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Performance Analysis of Zhao and Durbin Numerical Inversion Methods of Laplace Transform

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Abstract

The Laplace transform is essential to satisfy the independence of time for the analysis of the transient response of the composite or functionally-graded materials. The time independent boundary value problem may be solved then either by numerically or analytically. The solutions should be inverted to the physical plane using inverse Laplace transform. Therefore, the selected numerical inversion method may be crucial to obtain the high accuracy throughout the whole analysis steps. In the present study, Zhao's Method I, Zhao's Method II, Durbin's Method and Modified Durbin's Method are applied to dynamic loading conditions. The analysis results show that the accurate and stable solutions even for long time inversion have been obtained by Modified Durbin's Method and Zhao's Methods. However, compared with the methods of Zhao's, the computational and programing load of Durbin's Methods are minimum.

Keywords: Numerical inversion, Laplace transform, Transient analysis, Composite materials

Sayısal Ters Laplace Dönüşümü için Zhao ve Durbin Methodlarının Performans Analizi

Öz

Laplace dönüşümü kompozit ya da fonksiyonel derecelendirilmiş malzemelerin dinamik analizlerinde, zamana olan bağımlılığı ortadan kaldırmak için önemli bir yöntemdir. Zamandan bağımsız sınır değer problemleri analitik ya da sayısal olarak çözülebilir. Elde edilen sonuçlar fiziksel uzaya ters Laplace dönüşümü ile çevirilir. Bu yüzden seçilecek olan ters dönüşüm yöntemi tüm analiz adımlarında yüksek doğruluk elde edilmesi bakımından oldukça önemli olabilir. Bu çalışmada, Zhao Method I, Zhao Method II, Durbin Method ve Düzeltilmiş Durbin Methodları dinamik yükleme koşulları için uygulanmıştır. Analiz sonuçları, uzun süreli çözümlemelerde bile Düzeltilmiş Durbin ve Zhao Methodları ile güvenli ve stabil sonuçlar elde edildiğini göstermiştir. Fakat, Zhao Methodları ile kıyaslandığında Düzeltilmiş Durbin Methodu'nun programlama ve hesaplama yükü çok daha azdır.

Anahtar Kelimeler: Sayısal ters dönüşüm, Laplace dönüşümü, Transient analiz, Kompozit malzemeler

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1. INTRODUCTION

The functionally-graded materials (FGM) which are one of the advanced composite materials have started to be used especially in the aerospace, automotive, marine etc. industries. The dynamic analysis of such materials requires the infusion of Laplace transform method to the governing differential equations. The accuracy and efficiency of the analysis are highly depended on inversion from the Laplace domain to time domain. Closed form inversion may not be possible except for certain loading cases and many methods are proposed in the literature for numerical Laplace inversion [1-4]. Recently, two novel and powerful numerical inversion methods have been proposed by Zhao [5] based on irregularly spaced intervals. Also accelerations of these two algorithms are carried out by Chen and Mei [6] by sub-dividing the entire integrating range into sub-bands while sampling interval in each sub-band remaining constant.

Zhao [5] has reported that Durbin's Method gives an unstable results for long time inversion of functions. This drawback can be overcome by adding a correction factor to the Durbin's Method. Also, compared with the Durbin's Method, programming and computational load are much heavier and selections of sub-spaces and parameters effect the accuracy significantly.

2. MATERIAL AND METHOD

If a real function of time is considered, the inverse Laplace transform and Laplace transform of $f(t)$ are given as respectively:

$$\overline{f}(s) = \int_{0}^{\infty} f(t)\,\mathrm{e}^{-st}\,dt \tag{1}$$

$$f(t) = \frac{1}{2\pi} \int_{a-i\infty}^{a+i\infty} \overline{f}(s)\,\mathrm{e}^{-st}\,ds \tag{2}$$

where $\overline{f}(s)$, s and a, respectively are the Laplace transform of $f(t)$, Laplace parameter and reel number. In the proceeding section, the theoretical backgrounds of four different numerical inversion of Laplace transform based on Zhao and Durbin's algorithms will be presented.

