nabla_X \omega)(Y) + \frac{1}{2}\omega(X)\omega(Y) = g(\nabla_X B, Y) + \frac{1}{2}\omega(X)\omega(Y), \tag{28}$$

and X, Y and Z are vector fields on M.

As ω is closed and L is a symmetric 2-tensor, we have from (27) that

$$\begin{aligned} e^{\sigma_i} R^i(X,Y,Z,W) = R(X,Y,Z,W) & \\ -\frac{1}{2}\Big\{ L(X,Z)g(Y,W) - L(Y,Z)g(X,W) & \\ + L(Y,W)g(X,Z) - L(X,Z)g(Y,W)\Big\} & \\ -\frac{||\omega||^2}{4}\Big\{ g(Y,Z)g(X,W) - L(X,Z)g(Y,W)\Big\}, & \end{aligned} \tag{29}$$

where R^i is the curvature tensor of the locally conformal Kaehler metric g_i.

If $\pi : (M, J, g) \to (N, J', g')$ is a l.c.K. submersion, then (28) takes the form

$$\begin{aligned} L(U,X) = & \; g(\nabla_U B, X) + \tfrac{1}{2}\omega(U)\omega(X) \\ = & \; Ug(B,X) - g(B, \nabla_U X) + \tfrac{1}{2}\omega(U)\omega(X) \\ = & \; U\omega(X) - g(B, \mathcal{T}_U X) - g(B, (\nabla_U X)^h) + \tfrac{1}{2}\omega(U)\omega(X) \\ = & \; U\omega(X) - \omega(\mathcal{T}_U X) - \omega((\nabla_U X)^h) + \tfrac{1}{2}\omega(U)\omega(X), \end{aligned} \tag{30}$$

where U is a vertical and X is a horizontal vector field of M. Similarly, we obtain

$$L(X,Y) = X\omega(Y) - \omega(\mathcal{A}_X Y) - \omega((\nabla_X Y)^h) + \frac{1}{2}\omega(X)\omega(Y), \tag{31}$$

$$L(U,V) = U\omega(V) - \omega(\mathcal{T}_U V) - \omega((\nabla_U V)^v) + \frac{1}{2}\omega(U)\omega(V), \tag{32}$$

where U and V are vertical, and X and Y are horizontal vector fields on M.

Theorem 4.4. *Let $\pi : (M, J, g) \to (N, J', g')$ be a l.c.K. submersion. Then the Riemannian curvature tensor R^i is given by*

$$\begin{aligned} e^{\sigma_i} R^i(U,V,W,W') = & \; \hat{R}(U,V,W,W') + g(\mathcal{T}_U W, \mathcal{T}_V W') - g(\mathcal{T}_V W, \mathcal{T}_U W') \\ & -\frac{1}{2}\Big\{\big(U\omega(W) - \omega(\mathcal{T}_U W) - \omega((\nabla_U W)^v)\big) + \frac{1}{2}\omega(U)\omega(W)\big)g(V,W') \\ & -\big(V\omega(W) - \omega(\mathcal{T}_V W) - \omega((\nabla_V W)^v) + \frac{1}{2}\omega(V)\omega(W)\big)g(U,W') \\ & -\big(U\omega(W') - \omega(\mathcal{T}_U W') - \omega((\nabla_U W')^v) + \frac{1}{2}\omega(U)\omega(W')\big)g(V,W) \\ & +\big(V\omega(W') - \omega(\mathcal{T}_V W') - \omega((\nabla_V W')^v) + \frac{1}{2}\omega(V)\omega(W')\big)g(U,W)\Big\} \\ & -\frac{||\omega||^2}{4}\Big\{g(V,W)g(U,W') - g(U,W)g(V,W')\Big\}, \end{aligned} \tag{33}$$

$$\begin{aligned} e^{\sigma_i} R^i(U,V,W,X) = & \; g((\nabla_U \mathcal{T})(V,W),X) - g((\nabla_V \mathcal{T})(U,W),X) \\ & -\frac{1}{2}\Big\{\big(V\omega(X) - \omega(\mathcal{T}_V X) - \omega((\nabla_V X)^h) + \frac{1}{2}\omega(V)\omega(X)\big)g(U,W) \\ & -\big(U\omega(X) - \omega(\mathcal{T}_U X) - \omega((\nabla_U X)^h) + \frac{1}{2}\omega(U)\omega(X)\big)g(V,W)\Big\}, \end{aligned} \tag{34}$$

