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THE GRADIENT AND PARTIAL DERIVATIVES OF BICOMPLEX NUMBERS: A COMMUTATIVE-QUATERNION APPROACH

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Abstract. The study of bicomplex numbers, specifically commutative-quaternions, offers a fascinating exploration into the properties of complexified quaternions with commutative multiplication. Understanding the gradient and partial derivatives within this mathematical framework is crucial for analyzing the behavior of bicomplex functions. Real quaternions are not commutative but bicomplex numbers are commutative by multiplication. Bicomplex numbers are the special case of real quaternions. In this study, gradient and partial derivatives are obtained for bicomplex number valued functions.

1. Introduction

Commutative-quaternions, a specific subset of bicomplex numbers, have gained significant interest in mathematical research due to their commutative multiplication property. Unlike traditional quaternions, which are non-commutative, commutative-quaternions provide a unique algebraic structure for studying the behavior of bicomplex functions. For detailed information, see [5, 10, 11].

A real quaternion Q is defined by

$$Q = a + bi + cj + dk$$

where a, b, c, d are real numbers and

$$i^2 = j^2 = k^2 = ijk = -1$$

$$\begin{aligned} ij &= k,\ jk = i,\ ki = j, \\ ji &= -k,\ kj = -i,\ ik = -j. \end{aligned}$$

The conjugate of a real quaternion Q

$$\bar{Q} = a - bi - cj - dk$$

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and the norm of Q is

$$\begin{aligned} |Q| &= \sqrt{|Q|^2} \\ &= \sqrt{Q\overline{Q}} \\ &= \sqrt{a^2+b^2+c^2+d^2}. \end{aligned}$$

The set of quaternions is denoted by H [1, 6, 7].

A bicomplex number q is defined by

$$q = t + xi + yj + zk$$

where w, x, y, z are real numbers and

$$i^2 = j^2 = -1$$

$$\begin{aligned} ij &= ji = k, \\ ki &= ik = -j, \\ kj &= jk = -i. \\ k^2 &= ijij = iijj = i^2j^2 = 1 \end{aligned}$$

For detailed information about bicomplex numbers, we refer the reader to [5, 10].

The gradient of a scalar-valued function in bicomplex analysis allows us to determine the direction and magnitude of the steepest ascent or descent at any point. Similarly, partial derivatives provide a measure of how a function changes concerning each variable in a multidimensional space.

2. Preliminaries

Consider the bicomplex number function $f = f_1 + if_2 + jf_3 + kf_4$, whose components are bicomplex number valued functions. We can give the definition of derivative that

$$f'(q) = \frac{df}{dq} = \lim_{\Delta q \to 0} [f(q+\Delta q) - f(q)](\Delta q)^{(-1)}$$

where $q = t + xi + yj + zk$ is a bicomplex number. Then, $f(q) = f_1(q) + if_2(q) + jf_3(q) + kf_4(q)$.

In complex numbers algebra,

$$df/dz = \begin{bmatrix} \partial f_1/\partial x & \partial f_2/\partial x \\ \partial f_1/\partial y & \partial f_2/\partial y \end{bmatrix}$$

where $z = x + iy$ complex number and $f = f_1 + if_2$ complex function. So, $f(z) = f_1(z) + if_2(z)$ is written.

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial f}{\partial z}1 = f\,'(z) \\ &\Longrightarrow f\,'(z) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial x}i \\ \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial f}{\partial z}i = f\,'(z)i \\ &\Longrightarrow f\,'(z) = -\frac{\partial f}{\partial y}i = \frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}i \end{aligned}$$

Real parts and the coefficient of i are equal. Also,

$$T_z = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

is the matrix representation of z complex number and

$$\frac{\partial f}{\partial x} = \frac{\partial f_1}{\partial x} + i\frac{\partial f_2}{\partial x}$$

is a complex derivative. We can write that

$$\begin{aligned} T_{z\,'} &= \begin{bmatrix} \partial f_1/\partial x & \partial f_2/\partial x \\ -\partial f_2/\partial x & \partial f_1/\partial x \end{bmatrix} \\ &= \begin{bmatrix} \partial f_1/\partial x & \partial f_2/\partial x \\ \partial f_1/\partial y & \partial f_2/\partial y \end{bmatrix} \end{aligned}$$

by considering the matrix representation of z. Here,

$$\partial f_1/\partial x = \partial f_2/\partial y,\ \partial f_2/\partial x = -\partial f_1/\partial y$$

are Cauchy-Riemann terms [3].