2.1. Durbin's Methods

Durbin's inverse Laplace transform method is based on the Fast Fourier Transform which calculates the integral in Equation (2) numerically. For the real time function $f(t_j)$ obtained from corresponding function $\overline{f}(s_j)$ in Laplace domain, Durbin's inverse Laplace transform formula is given as follows [7]:

$$f(t_j) \cong \frac{2e^{aj\Delta t}}{T}\left[\begin{array}{l} -\dfrac{1}{2}Re\left\{\overline{f}(\mathrm{a})\right\} + \\ Re\left\{\displaystyle\sum_{k=0}^{N-1}\left(A(k)+iB(k)\right)e^{i\frac{2\pi}{N}jk}\right\}\end{array}\right] \tag{3}$$

$j = 0, 1, 2, ..., N - 1$

here

$$A(\mathrm{k}) = \sum_{l=0}^{L} \mathrm{Re}\left\{\overline{f}\left(a + i(k+lN)\frac{2\pi}{T}\right)\right\}$$

$$B(\mathrm{k}) = \sum_{l=0}^{L} \mathrm{Im}\left\{\overline{f}\left(a + i(k+lN)\frac{2\pi}{T}\right)\right\}$$

where i is the complex number, N is the number of equal time intervals and T is the solution interval. The time increment Δt is expressed as $\Delta t = T / N$. The selection of constant a is done by giving a value to aT in the range of 5-10 as suggested by Durbin [7].

The Modified Durbin's Method is obtained by multiplying each term by Fejer (F_k) or Lanczos (L_k) factors [8]. As given by Narayanan [8], the Equation (3) is modified using Lanczos correction term:

$$f(t_j) \cong \frac{2e^{aj\Delta t}}{T}\begin{bmatrix} -\frac{1}{2} Re\left\{\bar{f}(\mathrm{a})\right\} + \\ Re\left\{\sum_{k=0}^{N-1}\left(\bar{f}(s_k)L_k\right)e^{i\frac{2\pi}{N}jk}\right\}\end{bmatrix} \tag{4}$$

where *kth* Laplace parameter is expressed as $s_k = a + i\omega_k$ and the Lanczos correction factor is[8]:

$$L_k = \begin{Bmatrix} \dfrac{\sin\frac{k\pi}{N}}{\frac{k\pi}{N}} & for\ k > 1 \\ 1 & for\ k = 1 \end{Bmatrix} \tag{5}$$

Note that $\omega_k = k\,2\pi/T$ for Durbin's Methods.

2.2. Zhao's Method

In this section Zhao's Method I and Zhao's Method II will be discussed entirely. The idea behind the Zhao's algorithms is to divide the integration interval into small sub-spaces [5].

The formulation of first method can be summarized as follows:

$$f(0) \cong \sum_{k=1}^{N}\frac{(F_k + F_{k+1})\Delta_k}{2\pi} \tag{6}$$

$$f(t) \cong \frac{\mathrm{e}^{at}}{\pi t^2}\sum_{k=1}^{N}\begin{bmatrix} \frac{F_k + F_{k+1}}{\Delta_k}(\cos w_{k+1}t - \cos w_k t) \\ -\frac{G_k + G_{k+1}}{\Delta_k}(\sin w_{k+1}t - \sin w_k t)\end{bmatrix} \tag{7}$$

where

$$F_k = \mathrm{Re}\left[\bar{f}(\mathrm{a} + \mathrm{i}\,\omega_k)\right]$$
$$G_k = \mathrm{Im}\left[\bar{f}(\mathrm{a} + \mathrm{i}\,\omega_k)\right]$$
$$\Delta_k = \omega_{k+1} - \omega_k$$

The second algorithm proposed by Zhao are given below:

$$f(0) \cong \frac{1}{\pi}\sum_{k=1}^{N}\left[F_k + \frac{\Delta_k(Z_{k+1} - Z_k)}{24}\right]\Delta_k \tag{8}$$

$$f(t) \cong \frac{\mathrm{e}^{at}}{\pi}\begin{Bmatrix} \frac{1}{t}g_1(\mathrm{t}) + \frac{1}{t^2}g_2(\mathrm{t}) - \\ \sum_{k=1}^{N}\frac{1}{t^3\Delta_k}\begin{bmatrix}\left(Y_{k+1} - Y_k\right)(\cos w_{k+1}t - \cos w_k t) \\ +\left(Z_{k+1} - Z_k\right)(\sin w_{k+1}t - \sin w_k t)\end{bmatrix}\end{Bmatrix} \tag{9}$$

where

$$F_k = \mathrm{Re}\left[\bar{f}(\mathrm{a} + \mathrm{i}\,\tau_k)\right]$$
$$G_k = \mathrm{Im}\left[\bar{f}(\mathrm{a} + \mathrm{i}\,\tau_k)\right]$$
$$\tau_k = \frac{1}{2}(\omega_{k+1} + \omega_k)$$