$$
\begin{aligned}
e^{\sigma_i} R^i(X,Y,Z,V) = \; & g(\mathcal{A}_Y Z, \mathcal{T}_V X) + g(\mathcal{A}_Z X, \mathcal{T}_V Y) \\
& - g((\nabla_Z \mathcal{A})(X,Y), V) - g(\mathcal{A}_X Y, \mathcal{T}_V Z) \\
& - \frac{1}{2}\Big\{\big(Y\omega(V) - \omega(\mathcal{A}_Y V) - \omega((\nabla_Y V)^v) + \frac{1}{2}\omega(Y)\omega(V)\big) g(X,Z) \\
& - \big(V\omega(X) - \omega(\mathcal{T}_V X) - \omega((\nabla_V X)^h) + \frac{1}{2}\omega(V)\omega(X)\big) g(Y,Z)\Big\},
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
e^{\sigma_i} R^i(X,Y,Z,H) = \; & R^*(X,Y,Z,H) + 2g(\mathcal{A}_X Y, \mathcal{A}_Z H) \\
& - g(\mathcal{A}_Y Z, \mathcal{A}_X H) + g(\mathcal{A}_X Z, \mathcal{A}_Y H) \\
& - \frac{1}{2}\Big\{\big(X\omega(Z) - \omega(\mathcal{A}_X Z) - \omega((\nabla_X Z)^h) + \frac{1}{2}\omega(X)\omega(Z)\big) g(Y,H) \\
& - \big(Y\omega(Z) - \omega(\mathcal{A}_Y Z) - \omega((\nabla_Y Z)^h) + \frac{1}{2}\omega(Y)\omega(Z)\big) g(X,H) \\
& - \big(X\omega(H) - \omega(\mathcal{A}_X H) - \omega((\nabla_X H)^h) + \frac{1}{2}\omega(X)\omega(H)\big) g(Y,Z) \\
& + \big(Y\omega(H) - \omega(\mathcal{A}_Y H) - \omega((\nabla_Y H)^h) + \frac{1}{2}\omega(Y)\omega(H)\big) g(X,Z)\Big\} \\
& - \frac{||\omega||^2}{4}\Big\{g(Y,Z)g(X,H) - g(X,Z)g(Y,H)\Big\},
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
e^{\sigma_i} R^i(X,Y,V,W) = \; & g((\nabla_W \mathcal{A})(X,Y), V) - g((\nabla_V \mathcal{A})(X,Y), W) - g(\mathcal{A}_X V, \mathcal{A}_Y W) \\
& + g(\mathcal{A}_X W, \mathcal{A}_Y V) + g(\mathcal{T}_V X, \mathcal{T}_W Y) - g(\mathcal{T}_W X, \mathcal{T}_V Y)
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
e^{\sigma_i} R^i(X,V,Y,W) = \; & g(\mathcal{T}_V X, \mathcal{T}_W Y) - g((\nabla_V \mathcal{A})(X,Y), W) \\
& - g((\nabla_X \mathcal{T})(V,W), Y) - g(\mathcal{A}_X V, \mathcal{A}_Y W) \\
& - \frac{1}{2}\Big\{\big(X\omega(Y) - \omega(\mathcal{A}_X Y) - \omega((\nabla_X Y)^h) + \frac{1}{2}\omega(X)\omega(Y)\big) g(V,W) \\
& + \big(V\omega(W) - \omega(\mathcal{T}_V W) - \omega((\nabla_V W)^v) + \frac{1}{2}\omega(V)\omega(W)\big) g(X,Y)\Big\} \\
& + \frac{||\omega||^2}{4} g(X,Y)g(V,W),
\end{aligned} \tag{38}
$$

where U, V, W and W' are vertical, and X, Y, Z and H are horizontal vector fields on M, $\hat{R}$ is Riemannian curvature tensor of any fiber, and R^* is the horizontal lift of Riemannian curvature tensor of N.

Proof. *(33) ∼ (38) can be obtained from (29) ∼ (32) by direct computation.* □

Using (33) ∼ (38), we have the following proposition.

