ON DISTRIBUTION OF A SEMI-MARKOV RANDOM WALK PROCESS WITH TWO DELAYING BARRIERS

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ABSTRACT

In this paper, a step process of semi-Markovian random walk with delaying barrier on the zero-level from below and $a(a > 0)$-level from upper is constructed mathematically and the Laplace transformation for the distribution function of this is given. Also, the expectation and standard diversion of a boundary functional of the process are given.

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1. Introduction

In recent years, random walks with one or two barriers are being used to solve a number of very interesting problems in the fields of inventory, queues and reliability theories, mathematical biology etc. Many good monographs in this field exist in literature (see references Afanas'eva et al. 1983; Feller 1968 and etc.).

In particular, a number of very interesting problems of stock control, queues and reliability theories can be expressed by means of random walks with two barriers. These barriers can be reflecting, delaying, absorbing, elastic, etc., depending on concrete problems at hand. For instance, it is possible to express random levels of stock in a warehouse with finite volumes or queuning systems with finite waiting time or sojourn time by means of random walks with two delaying barriers. Furthermore, the functioning of stochastic systems with spare equipment can be given by random walks with two barriers, one of them is delaying and the other one is any type barrier.

It is known that the most of the problems of stock control theory is often given by means of random walks or random walks with delaying barriers (see Afanas'eva et al. 1983; Borovkov 1972 , etc.). Numerous studies have been done about step processes of

semi-Markovian random walk with two barriers of their practical and theoretical importance. But in the most of these studies the distribution of the process has free distribution. Therefore the obtained results in this case are cumbersome and they will not be useful for applications (Afanas'eva et al. 1983; Gihman 1975 and etc.). For the problem considered in this study, it is considered a semi-Markov random walk with two delaying barriers, and the process representing the quantity of the stock has been given by using a random walk and a renewal process. Such models were rarely considered in literature. The practical state of the problem mentioned above is as follows.

Suppose that some quantity of a stock in a certain warehouse is increasing or decreasing in random discrete portions depending to the demands at discrete times. Then, it is possible to characterize the level of stock by a process called the semi-Markovian random walk process. But sometimes some problems occur in stock control theory such that in order to get an adequate solution we have to consider some processes which are more complex than semi-Markovian random walk processes. For example, if the borrowed quantity is demanded to be added to the warehouse immediately when the quantity of demanded stock is more than the total quantity of stock in the warehouse then, it is possible to characterize the level of stock in the warehouse by a stochastic process called as semi-Markovian random walk processes with delaying barrier at zero-level. Also since the volume of warehouse is finite in real cases, the supply coming to the warehouse is stopped until the next demand when the warehouse becomes full. In order to characterize the quantity of stock in the warehouse under these conditions it is necessary to use a stochastic process called as semi-Markovian random walk process with two delaying barriers. Note that semi-Markovian random walk processes with two delaying barriers, have not been considered enough in literature. This type problems may occur, for example, in the control of military stocks, refinery stocks, reserve of oil wells, and etc.

2. Construction of the Process

Suppose $\{(\xi_i, \eta_i)\}$, $i = 1,2,3,\dots$ is a sequence of identically and independently distributed pairs of random variables, defined on any probability space $(\Omega, \mathcal{F}, P)$ such that ξ_i's are positive valued, i.e., $P\{\xi_i > 0\} = 1$, $i = 1,2,3,\dots$ In addition, the random

variables ξ_i and η_i are mutually independent as well. Also let us denote the distribution function of ξ_1 and η_1

$$\Phi(t) = P\{\xi_1 < t\}, \mathrm{F}(x) = P\{\eta_1 < x\}, t \in \mathbb{R}^+, x \in \mathbb{R},$$

respectively.