Z_k can be determined by the following set of equations:

$$3\Delta_1 Z_1 + \Delta_1 Z_2 = 8(F_1 - F_0) \tag{10}$$

$$\Delta_{k-1}Z_{k-1} + 3Z_k(\Delta_k + \Delta_{k-1}) + \Delta_k Z_{k+1} = 8(F_k - F_{k-1}) \quad (k = 2,3,...,n) \tag{11}$$

$$3\Delta_n Z_{n+1} + \Delta_n Z_n = 8(F_{n+1} - F_n) \tag{12}$$

where $F_0 = \mathrm{Re}[\bar{f}(\mathrm{a} + \mathrm{i}\,\omega_1)]$ and $F_{n+1} = \mathrm{Re}[\bar{f}(\mathrm{a} + \mathrm{i}\,\omega_{n+1})]$

Y_k can be determined by the following set of equations:

$$3\Delta_1 Y_1 + \Delta_1 Y_2 = 8(G_1 - G_0) \tag{13}$$

$$\Delta_{k-1}Y_{k-1} + 3Y_k(\Delta_k + \Delta_{k-1}) + \Delta_k Y_{k+1} = 8(G_k - G_{k-1}) \quad (k = 2,3,...,n) \tag{14}$$

$$3\Delta_n Y_{n+1} + \Delta_n Y_n = 8(G_{n+1} - G_n) \tag{15}$$

where $G_0 = \mathrm{Im}[\bar{f}(\mathrm{a} + \mathrm{i}\,\omega_1)]$ and $G_{n+1} = \mathrm{Im}[\bar{f}(\mathrm{a} + \mathrm{i}\,\omega_{n+1})]$.

Consequently, general expressions for the Zhao's Method II are given below:

$$f(0) \cong \frac{1}{\pi}\sum_{k=1}^{n}\left[F_k + \frac{\Delta_k}{24}(Z_{k+1} - Z_k)\right]\Delta_k \tag{16}$$

$$f(\mathrm{t}) \cong \frac{e^{at}}{\pi}\left[\begin{array}{l}\frac{1}{t}g_1(\mathrm{t}) + \frac{1}{t^2}g_2(\mathrm{t}) - \\ \sum_{k=1}^{n}\frac{1}{t^3\Delta_k}\left[\begin{array}{l}(Z_{k+1} - Z_k)(\sin\omega_{k+1}\mathrm{t} - \sin\omega_k\mathrm{t}) \\ +(Y_{k+1} - Y_k)(\cos\omega_{k+1}\mathrm{t} - \cos\omega_k\mathrm{t})\end{array}\right]\end{array}\right] \tag{17}$$

where

$$g_1(\mathrm{t}) = \left[F_n + \frac{1}{8}(Z_n + 3Z_{n+1})\Delta_n\right]\sin\omega_{n+1} +$$

$$\frac{1}{8}(Y_2 + 3Y_1)\Delta_1 - G_1 + \left[G_n + \frac{1}{8}(Y_n + 3Y_{n+1})\Delta_n\right]\cos\omega_{n+1}t$$

$$g_2(\mathrm{t}) = -Z_1 + Z_{n+1}\cos\omega_{n+1}t - Y_{n+1}\sin\omega_{n+1}t$$

3. RESULTS AND DISCUSSIONS

Two different loading case given by Zhao [5] are used to study of convercengy and accuracy of four proposed methods. The sample loading functions are $\overline{f}(s) = s^{-1}$ and $\overline{f}(s) = s(s^2+1)^{-2}$ which correspond the Heaviside function $f(t) = H(t)$ and sinusoidal function $f(t) = (t/2)sin(t)$ in the time domain, respectively. The results are tabulated in Table 1-4. The time length is defined as $T=20$ and the range of integration for Zhao' Methods is divided into three intervals as [0, 2.5], [2.5, 37.5] and [37.5, 897.5] where Δ_k is 0.005, 0.5 and 2.0, respectively. For Durbin's Methods, aT is chosen to be 6 as Ref. [9]. The Lanczos correction factor is used for Modified Durbin approach.