Corollary 4.5. Let $\pi : (M, J, g) \to (N, J', g')$ be a l.c.K submersion. Then the Ricci tensor S^i is given by

$$
\begin{aligned}
S^i(U,V) = \; & e^{-\sigma_i}\Big\{\hat{S}(U,V) - \sum_{i=1}^{k} g(\mathcal{T}_{U_i} U_i, \mathcal{T}_U V) + \sum_{i=1}^{k} g(\mathcal{T}_U U_i, \mathcal{T}_V U_i) \\
& + \sum_{j=1}^{l} g((\nabla_{E_j} \mathcal{T})(U,V), E_j) - \sum_{j=1}^{l} g(\mathcal{T}_U E_j, \mathcal{T}_V E_j) + \sum_{j=1}^{l} g(\mathcal{A}_{E_j} U, \mathcal{A}_{E_j} V) \\
& + \Big(\frac{k+l-2}{2}\Big)(U\omega(V) - \omega(\mathcal{T}_U V) - \omega((\nabla_U V)^v) + \frac{1}{2}\omega(U)\omega(V)) \\
& + g(U,V)\Big[\frac{1}{2}\sum_{i=1}^{k+l} g(\nabla_{E_i} B, E_i) - \frac{||\omega||^2}{4}(k+l-2)\Big]\Big\},
\end{aligned}
$$

$$S^i(X,Y) = e^{-\sigma_i}\Big\{S^*(X,Y) + \sum_{i=1}^{k} g((\nabla_X T)(U_i,U_i),Y) - \sum_{i=1}^{k} g(T_{U_i}X, T_{U_i}Y) + \sum_{i=1}^{k} g((\nabla_{U_i}\mathcal{A})(X,Y),U_i) + \sum_{i=1}^{k} g(\mathcal{A}_X U_i, \mathcal{A}_Y U_i) - 3\sum_{j=1}^{l} g(\mathcal{A}_{E_j}X, \mathcal{A}_{E_j}Y) + \Big(\frac{k+l-2}{2}\Big)(X\omega(Y) - \omega(\mathcal{A}_X Y) - \omega((\nabla_X Y)^h) + \frac{1}{2}\omega(X)\omega(Y)) + g(X,Y)\Big[\frac{1}{2}\sum_{i=1}^{k+l} g(\nabla_{E_i}B, E_i) - \frac{||\omega||^2}{4}(k+l-2)\Big]\Big\},$$

$$S^i(U,X) = e^{-\sigma_i}\Big\{\sum_{i=1}^{k} g((\nabla_U \mathcal{T})(U_i,U_i),X) - \sum_{i=1}^{k} g((\nabla_{U_i}\mathcal{T})(U,U_i),X) + \sum_{j=1}^{l} g((\nabla_{E_j}\mathcal{A})(X,E_j),U) - 2\sum_{j=1}^{l} g(\mathcal{A}_X E_j, \mathcal{T}_U E_j) + \Big(\frac{k+l-2}{2}\Big)(U\omega(X) - \omega(\mathcal{T}_U X) - \omega((\nabla_U X)^h) + \frac{1}{2}\omega(U)\omega(X))\Big\},$$

and the scalar curvature r^i is given by

$$r^i = e^{-\sigma_i}\Big\{r + (k+l-1)\Big[\sum_{i=1}^{k+l} g(\nabla_{E_i}B, E_i) - \frac{||\omega||^2}{4}(k+l-2)\Big]\Big\},$$

where $\{E_1, ..., E_{k+l}\}$ is an orthonormal frame field of tangent bundle of M.

Theorem 4.6. *Let $\pi : (M,J,g) \to (N,J',g')$ be a l.c.K. submersion. Then the curvature tensor R^i has the relation*

$$R^i(X,Y,Z,W) = R^i(JX,JY,JZ,JW) + \frac{1}{2}\Big\{\delta(X,Z)g(Y,W) - \delta(Y,Z)g(X,W) - \delta(X,W)g(Y,Z) + \delta(Y,W)g(X,Z)\Big\}, \tag{39}$$

where

$$\delta(X,Y) = L(X,Y) - L(JX,JY).$$

and X, Y, Z and W are vector fields on M.

Proof. *Since g^i is a Kaehler metric then $R^i(X,Y,Z,W) = R^i(JX,JY,JZ,JW)$. If we write the last equation in (29), we get (39).* □

Using (33), (38) and (36), we get the following equations, respectively.