By using the random pairs (ξ_i, η_i), we can construct a step process of a semi-Markov random walk

$$X_1(t) = \sum_{i=1}^{n} \eta_i \text{, if } \quad \sum_{i=0}^{n-1} \xi_i \leq t < \sum_{i=0}^{n} \xi_i \text{ , } n \geq 1$$

where $\xi_0 = 0$ and $\eta_0 = z \in (0, a)$. Now, let us delay this process by a delaying barrier at zero level as follows:

$$X_2(t) = X_1(t) - \inf_{0 \leq s \leq t} \{0, \ X_1(s)\}.$$

This process forms a step process of semi-Markovian random walk with delaying barrier on the zero-level. Then, the process $X_2(t)$ is delayed by a delaying barrier on $a(a > 0)$-level:

$$X(t) = X_2(t) - \sup_{0 \leq s \leq t} \{0, \ X_2(s) - a\}.$$

The process $X(t)$ forms a step process of semi-Markovian random walk with delaying barriers on the zero-level and on the $a(a > 0)$.
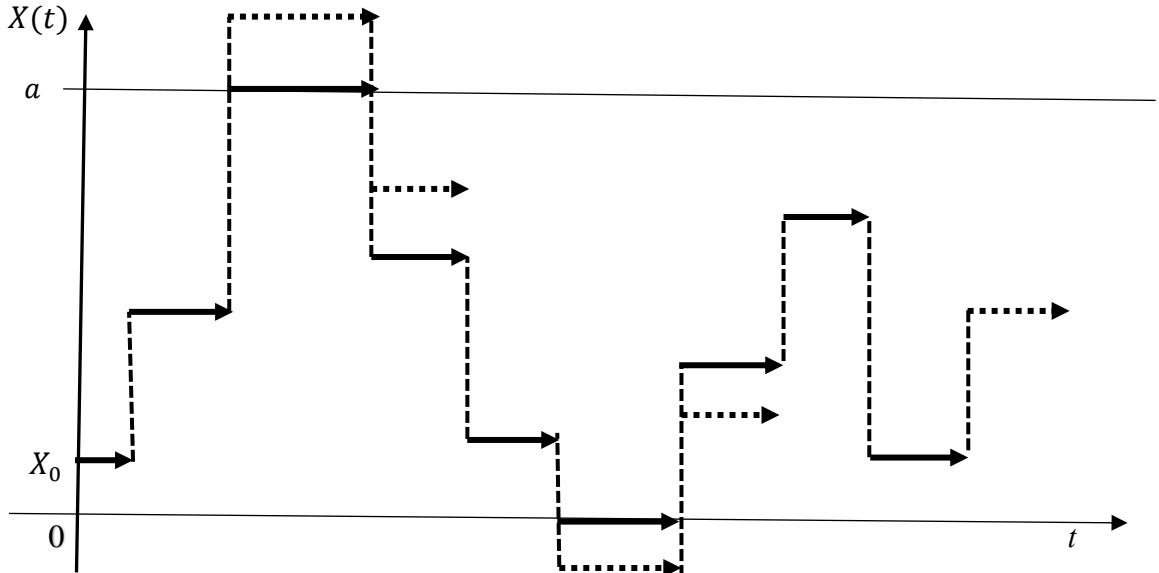

Fig. 1. A View of a Step Process of Semi-Markov Random Walk with Two Delaying Barriers

Now, the process $X(t)$ is restated by means of the sequence of identically and independently distributed random pairs $\{(\xi_i, \eta_i)\}$, $i = 1,2,3, \dots$, as follows:

$$X(t) = X_n, \text{if} \quad \sum_{i=0}^{n-1} \xi_i \le t < \sum_{i=0}^{n} \xi_i, n \ge 1,$$

where

$$X_n = min\{a, max\{0, X_{n-1} + \eta_n\}\}, n \ge 1,$$

and $X_0 = z \in (0, a)$.

Note that a lot of very interesting problems of stock control theory are expressed and solved by using this type processes.