It has been seen from Table 1 and 2, the divergency of the result for the last time sequence of first function is very high for the Durbin Method. As the considering Table 2 and 4, the relative error values of both Zhao Methods are lower than Durbin Methods, however, Modified Durbin Method also provides very good relative errors for both loading function which means that Lanczos correction factor has increased the stability. Also, time interval selection is not the case for Durbin Methods as both Zhao method.

Table 1. Inverse Laplace transform of $\overline{F}(s) = s^{-1}$

Time	Modified Durbin's Method	Durbin's Method [5]	Zhao's Method 1 [5]	Zhao's Method 2 [5]	Exact
0.0	0.501753412	0.50653049	0.50002454	0.49998200	1.0000
1.0	1.00248651	1.00549358	0.99984388	0.99999402	1.0000
2.0	1.00248643	1.00597629	0.99995698	1.00000370	1.0000
3.0	1.00248642	1.00612267	1.00016226	1.00002421	1.0000
4.0	1.00248642	1.00618855	0.99973049	1.00001323	1.0000
5.0	1.00248642	1.00622859	1.00017907	0.99999158	1.0000
6.0	1.00248642	1.00626599	0.99975946	0.99998448	1.0000
7.0	1.00248642	1.00631770	0.99995896	0.99996924	1.0000
8.0	1.00248642	1.00640245	0.99992645	1.00001701	1.0000
9.0	1.00248642	1.00654564	0.99966112	1.00001107	1.0000
10.0	1.00248642	1.00678511	1.00004361	1.00003587	1.0000
11.0	1.00248641	1.00717982	0.99944320	0.99999743	1.0000
12.0	1.00248641	1.00782468	1.00009414	0.99997066	1.0000
13.0	1.00248641	1.00887807	0.99923983	0.99996375	1.0000
14.0	1.00248641	1.01061669	0.99977563	0.99999187	1.0000
15.0	1.00248641	1.01355593	0.99945386	1.00003359	1.0000
16.0	1.00248641	1.01874996	0.99948358	1.00002965	1.0000
17.0	1.00248640	1.02868949	0.99940670	1.00002235	1.0000
18.0	1.00248639	1.05088628	0.99931966	0.99996898	1.0000
19.0	1.00248637	1.12293998	0.99923481	0.99995546	1.0000
20.0	1.00248631	75.17073028	0.99920029	0.99997784	1.0000

Table 2. Relative error values of inverse Laplace transform methods for $\overline{F}(s) = s^{-1}$

Time	Modified Durbin's Method	Durbin's Method	Zhao's Method 1	Zhao's Method 2
0.0	0.49824659	0.49346951	0,49997546	0,50001800
1.0	0.00248651	0.00549358	0,00015612	0,00000598
2.0	0.00248643	0.00597629	0,00004302	0,00000370
3.0	0.00248642	0.00612267	0,00016226	0,00002421
4.0	0.00248642	0.00618855	0,00026951	0,00001323
5.0	0.00248642	0.00622859	0,00017907	0,00000842
6.0	0.00248642	0.00626599	0,00024054	0,00001552
7.0	0.00248642	0.00631770	0,00004104	0,00003076
8.0	0.00248642	0.00640245	0,00007355	0,00001701
9.0	0.00248642	0.00654564	0,00033888	0,00001107
10.0	0.00248642	0.00678511	0,00004361	0,00003587
11.0	0.00248641	0.00717982	0,00055680	0,00000257
12.0	0.00248641	0.00782468	0,00009414	0,00002934
13.0	0.00248641	0.00887807	0,00076017	0,00003625
14.0	0.00248641	0.01061669	0,00022437	0,00000813
15.0	0.00248641	0.01355593	0,00054614	0,00003359
16.0	0.00248641	0.01874996	0,00051642	0,00002965
17.0	0.00248640	0.02868949	0,00059330	0,00002235
18.0	0.00248639	0.05088628	0,00068034	0,00003102
19.0	0.00248637	0.12293998	0,00076519	0,00004454
20.0	0.00248631	74.17073028	0,00079971	0,00002216