Theorem 4.7. *Let $\pi : (M,J,g) \to (N,J',g')$ be a l.c.K. submersion. Then the sectional curvature tensor K^i is given by*

$$K^i(U,V) = e^{-\sigma_i}\Big\{\hat{K}(U,V) + \frac{g(\mathcal{T}_U U, \mathcal{T}_V V) - ||\mathcal{T}_U V||^2}{||U\wedge V||^2} - \frac{1}{2||U\wedge V||^2}\Big[\Big(U\omega(U) - \omega(\mathcal{T}_U U) - \omega((\nabla_U U)^v) + \frac{1}{2}(\omega(U))^2\Big)g(V,V) - 2\Big(U\omega(V) - \omega(\mathcal{T}_U V) - \omega((\nabla_U V)^v) + \frac{1}{2}\omega(U)\omega(V)\Big)g(U,V) + \Big(V\omega(V) - \omega(\mathcal{T}_V V) - \omega((\nabla_V V)^v) + \frac{1}{2}(\omega(V))^2\Big)g(U,U)\Big] + \frac{||\omega||^2}{4}\Big\},$$

$$
\begin{aligned}
K^i(X,U) = \quad & e^{-\sigma_i}\Bigg\{\frac{-g((\nabla_X\mathcal{T})(U,U),X) - ||\mathcal{A}_XU||^2 + ||\mathcal{T}_UX||^2}{||X||^2||U||^2} \\
& -\frac{1}{2||X||^2||U||^2}\Big[\Big(X\omega(X) - \omega((\nabla_XX)^h) + \frac{1}{2}(\omega(X))^2\Big)g(U,U) \\
& +\Big(U\omega(U) - \omega(\mathcal{T}_UU) - \omega((\nabla_UU)^v) + \frac{1}{2}(\omega(U))^2\Big)g(X,X)\Big] + \frac{||\omega||^2}{4}\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
K^i(X,Y) = \quad & e^{-\sigma_i}\Bigg\{K^*(X,Y) + \frac{3||\mathcal{A}_XY||^2}{||X\wedge Y||^2} \\
& -\frac{1}{2||X\wedge Y||^2}\Big[\Big(X\omega(X) - \omega((\nabla_XX)^h) + \frac{1}{2}(\omega(X))^2\Big)g(Y,Y) \\
& -2\Big(X\omega(Y) - \omega(\mathcal{A}_XY) - \omega((\nabla_XY)^h) + \frac{1}{2}\omega(X)\omega(Y)\Big)g(X,Y) \\
& +\Big(Y\omega(Y) - \omega((\nabla_YY)^h) + \frac{1}{2}(\omega(Y))^2\Big)g(X,X)\Big] + \frac{||\omega||^2}{4}\Bigg\},
\end{aligned}
$$

where U,V are vertical and X,Y are horizontal vector fields on M.

Definition 4.8. Let $\pi : (M,J,g) \longrightarrow (N,J',g')$ be a l.c.K. submersion. The holomorphic bisectional curvature is defined for any pair of nonzero vector fields E and F by

$$
\mathcal{B}(E,F) = \frac{R(E,JE,F,JF)}{||E||^2||F||^2}, \tag{40}
$$

and the holomorphic sectional curvature of the 2-plane spanned by E and JE is

$$
\mathcal{H}(E) = \mathcal{B}(E,E). \tag{41}
$$

Using (40) and (41) we get the following two propositions.

Proposition 4.9. *Let $\pi : (M,J,g) \longrightarrow (N,J',g')$ be a l.c.K. submersion. Then the holomorphic bisectional curvature $\mathcal{B}^i$ is given by*