The aim of this study is to determine the Laplace transform for the distribution function with one dimensional of this process when the random variable η_1 has the Laplace distribution. For this reason, let us introduce the following notations: Let us denote the conditional distribution function with one dimensional of this process by

$$R(t, x|z) = P[X(t) < x | X(0) = z].$$

The Laplace transformation with the time of the conditional distribution function of the process is denoted by

$$\tilde{R}(\theta, x|z) = \int_{t=0}^{\infty} e^{-\theta t} R(t, x|z) dt, \ \theta > 0.$$

Also, let us denote the Laplace transformation with the time and the Laplace - Stieltjes transformation by the phase of the conditional distribution function of the process is denoted by

$$\tilde{\tilde{R}}(\theta, \alpha|z) = \int_{t=0}^{\infty} e^{-\alpha x} d_x \tilde{R}(\theta, x|z).$$

In order to convenient, let us denote the Laplace transformation of random variable ξ_1 by

$$\varphi(\theta) = E[e^{-\theta \xi_1}].$$

Finally, let us denote that $\mathcal{E}(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0 \end{cases}$.

3. The Integral Equation for $\tilde{\tilde{R}}(\theta, \alpha|z)$

In this section we give an integral equation for the Laplace transformation with the time and the Laplace - Stieltjes transformation by the phase of the conditional distribution function $R(t, x|z)$ of the process $X(t)$. For this, we can state the following theorem:

Theorem 1. Under above representations, we have

$$\tilde{\tilde{R}}(\theta, \alpha|z) = \frac{1-\varphi(\theta)}{\theta} \mathcal{E}(x - z) + \varphi(\theta) \left[\tilde{\tilde{R}}(\theta, \alpha|a) P\{\eta_1 > a - z\} \right.$$

$$\left. + \tilde{\tilde{R}}(\theta, \alpha|0) P\{\eta_1 < -z\} + \int_{y=0}^{a} \tilde{\tilde{R}}(\theta, \alpha|y) \, d_y P\{\eta_1 < y - z\} \right] \quad (3.1)$$

Proof: According to the total probability formula, it is obvious that

$$R(t, x|z) = P[X(t) < x|X(0) = z]$$

$$= P\{X(t) < x; \xi_1 > t|X(0) = z\} + P\{X(t) < x; \xi_1 < t|X(0) = z\}$$

$$= P\{z < x; \xi_1 > t\} + \int_{s=0}^{t} P\{\xi_1 \in ds\}$$

$$\times \int_{y=0}^{a} P\{\min\{a, \max\{0, z + \eta_1\}\} \in dy\} P\{X(t - s) < x|X(0) = y\}$$

$$= \mathcal{E}(x - z) P\{\xi_1 > t\} + \int_{s=0}^{t} P\{\xi_1 \in ds\}$$

$$\times \int_{y=0}^{a} P\{\min\{a, \max\{0, z + \eta_1\}\} \in dy\} R(t - s, x|y) \quad (3.2)$$

By applying the Laplace transformation on the both side of (3.2) with respect to t, i.e., multiplying both sides of (3.1) by $e^{-\theta t}$, integrating with respect to t from 0 to ∞, and taking into account the definition of $\tilde{N}(\theta|z)$, we have

$$\tilde{R}(\theta, x|z) = \int_{t=0}^{\infty} e^{-\theta t} R(t, x|z) dt$$

$$= \mathcal{E}(x - z) P\{\xi_1 > t\} + \int_{t=0}^{\infty} e^{-\theta t} P\{\xi_1 > t\} dt$$

$$+ \int_{y=0}^{a} P\{\min\{a, \max\{0, z + \eta_1\}\} \in dy\}$$

$$\times \int_{t=0}^{\infty} e^{-\theta t} \int_{s=0}^{t} P\{\xi_1 \in ds\} R(t - s, x|y) dt. \quad (3.3)$$