For real quaternions q_1 and q_2

$$q_1 = \mu q_2 \mu^{-1}$$

considering that the real quaternions q_1 and q_2 are similar if there is at least one μ real quaternion satisfying the equation. We can apply this feature for bicomplex numbers, which is the special case of real quaternion. Similar calculates are in [2, 8, 9] for quaternions. Hence,

$$\begin{aligned} q^i &= -iqi = -i(t + ix + jy + kz)i \\ &= -i(ti - x + ky - jz) \\ &= t + ix + jy + kz \\ q^j &= -jqj = -j(t + ix + jy + kz)j \\ &= -j(tj + kx - y - iz) \\ &= t + ix + jy + kz \\ q^k &= -kqk = -k(t + ix + jy + kz)k \\ &= -k(tk - jx - iy + z) \\ &= -t - ix - jy - kz \end{aligned}$$

involutions are obtained. Then, it is written

$$\begin{aligned} q &= t + ix + jy + kz \\ q^i &= t + ix + jy + kz \\ q^j &= t + ix + jy + kz \\ q^k &= -t - ix - jy - kz \end{aligned}$$

equation system. So,

$$\begin{aligned} t &= \frac{1}{4}(q + q^i + q^j - q^k) \\ x &= \frac{1}{4i}(q + q^i + q^j - q^k) \\ y &= \frac{1}{4j}(q + q^i + q^j - q^k) \\ z &= \frac{1}{4k}(q + q^i + q^j - q^k) \end{aligned}$$

are obtained. Hence,

$$\begin{aligned} dt &= \frac{1}{4}(dq + dq^i + dq^j - dq^k) \\ dx &= \frac{-i}{4}(dq + dq^i + dq^j - dq^k) \\ dy &= \frac{-j}{4}(dq + dq^i + dq^j - dq^k) \\ dz &= \frac{k}{4}(dq + dq^i + dq^j - dq^k) \end{aligned}$$

are written.

3. The Partial Derivatives of Bicomplex Functions

Partial derivatives in bicomplex analysis extend the concept from standard calculus to four dimensions. For a bicomplex function, the partial derivatives can be calculated by differentiating the function with respect to each variable while holding others constant.

We can give the following theorem similar to the case with complex numbers and by considering the theorem given in [4].

Theorem 3.1. *We can write that*

$$\frac{\partial f}{\partial q} = \begin{bmatrix} \partial f_1/\partial t & \partial f_2/\partial t & \partial f_3/\partial t & \partial f_4/\partial t \\ \partial f_1/\partial x & \partial f_2/\partial x & -\partial f_3/\partial x & -\partial f_4/\partial x \\ -\partial f_1/\partial y & \partial f_2/\partial y & \partial f_3/\partial y & \partial f_4/\partial y \\ \partial f_1/\partial z & -\partial f_2/\partial z & \partial f_2/\partial z & \partial f_4/\partial z \end{bmatrix}$$

where $f = f_1 + if_2 + jf_3 + kf_4$ *is a bicomplex function whose components are bicomplex number valued functions and* $q = t + ix + jy + kz$ *is a bicomplex number.*

Proof. We can write

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial q}\frac{\partial q}{\partial t} = \frac{\partial f}{\partial q}1 = f\,'(q)\\ &\Longrightarrow \;f\,'(q) = \frac{\partial f_1}{\partial t} + i\frac{\partial f_2}{\partial t} + j\frac{\partial f_3}{\partial t} + k\frac{\partial f_4}{\partial t}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = \frac{\partial f}{\partial q}i = f\,'(q)i\\ &\Longrightarrow \;f\,'(q) = \frac{-\partial f}{\partial x}i = \frac{\partial f_2}{\partial x} - i\frac{\partial f_1}{\partial x} - j\frac{\partial f_4}{\partial x} + k\frac{\partial f_3}{\partial x}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = \frac{\partial f}{\partial q}j = f\,'(q)j\\ &\Longrightarrow \;f\,'(q) = \frac{-\partial f}{\partial y}j = \frac{\partial f_3}{\partial y} + i\frac{\partial f_4}{\partial y} - j\frac{\partial f_1}{\partial y} - k\frac{\partial f_2}{\partial y}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{\partial f}{\partial q}\frac{\partial q}{\partial z} = \frac{\partial f}{\partial q}k = f\,'(q)k\\ &\Longrightarrow \;f\,'(q) = \frac{\partial f}{\partial z}k = \frac{\partial f_4}{\partial z} - j\frac{\partial f_2}{\partial z} - i\frac{\partial f_3}{\partial z} + k\frac{\partial f_1}{\partial z}\end{aligned}$$

equations. Here, coefficients are equal. Also,

$$T_q = \begin{bmatrix} t & x & y & z \\ -x & t & -z & y \\ -y & -z & t & x \\ z & -y & -x & t \end{bmatrix}$$

is the matrix representation of bicomplex number and

$$\frac{\partial f}{\partial t} = \frac{\partial f_1}{\partial t} + i\frac{\partial f_2}{\partial t} + j\frac{\partial f_3}{\partial t} + k\frac{\partial f_4}{\partial t}$$

is bicomplex number derivative. We can write that

$$\begin{aligned}T_{f\,'} &= \begin{bmatrix} \partial f_1/\partial t & \partial f_2/\partial t & \partial f_3/\partial t & \partial f_4/\partial t \\ -\partial f_2/\partial t & \partial f_1/\partial t & -\partial f_4/\partial t & \partial f_3/\partial t \\ -\partial f_3/\partial t & -\partial f_4/\partial t & \partial f_1/\partial t & \partial f_2/\partial t \\ \partial f_4/\partial t & -\partial f_3/\partial t & -\partial f_2/\partial t & \partial f_1/\partial t \end{bmatrix}\\ &= \begin{bmatrix} \partial f_1/\partial t & \partial f_2/\partial t & \partial f_3/\partial t & \partial f_4/\partial t \\ \partial f_1/\partial x & \partial f_2/\partial x & -\partial f_3/\partial x & -\partial f_4/\partial x \\ -\partial f_1/\partial y & \partial f_2/\partial y & \partial f_3/\partial y & \partial f_4/\partial y \\ \partial f_1/\partial z & -\partial f_2/\partial z & \partial f_2/\partial z & \partial f_4/\partial z \end{bmatrix}\end{aligned}$$