$$
\begin{aligned}
\mathcal{B}^i(U,V) = \frac{e^{-\sigma_i}}{||U||^2||V||^2} \quad & \Bigg\{\hat{R}(U,JU,V,JV) + g(\mathcal{T}_UV,\mathcal{T}_{JU}JV) - g(\mathcal{T}_{JU}V,\mathcal{T}_UJV) \\
& -\frac{1}{2}\Big[\Big(U\omega(V) - \omega(\mathcal{T}_UV) - \omega((\nabla_UV)^v) + \frac{1}{2}\omega(U)\omega(V) \\
& +JU\omega(JV) - \omega(\mathcal{T}_{JU}JV) - \omega((\nabla_{JU}JV)^v) + \frac{1}{2}\omega(JU)\omega(JV)\Big)g(U,V) \\
& +\Big(U\omega(JV) - \omega(\mathcal{T}_UJV) - \omega((\nabla_UJV)^v) + \frac{1}{2}\omega(U)\omega(JV) \\
& -JU\omega(V) + \omega(\mathcal{T}_{JU}V) + \omega((\nabla_{JU}V)^v) - \frac{1}{2}\omega(JU)\omega(V)\Big)g(U,JV)\Big] \\
& +\frac{||\omega||^2}{4}\Big[(g(U,JV))^2 + (g(U,V))^2\Big]\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}^i(X,U) = \frac{e^{-\sigma_i}}{||X||^2||U||^2}\Bigg\{ & -g((\nabla_U\mathcal{A})(X,JX),JU) + g((\nabla_{JU}\mathcal{A})(X,JX),U) \\
& -g(\mathcal{A}_XU,\mathcal{A}_{JX}JU) + g(\mathcal{A}_XJU,\mathcal{A}_{JX}U) \\
& +g(\mathcal{T}_UX,\mathcal{T}_{JU}JX) - g(\mathcal{T}_{JU}X,\mathcal{T}_UJX)\Bigg\}
\end{aligned}
$$

$$\mathcal{B}^i(X,Y) = \frac{e^{-\sigma_i}}{||X||^2||Y||^2} \Big\{ R^*(X,JX,Y,JY) + 2g(\mathcal{A}_X JX, \mathcal{A}_Y JY) - g(\mathcal{A}_{JX}Y, \mathcal{A}_X JY) + g(\mathcal{A}_X Y, \mathcal{A}_{JX}JY) - \frac{1}{2}\Big[\Big(X\omega(Y) - \omega(\mathcal{A}_X Y) - \omega((\nabla_X Y)^h) + \frac{1}{2}\omega(X)\omega(Y) + JX\omega(JY) - \omega(\mathcal{A}_{JX}JY) - \omega((\nabla_{JX}JY)^h) + \frac{1}{2}\omega(JX)\omega(JY)\Big)g(X,Y) + \Big(X\omega(JY) - \omega(\mathcal{A}_X JY) - \omega((\nabla_X JY)^h) + \frac{1}{2}\omega(X)\omega(JY) - JX\omega(Y) + \omega(\mathcal{A}_{JX}Y) + \omega((\nabla_{JX}Y)^h) - \frac{1}{2}\omega(JX)\omega(Y)\Big)g(X,JY)\Big] + \frac{||\omega||^2}{4}\Big[(g(X,JY))^2 + (g(X,Y))^2\Big]\Big\},$$

where U,V are vertical and X,Y are horizontal vector fields.

Proposition 4.10. *Let $\pi : (M,J,g) \to (N,J',g')$ be a l.c.K. submersion. Then the holomorphic sectional curvature $\mathcal{H}^i$ is given by*

$$\mathcal{H}^i(U) = \frac{e^{-\sigma_i}}{||U||^4} \Big\{ \hat{R}(U,JU,U,JU) + g(\mathcal{T}_U U, \mathcal{T}_{JU}JU) - ||\mathcal{T}_U JU||^2 - \frac{1}{2}\Big(U\omega(U) - \omega(\mathcal{T}_U U) - \omega((\nabla_U U)^v) + \frac{1}{2}(\omega(U))^2 + JU\omega(JU) - \omega(\mathcal{T}_{JU}JU) - \omega((\nabla_{JU}JU)^v) + \frac{1}{2}(\omega(JU))^2\Big)||U||^2 + \frac{||\omega||^2||U||^4}{4}\Big\},$$

$$\mathcal{H}^i(X) = \frac{e^{-\sigma_i}}{||X||^4} \Big\{ R^*(X,JX,X,JX) + 3||\mathcal{A}_X JX||^2 - \frac{1}{2}\Big(X\omega(X) - \omega(\nabla_X X) + \frac{1}{2}(\omega(X))^2 + JX\omega(JX) - \omega(\nabla_{JX}JX) + \frac{1}{2}(\omega(JX))^2\Big)||X||^2 + \frac{||\omega||^2||X||^4}{4}\Big\},$$

where U is vertical and X is horizontal vector fields on M.

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