We note that

$$\varphi(\theta) = \int_{t=0}^{\infty} e^{-\theta t} P\{\xi_1 < t\} dt.$$

Thus, we can write

$$\tilde{\tilde{R}}(\theta, \alpha | z) = \int_{t=0}^{\infty} e^{-\alpha x} d_x \tilde{R}(\theta, x | z)$$

$$= \frac{1-\varphi(\theta)}{\theta} \mathcal{E}(x-z) + \varphi(\theta) \int_{y=0}^{a} P\{min\{a, max\{0, z+\eta_1\}\} \in dy\} \tilde{R}(\theta, x | y)$$

$$= \frac{1-\varphi(\theta)}{\theta} \mathcal{E}(x-z) + \varphi(\theta) \Big[\tilde{\tilde{R}}(\theta, \alpha | a) P\{\eta_1 > a - z\}$$

$$+ \tilde{\tilde{R}}(\theta, \alpha | 0) P\{\eta_1 < -z\} + \int_{y=0}^{a} \tilde{\tilde{R}}(\theta, \alpha | y) d_y P\{\eta_1 < y - z\}\Big],$$

and therefore, the Theorem is proved.

The integral equation (3.1) can be solved by method of successive approximations for arbitrarily distributed random variables ξ_i and η_i, $i \geq 1$, but it is unsuitable for applications. On the other hand, this equation has a solution in explicit form in the class of Laplace distributions. For example, let the random walk follow the compound Laplace distribution. We introduce random variable $\eta_1 = \eta_1^+ - \eta_1^-$ in which random variables η_i^+ and η_i^- are distributed as follows:

$$F_{\eta_1^+}(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda t}, & x > 0, \quad \lambda > 0, \end{cases}$$

and

$$F_{\eta_1^-}(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\mu x}, & x > 0, \quad \mu > 0, \end{cases}$$

respectively. Then it is easy to see that the distribution function of random variable η_1 has the form

$$F_{\eta_1}(x) = P\{\eta_1 < x\} = \begin{cases} \frac{\lambda}{\lambda+\mu} e^{\mu x}, & x < 0, \lambda > 0, \mu > 0, \\ 1 - \frac{\mu}{\lambda+\mu} e^{-\lambda x}, & x > 0, \lambda > 0, \mu > 0, \end{cases} \tag{3.4}$$

and the probability density function of its has the form

$$f_{\eta_1}(t) = \frac{d[F_{\eta_1}(t)]}{dt} = \begin{cases} \frac{\lambda\mu}{\lambda+\mu} e^{\mu t}, & t < 0, \\ \frac{\lambda\mu}{\lambda+\mu} e^{-\lambda t}, & t > 0, \end{cases} \tag{3.5}$$

(see, Nasirova et al. 2009). The distribution function given by (3.4) is called the compound Laplace distribution function of order $(1,1)$ and denote by $L(1^+, 1^-)$. In this case, the integral equation (3.1) can be rewritten as follows:

$$\tilde{\tilde{R}}(\theta, \alpha|z) = \int_{t=0}^{\infty} e^{-\alpha x} d_x \tilde{R}(\theta, x|z)$$

$$= \frac{1-\varphi(\theta)}{\theta} e^{-\alpha z} + \frac{\mu}{\lambda+\mu} \varphi(\theta) \tilde{\tilde{R}}(\theta, \alpha|a) e^{-\lambda(a-z)}$$

$$+ \frac{\lambda}{\lambda+\mu} \varphi(\theta) \tilde{\tilde{R}}(\theta, \alpha|0) e^{-\mu z} + \frac{\lambda\mu}{\lambda+\mu} \varphi(\theta) e^{-\mu z} \int_{y=0}^{z} \tilde{\tilde{R}}(\theta, \alpha|y) e^{\mu y} dy$$

$$+ \frac{\lambda\mu}{\lambda+\mu} \varphi(\theta) e^{\lambda z} \int_{z}^{a} \tilde{\tilde{R}}(\theta, \alpha|y) e^{-\lambda y} dy \qquad (3.6)$$