(See [2] for similar operations). Thus, proof is complete. □

4. Gradient for Bicomplex Number Valued Functions

To calculate the gradient of a bicomplex function, we differentiate the function with respect to each variable (a, b, c, and d) independently. The resulting gradient vector provides the directional derivative along each axis.

Now let's replace these values in partial derivatives of the function f. Using these values in the partial derivatives of the function f,

$$
\begin{aligned}
\frac{df}{dq} &= \frac{\partial f}{\partial t}\frac{\partial t}{\partial q} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial q} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial q} \\
&= \frac{\partial f}{\partial t}\frac{1}{4} + \frac{\partial f}{\partial x}\frac{(-i)}{4} + \frac{\partial f}{\partial y}\frac{(-j)}{4} + \frac{\partial f}{\partial z}\frac{k}{4} \\
&= \frac{1}{4}(\frac{\partial f}{\partial t} - i\frac{\partial f}{\partial x} - j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}) \\
\frac{df}{dq^i} &= \frac{\partial f}{\partial t}\frac{\partial t}{\partial q^i} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial q^i} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial q^i} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial q^i} \\
&= \frac{\partial f}{\partial t}\frac{1}{4} + \frac{\partial f}{\partial x}\frac{1}{4i} + \frac{\partial f}{\partial y}\frac{1}{4j} + \frac{\partial f}{\partial z}\frac{k}{4} \\
&= \frac{1}{4}(\frac{\partial f}{\partial t} - i\frac{\partial f}{\partial x} - j\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}k) \\
\frac{df}{dq^j} &= \frac{\partial f}{\partial t}\frac{\partial t}{\partial q^j} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial q^j} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial q^j} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial q^j} \\
&= \frac{\partial f}{\partial t}\frac{1}{4} + \frac{\partial f}{\partial x}\frac{1}{4i} + \frac{\partial f}{\partial y}\frac{1}{4j} + \frac{\partial f}{\partial z}\frac{k}{4} \\
&= \frac{1}{4}(\frac{\partial f}{\partial t} - i\frac{\partial f}{\partial x} - j\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}k) \\
\frac{df}{dq^k} &= \frac{\partial f}{\partial t}\frac{\partial t}{\partial q^k} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial q^k} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial q^k} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial q^k} \\
&= \frac{\partial f}{\partial t}\frac{(-1)}{4} - \frac{\partial f}{\partial x}\frac{1}{4i} - \frac{\partial f}{\partial y}\frac{1}{4j} - \frac{\partial f}{\partial z}\frac{k}{4} \\
&= \frac{1}{4}(-\frac{\partial f}{\partial t} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}k)
\end{aligned}
$$

equations can be written.It is obtained that

$$
\begin{bmatrix} \frac{\partial f(q,q^i,q^j,q^k)}{\partial q} \\ \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^i} \\ \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^j} \\ \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^k} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & -i & -j & k \\ 1 & -i & -j & k \\ 1 & -i & -j & k \\ -1 & i & j & -k \end{bmatrix}\begin{bmatrix} \frac{df}{dt} \\ \frac{df}{dx} \\ \frac{df}{dy} \\ \frac{df}{dz} \end{bmatrix}
$$

in matrix form. It can be written that

$$
\begin{bmatrix} \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^*} \\ \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^{i*}} \\ \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^{j*}} \\ \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^{k*}} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & i & j & -k \\ 1 & i & j & -k \\ 1 & i & j & -k \\ -1 & -i & -j & k \end{bmatrix}\begin{bmatrix} \frac{df}{dt} \\ \frac{df}{dx} \\ \frac{df}{dy} \\ \frac{df}{dz} \end{bmatrix}
$$

for conjugate. Here,

$$\nabla f = \begin{bmatrix} \frac{df}{dt} \\ \frac{df}{dx} \\ \frac{df}{dy} \\ \frac{df}{dz} \end{bmatrix}$$

is the gradient of f.

5. Conclusion

Bicomplex numbers extend the complex number system by introducing an additional imaginary unit, resulting in a four-dimensional algebraic structure. The commutative-quaternion algebra adds the property of commutativity to the quaternion algebra, allowing for a more versatile mathematical framework. The study of bicomplex numbers, specifically commutative-quaternions, offers a fascinating exploration into the properties of complexified quaternions with commutative multiplication. Understanding the gradient and partial derivatives within this mathematical framework is crucial for analyzing the behavior of bicomplex functions.

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