Table 3. Inverse Laplace transform of $\overline{F}(s) = s(s^2+1)^{-2}$

Time	Modified Durbin's Method	Durbin's Method [5]	Zhao's Method 1 [5]	Zhao's Method 2 [5]	Exact
0.0	-0.004128592	0.06218747	-0.00009131	0.00000218	0.00000000
1.0	0.445407616	0.47977090	0.42073715	0.42072545	0.42073549
2.0	0.941424649	0.90776061	0.90949537	0.90931095	0.90929743
3.0	0.221132725	0.14529924	0.21129895	0.21166887	0.21168001
4.0	-1.53750814	-1.58679915	-1.51309243	-1.51360309	-1.51360499
5.0	-2.43415229	-2.40758056	-2.39762968	-2.39729723	-2.39731069
6.0	-0.85384868	-0.77051018	-0.83785428	-0.83827882	-0.83824649
7.0	2.32132447	2.38657029	2.29907174	2.29950339	2.29945310
8.0	3.99840548	3.98214155	3.956842373	3.95736997	3.95743299
9.0	1.87692324	1.78863406	1.85447616	1.85460021	1.85453318
10.0	-2.73863399	-2.82025465	-2.71993636	-2.72016580	-2.72010555
11.0	-5.54423632	-5.54135701	-5.49808585	-5.49990224	-5.49994614
12.0	-3.24904812	-3.15882255	-3.21897264	-3.21946065	-3.21943751
13.0	2.74495541	2.84272087	2.73054498	2.73109266	2.73108574
14.0	6.98083258	6.99415956	6.93118737	6.93426442	6.93425149
15.0	4.91419067	4.82541788	4.87496126	4.87712892	4.87715880
16.0	-2.31116225	-2.42416899	-2.30199547	-2.30319122	-2.30322653
17.0	-8.21953369	-8.25152215	-8.16699679	-8.17190744	-8.17187868
18.0	-6.80328714	-6.71963241	-6.75430215	-6.75886561	-6.75888522
19.0	1.42464613	1.55131243	1.42278407	1.42382934	1.42383349
20.0	9.17680430	9.22943796	9.12178084	9.12944158	9.12945251

Table 4. Relative error values of inverse Laplace transform methods for $\overline{F}(s) = s(s^2+1)^{-2}$

Time	Modified Durbin's Method	Durbin's Method [ref]	Zhao's Method 1 [ref]	Zhao's Method 2 [ref]
0.0	-	-	-	-
1.0	0.05864047	0.14031479	0.00000395	0.00002386
2.0	0.03533191	0.00169012	0.00021768	0.00001487
3.0	0.04465568	0.31359017	0.00180017	0.00005263
4.0	0.01579220	0.04835750	0.00033864	0.00000126
5.0	0.01536789	0.00428391	0.00013306	0.00000561
6.0	0.01861289	0.08080715	0.00046789	0.00003857
7.0	0.00951155	0.03788605	0.00016585	0.00002187
8.0	0.01035330	0.00624358	0.00014924	0.00001592
9.0	0.01207315	0.03553407	0.00003075	0.00003614
10.0	0.00681166	0.03681809	0.00006220	0.00002215
11.0	0.00805284	0.00752932	0.00033824	0.00000798
12.0	0.00919745	0.01882781	0.00014439	0.00000719
13.0	0.00507845	0.04087573	0.00019800	0.00000253
14.0	0.00671754	0.00863944	0.00044188	0.00000186
15.0	0,00759292	0,01060882	0,00045058	0,00000613
16.0	0,00344548	0,05251002	0,00053449	0,00001533
17.0	0,00583159	0,00974604	0,00059740	0,00000352
18.0	0,00656941	0,00580759	0,00067808	0,00000290
19.0	0,00057074	0,08953220	0,00073704	0,00000291
20.0	0,00518671	0,01095197	0,00084032	0,00000120

4. CONCLUSION

In this study four different numerical inverse Laplace transform method are investigated. The main drawback of Zhao's algorithm is having heavy programming burden and progress time. The selection of intervals and parameters affects significantly to accuracy of method. The advantage of Zhao's algorithm is to give stable results for long time inversion on the contrary to Durbin's Method. However, adding a correction factor to the formulation of Durbin's algorithm increases stability of method. It can be induced that Modified Durbin's Method gives an accurate result even for long time investigations with less burden and progress time.

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