By using the integral equation (3.6), we obtain the ordinary differential equation with constant coefficients

$$\tilde{\tilde{R}}''(\theta, \alpha|z) - (\lambda - \mu)\tilde{\tilde{R}}'(\theta, \alpha|z) - \lambda\mu[1 - \varphi(\theta)]\tilde{\tilde{R}}(\theta, \alpha|z)$$

$$= \frac{1-\varphi(\theta)}{\theta} (\lambda - \mu)(\alpha + \lambda)e^{-\alpha z}. \qquad (3.7)$$

This equation has a characteristic equation

$$k^2(\theta) - (\lambda - \mu)k(\theta) - \lambda\mu[1 - \varphi(\theta)] = 0. \qquad (3.8)$$

and so, it has a general solution in the form

$$\tilde{\tilde{R}}(\theta, \alpha|z) = C_1(\theta)e^{k_1(\theta)z} + C_2(\theta)e^{k_2(\theta)z} + \frac{(1-\varphi(\theta))(\alpha-\mu)(\alpha+\lambda)}{\theta(\alpha+k_1(\theta))(\alpha+k_2(\theta))} e^{-\alpha z} \qquad (3.9)$$

where $k_1(\theta)$ and $k_2(\theta)$ are roots of the characteristic equation (3.8).

Now, we should find $C_i(\theta), i = 1,2$. The following system is obtained from the initially boundary conditions for differential equation (3.7) from integral equation (3.6), by taking zero instead of z:

$$\begin{cases} \tilde{\tilde{R}}(\theta, \alpha|0) = \frac{1-\varphi(\theta)}{\theta} + \frac{\mu}{\lambda+\mu} \varphi(\theta)\tilde{\tilde{R}}(\theta, \alpha|a)e^{-\lambda a} \\ \qquad + \frac{\lambda}{\lambda+\mu} \varphi(\theta) \tilde{\tilde{R}}(\theta, \alpha|0) + \frac{\lambda\mu}{\lambda+\mu} \varphi(\theta) \int_0^a \tilde{\tilde{R}}(\theta, \alpha|y) e^{-\lambda y} dy \\ \tilde{\tilde{R}}'(\theta, \alpha|0) = -\alpha \frac{1-\varphi(\theta)}{\theta} + \frac{\lambda\mu}{\lambda+\mu} \varphi(\theta)\tilde{\tilde{R}}(\theta, \alpha|a)e^{-\lambda a} \\ \qquad - \frac{\lambda\mu}{\lambda+\mu} \varphi(\theta) \tilde{\tilde{R}}(\theta, \alpha|0) + \frac{\lambda^2\mu}{\lambda+\mu} \varphi(\theta) \int_0^a \tilde{\tilde{R}}(\theta, \alpha|y) e^{-\lambda y} dy \end{cases} \qquad (3.10)$$

By substituting these expressions on the left hand of (3.9) instead of the desired function under the integral into (3.10), we obtain the system of algebraic equations in relation to $C_i(\theta), i = 1,2$. Thus we have

$$\begin{cases} \tilde{\tilde{R}}(\theta,\alpha|0) = C_1(\theta) + C_2(\theta) + \frac{(1-\varphi(\theta))(\alpha-\mu)(\alpha+\lambda)}{\theta(\alpha+k_1(\theta))(\alpha+k_2(\theta))} \\ \tilde{\tilde{R}}'(\theta,\alpha|0) = k_1(\theta)C_1(\theta) + k_2(\theta)C_2(\theta) - \alpha\frac{(1-\varphi(\theta))(\alpha-\mu)(\alpha+\lambda)}{\theta(\alpha+k_1(\theta))(\alpha+k_2(\theta))} \end{cases} \qquad (3.11)$$

Now, by multiplying the first equality by α and then adding these equalities side by side, we have

$$\tilde{\tilde{R}}'(\theta,\alpha|0) + \alpha\tilde{\tilde{R}}(\theta,\alpha|0) = \big(\alpha + k_1(\theta)\big)C_1(\theta) + \big(\alpha + k_2(\theta)\big)C_2(\theta). \qquad (3.12)$$

By using (3.12) and the definition of $\tilde{R}(\theta,\alpha)$, we get

$$\tilde{R}(\theta,\alpha) = \int_0^a \tilde{\tilde{R}}(\theta,\alpha|x)\, dP\{min\{a,\eta_1\} < x\}.$$

Therefore, we can write

$$\tilde{R}(\theta,\alpha) = \frac{1}{(k_1(\theta)-\lambda)}\Big[k_1(\theta)\, e^{-(\lambda-k_1(\theta))a} - \lambda\Big]C_1(\theta)$$

$$+ \frac{1}{(k_2(\theta)-\lambda)}\Big[k_2(\theta)\, e^{-(\lambda-k_2(\theta))a} - \lambda\Big]C_2(\theta)$$

$$+ \frac{(1-\varphi(\theta))(\alpha-\mu)}{\theta(\alpha+k_1(\theta))(\alpha+k_2(\theta))}\Big[\alpha\, e^{-(\alpha+\lambda)a} + \lambda\Big]. \qquad (3.13)$$

and from this, we have

$$\underset{\theta\to 0}{Lim}\, \theta\, \tilde{R}(\theta,\alpha) = \frac{(\lambda-\mu)}{\lambda^2-\mu^2 e^{-(\lambda-\mu)a}}\Big[\frac{\lambda(\alpha+\lambda)}{\alpha+\lambda-\mu} + \frac{\mu(\alpha-\mu)}{\alpha+\lambda-\mu}e^{-(\alpha+\lambda-\mu)a}\Big].$$

On the other hand, the process $X(t)$ is ergodic since $P\{\eta_1 > 0\} > 0$ and $P\{\eta_1 < 0\} > 0$ (see Nasirova 1984.). Hence the Tauberian Theorem can be used:

$$\tilde{R}(\alpha) = \underset{\theta\to 0}{Lim}\, \theta\, \tilde{R}(\theta,\alpha).$$

So, we get

$$\tilde{R}(\alpha) = \frac{(\lambda-\mu)}{\lambda^2-\mu^2 e^{-(\lambda-\mu)a}}\Big[\frac{\lambda(\alpha+\lambda)}{\alpha+\lambda-\mu} + \frac{\mu(\alpha-\mu)}{\alpha+\lambda-\mu}e^{-(\alpha+\lambda-\mu)a}\Big], \quad \lambda > \mu. \qquad (3.14)$$

Now we can derive $E[X]$ and $D[X]$, expectation and standard diversion of the the ergodic distribution of the process $X(t)$ from (3.14). We use X to denote the random variable, for which the following equality holds:

$$\underset{t\to\infty}{Lim} P\{X(t) < x\} = P\{X < x\}.$$

Since $\tilde{R}(\alpha)$ is a Laplace transformation, it is known that

$$E[X] = -\tilde{R}'(0) \text{ and } D[X] = \tilde{R}''(0) - [\tilde{R}'(0)]^2.$$

Thus, by considering (3.14), it is obtained that

$$E[X] = -\frac{\mu}{(\lambda-\mu)(\lambda^2-\mu^2 e^{-(\lambda-\mu)a})}\left[\lambda - \lambda\, e^{-(\lambda-\mu)a} - \mu a(\lambda-\mu)e^{-(\lambda-\mu)a}\right],\ \lambda > \mu,$$

and

$$D[X] = \frac{1}{(\lambda^2-\mu^2 e^{-(\lambda-\mu)a})^2}\left[-\mu^2 a^2\lambda^2\, e^{-(\lambda-\mu)a} - 2\mu a\lambda(\lambda+\mu)e^{-(\lambda-\mu)a}\right]$$

$$-\frac{2\lambda\mu^2}{(\lambda-\mu)}e^{-(\lambda-\mu)a} - \frac{2\lambda\mu(\lambda^2+\mu^2+\lambda\mu)}{(\lambda-\mu)^2}e^{-(\lambda-\mu)a} - \frac{\lambda^2\mu(2\lambda-\mu)}{(\lambda-\mu)^2}.$$

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